p-adic Beilinson conjecture for ordinary Hecke motives associated to imaginary quadratic fields

By

Kenichi BANNAI* and GUIDO KINGS**

Abstract

The purpose of this article is to show that the result of [BK] may be used to prove the p-adic Beilinson conjecture at non-critical points of motives associated to Hecke characters of an imaginary quadratic field K, for a prime p which splits in K. For simplicity, we assume in this article that the imaginary quadratic field K has class number *one* and that the Hecke character ψ we consider corresponds to an elliptic curve with complex multiplication defined over $\mathbb{Q}$.

§ 1. Introduction

The purpose of this article is to show that the result of [BK] may be used to prove the p-adic Beilinson conjecture of motives associated to Hecke characters of an imaginary quadratic field K, for a prime p which splits in K. The p-adic L-function for such p interpolating *critical values* of L-functions of Hecke characters associated to imaginary quadratic fields was first constructed by Vishik and Manin [VM], and a different construction using p-adic Eisenstein series was given by Katz [Katz]. The p-adic Beilinson conjecture, as formulated by Perrin-Riou in [PR], gives a precise conjecture concerning the *non-critical values* of p-adic L-functions associated to general motives. The purpose of our research is to investigate the interpolation property at non-critical points of the p-adic L-function constructed by Vishik-Manin and Katz.

For simplicity, we assume in this article that the imaginary quadratic field K has class number *one* and that the Hecke character ψ we consider corresponds to an elliptic

curve with complex multiplication defined over $\mathbb{Q}$. Let a be an integer > 0. The main theorem of this article (Theorem 6.8) is a proof of the p-adic Beilinson conjecture for ψ^a (see Conjecture 2.2), when the prime $p \geq 5$ is an *ordinary* prime. A more general case will be treated in a future article. The authors would like to thank the organizers Takashi Ichikawa, Masanari Kida and Takao Yamazaki for the opportunity to present our research at the RIMS "Algebraic Number Theory and Related Topics 2009" conference.

§2. The p-adic Beilinson conjecture

Assume that K is an imaginary quadratic field of *class number one*. Let E be an elliptic curve defined over $\mathbb{Q}$. We assume in addition that E has complex multiplication by the ring of integers $\mathcal{O}_K$ of K. We let $\psi := \psi_{E/K}$ be the Grossencharacter of K associated to $E_K := E \otimes_{\mathbb{Q}} K$ by the theory of complex multiplication, and we denote by $\mathfrak{f}$ the conductor of ψ.

We let $M(\psi)$ be the motive over K with coefficients in K associated to the Grossencharacter ψ. Then we have $M(\psi) = H^1(E_K)$, where $H^1(E_K)$ is the motive associated to E_K. The Hasse-Weil L-function of $M(\psi)$ is a function with values in $K \otimes_{\mathbb{Q}} \mathbb{C}$ given by

$$L(M(\psi), s) = (L(\psi_\tau, s))_{\tau : K \hookrightarrow \mathbb{C}},$$

where $\tau : K \hookrightarrow \mathbb{C}$ are the embeddings of the coefficient K of $M(\psi)$ into $\mathbb{C}$ and $L(\psi_\tau, s)$ is the Hecke L-function

$$L(\psi_\tau, s) = \prod_{(\mathfrak{q}, \mathfrak{f}) = 1} \left(1 - \frac{\psi_\tau(\mathfrak{q})}{N\mathfrak{q}^s} \right)^{-1}$$

associated to the character $\psi_\tau : \mathbb{A}_K^\times \xrightarrow{\psi} K \xrightarrow{\tau} \mathbb{C}$. Here, the product is over the prime ideals $\mathfrak{q}$ of K which are prime to $\mathfrak{f}$.

For integers $a > 0$ and n, we let $M^a = M(\psi^a) := M(\psi)^{\otimes_K a}$, which is a motive over K with coefficients in K. Then the Hasse-Weil L-function $L(M^a, s)$ is given by the Hecke L-function

$$L(M^a, s) = (L(\psi_\tau^a, s))_{\tau : K \hookrightarrow \mathbb{C}}$$

with values in $K \otimes_{\mathbb{Q}} \mathbb{C}$. We let M_B^a be the Betti realization of M^a, which is a K-vector space of dimension *one*. We fix a K-basis ω_B^a of M_B^a. The de Rham realization $M_{\mathrm{dR}}^a(n)$ of $M^a(n)$ is the rank one $K \otimes_{\mathbb{Q}} K$-module

$$M_{\mathrm{dR}}^a(n) = K\omega^{n-a,n} \bigoplus K\omega^{n,n-a},$$

with Hodge filtration given by

$$F^m M_{\mathrm{dR}}^a(n) = \begin{cases} M_{\mathrm{dR}}^a(n) & m \le -n \\ K\omega^{n,n-a} & -n < m \le a-n \\ 0 & \text{otherwise.} \end{cases}$$

In what follows, we consider the case when $n > a$, which implies in particular that our motive is *non-critical*. We have in this case $F^0 M_{\mathrm{dR}}^a(n) = 0$. The tangent space of our motive is given by

$$t_n^a := M_{\mathrm{dR}}^a(n)/F^0 M_{\mathrm{dR}}^a(n) \cong M_{\mathrm{dR}}^a(n),$$

which is again a $K \otimes_{\mathbb{Q}} K$-module of rank one. Note that $\omega_{\mathrm{tg},n}^a := \omega^{n-a,n} + \omega^{n,n-a}$ gives a basis of t_n^a as a $K \otimes_{\mathbb{Q}} K$-module.

We denote by $V_\infty^a(n)$ the $\mathbb{R}$-Hodge realization of $M^a(n)$. The Beilinson-Deligne cohomology $H_{\mathscr{D}}^1(K \otimes_{\mathbb{Q}} \mathbb{R}, V_\infty^a(n))$ is given as the cokernel of the natural inclusion

$$M_B^a(n) \otimes_{\mathbb{Q}} \mathbb{R} \to t_n^a \otimes_{\mathbb{Q}} \mathbb{R}.$$

The Beilinson regulator map gives a homomorphism

$$(2.1) \qquad r_\infty : H_{\mathrm{mot}}^1(K, M^a(n)) \to H_{\mathscr{D}}^1(K \otimes_{\mathbb{Q}} \mathbb{R}, V_\infty^a(n)),$$

from the motivic cohomology $H_{\mathrm{mot}}^1(K, M^a(n))$ of K with coefficients in $M^a(n)$ to $H_{\mathscr{D}}^1(K \otimes_{\mathbb{Q}} \mathbb{R}, V_\infty^a(n))$. Then $r_\infty \otimes_{\mathbb{Q}} \mathbb{R}$ is known to be surjective and is conjectured to be an isomorphism. We let c_n^a be an element of $H_{\mathrm{mot}}^1(K, M^a(n))$ such that $r_\infty(c_n^a)$ generates $H_{\mathscr{D}}^1(K_{\mathfrak{p}}, V_\infty^a(n))$ as a $K_\infty := K \otimes_{\mathbb{Q}} \mathbb{R}$-module. We define the complex period $\Omega_\infty(n)$ of $M^a(n)$ to be the determinant of the exact sequence

$$(2.2) \qquad 0 \to M_B^a(n) \otimes_{\mathbb{Q}} \mathbb{R} \to t_n^a \otimes_{\mathbb{Q}} \mathbb{R} \to H_{\mathscr{D}}^1(K \otimes_{\mathbb{Q}} \mathbb{R}, V_\infty^a(n)) \to 0$$

for the basis $r_\infty(c_n^a)$, $\omega_{\mathrm{tg},n}^a$, and ω_B^a. The complex period is an element in K_∞ and is independent of the choice of the basis up to multiplication by an element in $K^\times$. The value $L(M^a, n)$ is in $K \otimes_{\mathbb{Q}} \mathbb{R}$, and the weak Beilinson conjecture for $M^a(n)$ as proved by Deninger [De1] gives the following (see Theorem 6.2 and Corollary 6.4 for the precise statement.)

Theorem 2.1. *For any integer $n > a$, the value*

$$\frac{L(M^a, n)}{\Omega_\infty(n)}$$

is an element in $K^\times$.

For any prime p, the étale realization $V_p^a(n)$ of our motive is a $K \otimes_{\mathbb{Q}} \mathbb{Q}_p$-vector space with continuous action of $\mathrm{Gal}(\overline{K}/K)$. We fix a prime $p \geq 5$ relatively prime to $\mathfrak{f}$ such that E has good *ordinary* reduction at p. In this case, the ideal generated by p splits as $(p) = \mathfrak{p}\mathfrak{p}^*$ in K. We fix a prime ideal $\mathfrak{p}$ of K above p. Then the Bloch-Kato exponential map gives an isomorphism

$$(2.3) \qquad \exp_p : t_n^a \otimes_K K_{\mathfrak{p}} \xrightarrow{\cong} H_f^1(K_{\mathfrak{p}}, V_p^a(n)),$$

and the inverse of this isomorphism is denoted by $\log_p$. The p-adic étale regulator map gives a homomorphism

$$(2.4) \qquad r_p : H_{\mathrm{mot}}^1(K, M^a(n)) \to H_f^1(K_{\mathfrak{p}}, V_p^a(n)),$$

and the map $r_p \otimes \mathbb{Q}_p$ is conjectured to be an isomorphism. We define the p-adic period $\Omega_p(n)$ of $M^a(n)$ to be an element in $K_p := K \otimes_{\mathbb{Q}} K_{\mathfrak{p}} \cong K_{\mathfrak{p}} \bigoplus K_{\mathfrak{p}^*}$ satisfying

$$(2.5) \qquad \log_p \circ r_p(c_n^a) = \Omega_p(n)\, \omega_{\mathrm{tg},n}^a.$$

The p-adic period $\Omega_p(n)$ is independent of the choice of basis up to multiplication by an element in $K^{\times}$. If we assume that the p-adic regulator map r_p is *injective*, then $r_p(c_n^a)$ is non-zero, hence the p-adic period may be interpreted in this case as the determinant of the map $\log_p$ for the basis $r_p(c_n^a)$ and $\omega_{\mathrm{tg},n}^a$.

Remark. It is conjectured that the p-adic regulator r_p is injective, which would imply that the p-adic period $\Omega_p(n)$ is non-zero. Kato has proved in [Kato] 15.15 the weak Leopoldt conjecture for any Hecke character of K. Hence by a result of Jannsen ([Jan], Lemma 8), we may then conclude that

$$H^2(\mathcal{O}_K[1/p\mathfrak{f}_a], V_p^a(n)) = 0$$

for almost all n. This implies that for such n, the map from global to local étale cohomology groups is injective, hence that $\Omega_p(n)$ is non-zero.

The p-adic Beilinson conjecture as formulated by Perrin-Riou (see [Col] Conjecture 2.7) specialized to our setting is given as follows.

Conjecture 2.2. *Let a be an integer > 0. Then there exists a p-adic pseudo-measure μ^a on $\mathbb{Z}_p$ with values in K_p such that the value*

$$L_p(\psi^a \otimes \chi_{\mathrm{cyc}}^n) := \int_{\mathbb{Z}_p^{\times}} w^n \mu^a(w)$$

in K_p for any integer $n > a$ satisfies

$$\frac{L_p(\psi^a \otimes \chi_{\mathrm{cyc}}^n)}{\Omega_p(n)} = \left(1 - \frac{\psi(\mathfrak{p})^a}{p^n}\right)\left(1 - \frac{\overline{\psi}(\mathfrak{p}^*)^a}{p^{a+1-n}}\right)\frac{\Gamma(n)L(\psi^a, n)}{\Omega_{\infty}(n)},$$

where $\mathfrak{p}$ is a fixed prime in K above p.

If $\mathfrak{f}_a \neq (1)$ for the conductor $\mathfrak{f}_a$ of ψ^a, then μ^a should in fact be a p-adic measure. Note that the dependence of the pseudo-measure on the choices of the basis $\omega^a_{\mathrm{tg},n}$ and c^a_n cancel, where as the pseudo-measure depends on the choice of the basis ω^a_B. The main goal of our research is to prove that the p-adic measure constructed by Vishik-Manin and Katz gives the pseudo-measure of the above conjecture when the prime p is split in K.

The main theorem of this article (Theorem 6.8) is the proof of the above conjecture for integers $n > a$ such that the corresponding p-adic period $\Omega_p(n)$ is non-zero.

§3.   Construction of the Eisenstein class

The main difficulty in the proof of the Beilinson and p-adic Beilinson conjectures is to construct the element $c^a_n \in H^1_{\mathrm{mot}}(K, M^a(n))$ for $M^a := M(\psi^a)$ and to calculate the images $r_\infty(c^a_n)$ and $r_p(c^a_n)$ with respect to the Beilinson-Deligne and p-adic regulator maps. We will use the Eisenstein symbol as constructed by Beilinson.

We fix an integer $N \geq 3$, and let $M(N)$ be the modular curve defined over $\mathbb{Z}[1/N]$ parameterizing for any scheme S over $\mathbb{Z}[1/N]$ the pair (E, ν), where E is an elliptic curve over S and

$$\nu : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\cong} E[N]$$

is a full level N-structure on E, where $E[N]$ is the group of N-torsion points of E. We let $\mathrm{pr} : \widetilde{E} \to M$ be the universal elliptic curve over M with universal level N-structure $\widetilde{\nu} : (\mathbb{Z}/N\mathbb{Z})^2 \cong \widetilde{E}[N]$, and consider the motivic sheaf $\mathbb{Q}(1)$ on $\widetilde{E}$. We let

$$(3.1) \qquad\qquad \mathscr{H} := R^1\mathrm{pr}_*\mathbb{Q}(1),$$

and we denote by $\mathrm{Sym}^k \mathscr{H}$ the k-th symmetric product of $\mathscr{H}$. Let $\varphi = \sum_{\rho \in \widetilde{E}[N] \setminus \{0\}} a_\rho [\rho]$ be a $\mathbb{Q}$-linear sum of non-zero elements in $\widetilde{E}[N]$. For any integer $k > 0$, the Eisenstein class $\mathrm{Eis}^{k+2}_{\mathrm{mot}}(\varphi)$ is an element

$$(3.2) \qquad\qquad \mathrm{Eis}^{k+2}_{\mathrm{mot}}(\varphi) \in H^1_{\mathrm{mot}}(M, \mathrm{Sym}^k \mathscr{H}(1)).$$

Although the formalism of mixed motivic sheaves or motivic cohomology with coefficients have not yet been fully developed, one can give meaning to the above sheaves and cohomology (see [BL], [BK] for details).

Then the class c^a_n may be constructed from the Eisesntein class as follows. Let K be an imaginary quadratic field of class number one, and let E be an elliptic curve defined over $\mathbb{Q}$ with complex multiplication by the ring of integers $\mathcal{O}_K$ of K. We denote again by ψ the Hecke character of K corresponding to E_K with conductor $\mathfrak{f}$. We take $N \geq 3$ such that N is divisible by $\mathfrak{f}$. For the extension $F := K(E[N])$ of K generated by the coordinates of the points in $E[N]$, we let $G_{F/K} := \mathrm{Gal}(F/K)$ the Galois group of

F over K. We fix a level N-structure $\nu : (\mathbb{Z}/N\mathbb{Z})^2 \cong E[N]$ of E over F, and we denote by ν^σ the composition of ν with the action of $\sigma \in G_{F/K}$. Then for any $\sigma \in G_{F/K}$, we denote by $\iota^{\sigma*}$ the pull-back with respect to the F-valued point $\iota^\sigma : \operatorname{Spec} F \to M$ of M corresponding to (E, ν^σ). Then the image of the sum $\iota^* := \sum_{\sigma \in G_{F/K}} \iota^{\sigma*}$ is invariant by the action of the Galois group, hence gives a pull-back morphism

$$H^1_{\mathrm{mot}}(M, \operatorname{Sym}^k \mathscr{H}(1)) \xrightarrow{\iota^*} H^1_{\mathrm{mot}}(K, \operatorname{Sym}^k \iota^* \mathscr{H}(1)).$$

Note that on $\operatorname{Spec} K$, the motivic sheaf $\iota^* \mathscr{H}$ is given by the motive $H^1(E)(1)$, which by definition corresponds to the motive $M(\psi)(1)$. The structure of K-coefficients on $\iota^* \mathscr{H}$ gives the following decomposition.

Lemma 3.1. *For integers j satisfying $0 \leq j \leq k/2$, we have the decomposition of motives*

$$\operatorname{Sym}^k \iota^* \mathscr{H} = \bigoplus_{0 \leq j \leq \frac{k}{2}} M(\psi^{k-2j})(k-j),$$

where we take the convention that for $k = 2j$, we let $M(\psi^0)(k/2)$ be the Tate motive $\mathbb{Q}(k/2)$ with coefficients in $\mathbb{Q}$.

Let $a > 0$ be an integer and we let $\mathfrak{f}_a$ be the conductor of ψ^a. We let $F_a := K(E[\mathfrak{f}_a])$ be the extension of K generated by the coordinates of the points in $E[\mathfrak{f}_a]$, and we let w_{F/F_a} be the order of the Galois group $\operatorname{Gal}(F/F_a)$. The Eisenstein classes $\operatorname{Eis}^{k+2}_{\mathrm{mot}}(\rho)$ are defined for points $\rho \in \widetilde{E}[N] \setminus \{0\}$ but is *not* defined for $\rho = 0$. Hence in defining c_n^a, we differentiate between the case when $\mathfrak{f}_a \neq (1)$ and $\mathfrak{f}_a = (1)$.

Definition 3.2. We define φ_a as follows.

1. If $\mathfrak{f}_a \neq (1)$, then we fix a primitive $\mathfrak{f}_a$-torsion point ρ_a of E and let

$$\varphi_a := \frac{1}{w_a w_{F/F_a}} [\rho_a],$$

where we denote again by ρ_a the N-torsion point of $\widetilde{E}$ corresponding to ρ_a through ν and $\widetilde{\nu}$, and w_a is the number of units in $\mathcal{O}_K$ which are congruent to *one* modulo $\mathfrak{f}_a$.

2. If $\mathfrak{f}_a = (1)$, then we let

$$\varphi_a := \frac{1}{w_a w_{F/F_a}} \sum_{\rho \in \widetilde{E}[N] \setminus \{0\}} [\rho].$$

We define the class c_n^a as follows.

Definition 3.3. For any integer a, n such that $n > a > 0$, we let $k = 2n - a - 2$. Then the motive $M^a(n) := M(\psi^a)(n)$ is a direct summand of $\mathrm{Sym}^k \iota^* \mathscr{H}(1)$. We define the motivic class c_n^a to be the image of $\mathrm{Eis}_{\mathrm{mot}}^{k+2}(\varphi_a)$ with respect to the projection

$$H_{\mathrm{mot}}^1(K, \mathrm{Sym}^k \iota^* \mathscr{H}(1)) \to H_{\mathrm{mot}}^1(K, M^a(n)),$$

where φ_a is as in Definition 3.2.

Let p be a rational prime which does not divide $\mathfrak{f}$, and we take $N \geq 3$ to be an integer divisible by $\mathfrak{f}$ and prime to p. In order to prove the p-adic Beilinson conjecture, it is necessary to calculate the images of c_n^a with respect to the Beilinson-Deligne and p-adic regulator maps. The image $r_\infty(c_n^a)$ by the Beilinson-Deligne regulator map was calculated by Deninger [De1]. We will calculate the image $r_p(c_n^a)$ by the p-adic regulator map using rigid syntomic cohomology.

Denote by $M_{\mathrm{cris}}^a(n)$ the crystalline realization of $M^a(n)$, which is a filtered module with a σ-linear action of Frobenius, and let $H_{\mathrm{syn}}^1(K_\mathfrak{p}, M_{\mathrm{cris}}^a(n))$ be the syntomic cohomology of $K_\mathfrak{p}$ with coefficients in $M_{\mathrm{cris}}^a(n)$. Then noting that $t_n^a \otimes_\mathbb{Q} \mathbb{Q}_p = M_{\mathrm{cris}}^a(n)$ in this case, there exists a canonical isomorphism

$$(3.3) \qquad t_n^a \otimes_\mathbb{Q} \mathbb{Q}_p \xrightarrow{\cong} H_{\mathrm{syn}}^1(K_\mathfrak{p}, M_{\mathrm{cris}}^a(n)).$$

If we let $V_p^a(n)$ be the p-adic étale realization of $M^a(n)$, then we have a canonical isomorphism

$$(3.4) \qquad H_{\mathrm{syn}}^1(K_\mathfrak{p}, M_{\mathrm{cris}}^a(n)) \xrightarrow{\cong} H_f^1(K_\mathfrak{p}, V_p^a(n)),$$

which combined with (3.3) gives the exponential map (2.3). The syntomic regulator map

$$r_{\mathrm{syn}} : H_{\mathrm{mot}}^1(K, M^a(n)) \to H_{\mathrm{syn}}^1(K_\mathfrak{p}, M_{\mathrm{cris}}^a(n))$$

defined by Besser ([Bes] §7) is compatible with the p-adic regulator r_p through the isomorphism (3.4) ([Bes] Proposition 9.9, see also [Ni]). Therefore, in order to calculate $\log_p \circ r_p(c_n^a)$, it is sufficient to calculate the image of $r_{\mathrm{syn}}(c_n^a)$ with respect to (3.3). We will calculate this image using the explicit determination of the syntomic Eisenstein class given in [BK].

§ 4. Eisenstein class and p-adic Eisenstein series

In this section, we review the explicit description of the syntomic Eisenstein class in terms of p-adic Eisenstein series given in [BK]. Let $M := M(N)$ be the modular curve over $\mathbb{Z}[1/N]$ given in the previous section. We will first describe a certain real analytic Eisenstein series $E_{k+2,l,\varphi}^\infty$.

Let $\Gamma \subset \mathbb{C}$ be a lattice, and we denote by A the area of the fundamental domain of Γ divided by $\boldsymbol{\pi} := 3.14159\cdots$. For any integer a and complex number s satisfying $\operatorname{Re}(s) > a/2 + 1$, the Eisenstein-Kronecker-Lerch series $K_a^*(z, w, s; \Gamma)$ to be the series

$$K_a^*(z, w, s; \Gamma) := \sum_{\gamma \in \Gamma}^* \frac{(\overline{z} + \overline{\gamma})^a}{|z + \gamma|^{2s}} \langle \gamma, w \rangle$$

where $\sum^*$ denotes the sum over $\gamma \in \Gamma$ satisfying $\gamma \neq -z$ and $\langle z, w \rangle := \exp((\overline{w}z - w\overline{z})/A)$. By [Wei] VIII §12 (see [BKT] Proposition 2.4 for the case $a < 0$), this series for s continues meromorphically to a function on the whole s-plane, holomorphic except for a simple pole at $s = 1$ when $a = 0$ and $w \in \Gamma$. This function satisfies the functional equation

$$(4.1) \qquad \Gamma(s)K_a^*(z, w, s; \Gamma) = A^{a+1-2s}\Gamma(a + 1 - s)K_a^*(w, z, a + 1 - s)\langle w, z \rangle.$$

We fix a level N-structure $\nu : (\mathbb{Z}/N\mathbb{Z})^2 \cong \frac{1}{N}\Gamma/\Gamma$, and let $\rho \in \frac{1}{N}\Gamma/\Gamma$. For integers k and l, we define the real analytic Eisenstein series $E_{k+2,l,\rho}^\infty$ to be the modular form on $M_\mathbb{C} := M(N) \otimes_\mathbb{Q} \mathbb{C}$ whose value at the test object $(\mathbb{C}/\Gamma, dz, \nu)$ is given by

$$(4.2) \qquad E_{k+2,l,\rho}^\infty(\mathbb{C}/\Gamma, dz, \nu) := A^{-l}\, \Gamma(s)K_{k+l+2}^*(0, \rho, s; \Gamma)\big|_{s=k+2}\,.$$

We let $E_{k+2,l,\varphi}^\infty := \sum_\rho a_\rho E_{k+2,l,\rho}^\infty$ for the $\mathbb{Q}$-linear sum $\varphi = \sum_\rho a_\rho[\rho]$.

When $l = 0$, then $E_{k+2,0,\varphi}^\infty$ is a holomorphic Eisenstein series of weight $k + 2$ on $M_\mathbb{C}$. From the q-expansion, we see in this case that this Eisenstein series is defined over $\mathbb{Q}$, and hence defines a section $E_{k+2,0,\varphi}$ in $\Gamma(M_\mathbb{Q}, \omega^{\otimes k} \otimes \Omega_{M_\mathbb{Q}}^1)$ for $\omega := \operatorname{pr}_*\Omega_{\widetilde{E}/M}^1$. Denote by $\mathscr{H}_{\mathrm{dR}}$ the de Rham realization of $\mathscr{H}$, which is the coherent $\mathcal{O}_{M_\mathbb{Q}}$-module $R^1\operatorname{pr}_*\Omega_{\widetilde{E}}^\bullet$ with Gauss-Manin connection

$$\nabla : \mathscr{H}_{\mathrm{dR}} \to \mathscr{H}_{\mathrm{dR}} \otimes \Omega_{M_\mathbb{Q}}^1,$$

and let $\operatorname{Sym}^k\mathscr{H}_{\mathrm{dR}}$ be the k-th symmetric product of $\mathscr{H}_{\mathrm{dR}}$ with the induced connection. From the natural inclusion $\omega^{\otimes k} \hookrightarrow \operatorname{Sym}^k\mathscr{H}_{\mathrm{dR}}$, we see that $E_{k+2,0,\varphi}$ defines a section in $\Gamma(M_\mathbb{Q}, \operatorname{Sym}^k\mathscr{H}_{\mathrm{dR}} \otimes \Omega_{M_\mathbb{Q}}^1)$.

Let p be a prime number not dividing N. We denote by $\mathscr{H}_{\mathrm{rig}}$ the filtered overconvergent F-isocrystal associated to $\mathscr{H}$ on $M_{\mathbb{Z}_p}$, which is given by $\mathscr{H}_{\mathrm{dR}}$ with an additional structure of Hodge filtration and Frobenius. Let $H_{\mathrm{syn}}^1(M_{\mathbb{Z}_p}, \operatorname{Sym}^k\mathscr{H}_{\mathrm{rig}}(1))$ be the rigid syntomic cohomology of $M_{\mathbb{Z}_p}$ with coefficients in $\operatorname{Sym}^k\mathscr{H}_{\mathrm{rig}}(1)$. The rigid syntomic regulator is a map

$$r_{\mathrm{syn}} : H_{\mathrm{mot}}^1(M, \operatorname{Sym}^k\mathscr{H}(1)) \to H_{\mathrm{syn}}^1(M_{\mathbb{Z}_p}, \operatorname{Sym}^k\mathscr{H}_{\mathrm{rig}}(1)),$$

and we define the syntomic Eisenstein class $\operatorname{Eis}_{\mathrm{syn}}^{k+2}(\varphi)$ to be the image by the syntomic regulator of the motivic Eisenstein class. We let $M_{\mathbb{Z}_p}^{\mathrm{ord}}$ be the ordinary locus in $M_{\mathbb{Z}_p}$, and

$M^{\mathrm{ord}}_{\mathbb{Q}_p} := M^{\mathrm{ord}}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. By [BK] Proposition A.16, a class in $H^1_{\mathrm{syn}}(M^{\mathrm{ord}}_{\mathbb{Z}_p}, \mathrm{Sym}^k \mathscr{H}_{\mathrm{rig}}(1))$ is given by a pair (α, ξ) of sections

$$
(4.3) \qquad
\begin{aligned}
\alpha &\in \Gamma(M^{\mathrm{ord}}_{\mathbb{Q}_p}, j^{\dagger}\mathrm{Sym}^k \mathscr{H}_{\mathrm{rig}}(1)) \\
\xi &\in \Gamma(M^{\mathrm{ord}}_{\mathbb{Q}_p}, \mathrm{Sym}^k \mathscr{H}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \Omega^1_{M^{\mathrm{ord}}})
\end{aligned}
$$

satisfying $\nabla(\alpha) = (1 - \phi^*)\xi$. The α for the class (α, ξ) corresponding to the restriction to the ordinary locus of the syntomic Eisenstein class $\mathrm{Eis}^{k+2}_{\mathrm{syn}}(\varphi)$ is given as follows.

We let $p \geq 5$ be a prime not dividing N, and we let $\mathcal{M}$ be the p-adic modular curve defined over $\mathbb{Z}_p$ parameterizing the triples (E_B, η, ν) consisting of an elliptic curve E_B over a p-adic ring B, an isomorphism

$$
(4.4) \qquad \eta : \widehat{\mathbb{G}}_m \cong \widehat{E}_B
$$

of formal groups over B, and a level N-structure ν. The ring of p-adic modular forms $V_p(\mathbb{Q}_p, \Gamma(N))$ is defined as the global section

$$
V_p(\mathbb{Q}_p, \Gamma(N)) := \Gamma(\mathcal{M}, \mathcal{O}_{\mathcal{M}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.
$$

The q-expansion gives an injection

$$
V_p(\mathbb{Q}_p, \Gamma(N)) \hookrightarrow \mathbb{Q}_p(\zeta_N)[[q]].
$$

There exists a Frobenius action ϕ^* on $V_p(\mathbb{Q}_p, \Gamma(N))$ given on the q-expansion as $\phi^* = \mathrm{Frob} \otimes \sigma$, where $\mathrm{Frob}(q) = q^p$ and σ is the the absolute Frobenius acting on $\mathbb{Q}_p(\zeta_N)$. The Eisenstein series $E_{k+2,0,\varphi}$ naturally defines an element in $V_p(\mathbb{Q}_p, \Gamma(N))$, and using the fact that the differential $\partial_{\log q} := q\frac{d}{dq}$ preserves the space of p-adic modular forms, we let for any integer $l \geq 0$

$$
E_{k+l+2,l,\varphi} := \partial^l_{\log q} E_{k+2,0,\varphi}.
$$

We let $E^{(p)}_{k+2,0,\varphi} := (1 - \phi^*)E_{k+2,0,\varphi}$ and $E^{(p)}_{k+l+2,l,\varphi} := \partial^l_{\log q} E^{(p)}_{k+2,0,\varphi}$ for any integer $l \geq 0$. Then the calculation of the q-expansion shows that we have

$$
(4.5) \qquad E^{(p)}_{k+l+2,l,\varphi} = (1 - p^l \phi^*)E_{k+l+2,l,\varphi}.
$$

Following the method of Katz [Katz], we may construct a p-adic measure on $\mathbb{Z}_p \times \mathbb{Z}_p^{\times}$ with values in $V_p(\mathbb{Q}_p, \Gamma(N))$ satisfying the following interpolation property.

Theorem 4.1.   *There exists a p-adic measure μ_φ on $\mathbb{Z}_p \times \mathbb{Z}_p^{\times}$ with values in $V_p(\mathbb{Q}_p, \Gamma(N))$ such that*

$$
\int_{\mathbb{Z}_p \times \mathbb{Z}_p^{\times}} x^{k+1} y^l \mu_\varphi(x, y) = E^{(p)}_{k+2,l,\varphi}
$$

for integers $k > 0$, $l \geq 0$.

Using this measure, we define $E^{(p)}_{k+2,l,\varphi}$ for $l < 0$ as follows.

Definition 4.2 (*p*-adic Eisenstein series).   Let k be an integer ≥ -1. We let

$$E^{(p)}_{k+2,l,\varphi} := \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} x^{k+1} y^l \mu_\varphi(x,y) \ \in V_p(\mathbb{Q}_p, \Gamma(N)),$$

where l is any integer in $\mathbb{Z}$.

The *p*-adic Eisenstein series satisfies the differential equation

$$\partial_{\log q} E^{(p)}_{k+2,l,\varphi} = E^{(p)}_{k+3,l+1,\varphi},$$

and the weight of $E^{(p)}_{k+2,l,\varphi}$ is $k+l+2$. The syntomic Eisenstein class may be described using these *p*-adic Eisenstein series. The moduli problem for $\mathcal{M}$ implies that there exists a universal trivialization

$$\eta : \widehat{\mathbb{G}}_m \cong \widehat{\widetilde{E}}$$

of the universal elliptic curve on $\mathcal{M}$, which gives rise to a canonical section $\widetilde{\omega}$ of $\underline{\omega} :=$ $\mathrm{pr}_* \Omega^1_{\widetilde{E}/\mathcal{M}}$ corresponding to the invariant differential $d\log(1+T)$ on $\widehat{\mathbb{G}}_m$. Since $\mathcal{M}$ is affine, there exists sections x and y of $\widetilde{E}$ such that the elliptic curve $\widetilde{E}_{\mathbb{Q}_p} := \widetilde{E} \otimes \mathbb{Q}_p$ is given by the Weierstrass equation

$$\widetilde{E}_{\mathbb{Q}_p} : y^2 = 4x^3 - g_2 x - g_3, \quad g_2, g_3 \in V_p(\mathbb{Q}_p, \Gamma(N))$$

satisfying $\widetilde{\omega} = dx/y$. Then the pull back of the F-isocrystal $\mathscr{H}_{\mathrm{rig}}$ to $\mathcal{M}_{\mathbb{Q}_p}$ is given as

$$\mathscr{H}_{\mathrm{rig}} = \mathcal{O}_{\mathcal{M}_{\mathbb{Q}_p}} \widetilde{\omega}^\vee \oplus \mathcal{O}_{\mathcal{M}_{\mathbb{Q}_p}} \widetilde{u}^\vee,$$

with connection $\nabla(\widetilde{u}^\vee) = \widetilde{\omega}^\vee \otimes d\log q$, $\nabla(\widetilde{\omega}^\vee) = 0$, Frobenius $\phi^*(\widetilde{\omega}^\vee) = p^{-1}\widetilde{\omega}^\vee$, $\phi^*(\widetilde{u}^\vee) = \widetilde{u}^\vee$ and Hodge filtration $\mathrm{Fil}^{-1}\mathscr{H}_{\mathrm{rig}} = \mathscr{H}_{\mathrm{rig}}$, $\mathrm{Fil}^0\mathscr{H}_{\mathrm{rig}} = \mathcal{O}_{\mathcal{M}_{\mathbb{Q}_p}}\widetilde{u}^\vee$, $\mathrm{Fil}^1\mathscr{H}_{\mathrm{rig}} = 0$ (See [BK] §4.3). If let $\widetilde{\omega}^{m,n} := \widetilde{\omega}^{\vee m}\widetilde{u}^{\vee n}$, then the filtered F-isocrystal $\mathrm{Sym}^k\mathscr{H}_{\mathrm{rig}}(1)$ on $\mathcal{M}_{\mathbb{Q}_p}$ is given by the coherent module

$$\mathrm{Sym}^k\mathscr{H}_{\mathrm{rig}}(1) = \bigoplus_{j=0}^{k} \mathcal{O}_{\mathcal{M}_{\mathbb{Q}_p}} \widetilde{\omega}^{k-j,j}(1)$$

with connection $\nabla(\widetilde{\omega}^{k-j,j}(1)) = j\,\widetilde{\omega}^{k-j+1,j-1}(1) \otimes d\log q$, Frobenius

$$\phi^*(\widetilde{\omega}^{,k-j,j}(1)) = p^{j-k-1}\widetilde{\omega}^{k-j,j}(1),$$

and Hodge filtration

$$\mathrm{Fil}^m(\mathrm{Sym}^k\mathscr{H}_{\mathrm{rig}}(1)) = \bigoplus_{j=m+k+1}^{k} \mathcal{O}_{\mathcal{M}_{\mathbb{Q}_p}}\widetilde{\omega}^{k-j,j}(1).$$

If we let $\widetilde{\alpha}_{\mathrm{Eis}}^{k+2}$ be the section

$$\widetilde{\alpha}_{\mathrm{Eis}}^{k+2}(\varphi) := \sum_{j=0}^{k} \frac{(-1)^{k-j}}{j!} E_{j+1,j-k-1,\varphi}^{(p)} \widetilde{\omega}^{k-j,j}(1),$$

then we have

$$\nabla(\widetilde{\alpha}_{\mathrm{Eis}}^{k+2}(\varphi)) = \frac{(1-\phi^*)E_{k+2,0,\varphi}}{k!} \widetilde{\omega}^{0,k}(1) \otimes d\log q.$$

The main result of [BK] is the following.

Theorem 4.3 ([BK] Theorem 5.11). *Let k be an integer > 0. The syntomic Eisenstein class*

$$\mathrm{Eis}_{\mathrm{syn}}^{k+2}(\varphi) \in H_{\mathrm{syn}}^1(M_{\mathbb{Z}_p}, \mathrm{Sym}^k \mathscr{H}_{\mathrm{rig}}(1))$$

restricted to the ordinary locus $H_{\mathrm{syn}}^1(M_{\mathbb{Z}_p}^{\mathrm{ord}}, \mathrm{Sym}^k \mathscr{H}_{\mathrm{rig}}(1))$ is represented by the pair (α, ξ) as in (4.3), where $\xi = E_{k+2,0,\varphi} \widetilde{\omega}^{0,k}(1)/k! \otimes d\log q$ and α is a section which maps to $\widetilde{\alpha}_{\mathrm{Eis}}^{k+2}(\varphi)$ in $\Gamma(\mathcal{M}_{\mathbb{Q}_p}, \mathrm{Sym}^k \mathscr{H}_{\mathrm{rig}}(1))$.

The main ingredient in the proof of the above theorem is the characterization of ξ by the residue, which by [BL] 2.2.3 (see also [HK] C.1.1) and the compatibility of the Beilinson-Deligne regulator map with the residue morphism shows that ξ represents the de Rham Eisenstein class in de Rham cohomology. See [BK] Proposition 3.6 and Proposition 4.1 for details concerning this point.

§5. Special values of Hecke L-functions

In this section, we give in Propositions 5.2 and 5.4 the precise relation between the special values of the Hecke L-function $L(\psi^a, s)$ and Eisenstein-Kronecker-Lerch series. Assume that K is an imaginary quadratic field of class number one, and let E be an elliptic curve over $\mathbb{Q}$ with good ordinary reduction at a prime p with complex multiplication by the ring of integers $\mathcal{O}_K$ of K. We let ψ be the Grossencharacter of K associated to $E_K := E \otimes_{\mathbb{Q}} K$, and we denote by $\mathfrak{f}$ the conductor of ψ. We fix an invariant differential ω of E defined over K. We fix once and for all a complex embedding $\tau : K \hookrightarrow \mathbb{C}$ of the base field K into $\mathbb{C}$, and we let Γ be the period lattice of $E := E \otimes_{K,\tau} \mathbb{C}$ with respect to ω. Then we have a complex uniformization

$$(5.1) \qquad\qquad \mathbb{C}/\Gamma \xrightarrow{\cong} E(\mathbb{C})$$

such that the pull-back of the invariant differential ω coincides with dz. Note that since E has complex multiplication, we have $\Gamma = \Omega \mathcal{O}_K$ for some complex period $\Omega \in \mathbb{C}^\times$.

By abuse of notation, we will denote by ψ and $\overline{\psi}$ the complex Hecke characters ψ_τ and $\overline{\psi_\tau}$ associated to ψ, where τ is the fixed embedding given above. Let $-d_K$ denote

the discriminant of K, so that $K = \mathbb{Q}(\sqrt{-d_K})$. The Hecke character ψ is of the form $\psi((u)) = \varepsilon(u)u$ for any $u \in \mathcal{O}_K$ prime to $\mathfrak{f}$, where $\varepsilon : (\mathcal{O}_K/\mathfrak{f})^\times \to K^\times$ is a primitive character on $(\mathcal{O}_K/\mathfrak{f})^\times$.

Let $\chi : (\mathcal{O}_K/\mathfrak{f}_\chi)^\times \to K^\times$ be a primitive character of conductor $\mathfrak{f}_\chi$, and let f_χ be a generator of $\mathfrak{f}_\chi$. Then for any u in $\mathcal{O}_K$, we define the *Gauss sum* $G(\chi, u)$ by

$$G(\chi, u) := \sum_{v \in \mathcal{O}_K/\mathfrak{f}_\chi} \overline{\chi}(v) \exp\left(2\pi i \mathrm{Tr}_{K/\mathbb{Q}} \left(uv/f_\chi \sqrt{-d_K} \right) \right)$$

(see [Lan] Chapter 22 §1), where we extend χ to a function on $\mathcal{O}_K/\mathfrak{f}_\chi$ by taking $\chi(u) := 0$ for any $u \in \mathcal{O}_K$ not prime to $\mathfrak{f}_\chi$. We let $G(\chi) := G(\chi, 1)$. Then the standard fact concerning Gauss sums are as follows (see for example [Lan] Chapter 22 §1.)

Lemma 5.1. *Let the notations be as above.*

1. We have $|G(\chi)|^2 = N(\mathfrak{f}_\chi)$.

2. For any $u \in \mathcal{O}_K$, we have $G(\chi, u) = \chi(u)G(\chi)$.

As in §3, we let $a > 0$ be an integer and $\mathfrak{f}_a$ be the conductor of ψ^a. Then the finite part ε^a of ψ^a is a primitive character $\varepsilon^a : (\mathcal{O}_K/\mathfrak{f}_a)^\times \to \mathbb{C}^\times$ of conductor $\mathfrak{f}_a$. We fix a generator f_a of $\mathfrak{f}_a$ and we denote by by $G(\varepsilon^a, u)$ the corresponding Gauss sum for any u in $\mathcal{O}_K$.

We let the notations be as in §3. In particular, we let w_a be the number of units in $\mathcal{O}_K$ which are congruent to *one* mod $\mathfrak{f}_a$, and we let w_{F/F_a} be the order of the Galois group $\mathrm{Gal}(F/F_a)$. We again let $N \geq 3$ be a rational integer divisible by $\mathfrak{f}$ and prime to p, and $F := K(E[N])$. We fix an isomorphism $\nu : (\mathbb{Z}/N\mathbb{Z})^2 \cong \frac{1}{N}\Gamma/\Gamma$.

We first consider the case when $\mathfrak{f}_a \neq (1)$. We let $\rho_a := \Omega/f_a$ be a primitive $\mathfrak{f}_a$-torsion point, which corresponds through the uniformization (5.1) to a point $\rho_a \neq 0 \in E(\overline{K})$. We then have the following.

Proposition 5.2. *Suppose $\mathfrak{f}_a \neq (1)$. Then we have*

$$\frac{(-1)^a}{w_a w_{F/F_a}} \sum_{\sigma \in \mathrm{Gal}(F/K)} E_{n, a-n, \rho_a^\sigma}^\infty(\mathbb{C}/\Gamma, dz, \nu) = \frac{G(\varepsilon^a)\overline{\Omega}^a}{A^{a-n}|\Omega|^{2n}} \Gamma(s) L(\psi^a, s)|_{s=n}.$$

Proof. Let w_0 be the number of units in $\mathcal{O}_K$. By definition, we have

$$L(\psi^a, s) = \frac{1}{w_0} \sum_{u \in \mathcal{O}_K} \frac{\psi^a(u)}{N(u)^s} = \frac{1}{w_0} \sum_{u \in \mathcal{O}_K} \frac{\varepsilon^a(u)u^a}{N(u)^s}$$

Then Lemma 5.1 (2) gives the equality $\varepsilon^a(u) = G(\varepsilon^a, u)/G(\varepsilon^a)$. If we expand the definition of the Gauss sum, we see that

$$L(\psi^a, s) = \frac{1}{w_0} \sum_{\substack{u \in \mathcal{O}_K \\ v \in \mathcal{O}_K/\mathfrak{f}_a}} \frac{\overline{\varepsilon}^a(v)u^a}{N(u)^s} \exp\left(2\pi i \mathrm{Tr}_{K/\mathbb{Q}} \left(\frac{uv}{f_a\sqrt{-d_K}} \right) \right).$$

Noting that $\mathcal{O}_K$ is preserved by complex conjugation, we see that the above is equal to

$$\frac{(-1)^a}{w_0} \sum_{v \in \mathcal{O}_K/\mathfrak{f}_a} \sum_{u \in \mathcal{O}_K} \frac{\overline{\varepsilon}^a(v)\overline{u}^a}{|u|^{2s}} \exp\left(\frac{2\pi}{\sqrt{d_K}}\left(\frac{u\overline{v}}{\overline{f}_a} - \frac{\overline{u}v}{f_a}\right)\right).$$

For any $\sigma \in \mathrm{Gal}(F/K)$, we have $\rho_a^\sigma = \rho_a^{\sigma'}$, where σ' is the class of σ in $\mathrm{Gal}(F_a/K)$. If $\sigma_v' := (v, F_a/F)$ is the element in $\mathrm{Gal}(F_a/K)$ corresponding to $v \in (\mathcal{O}_K/\mathfrak{f}_a)^\times$ through the *inverse* of the Artin map, then by the theory of complex multiplication, we have $\rho_a^{\sigma_v'} = \psi(v)\rho_a$. Hence

$$\sum_{\sigma \in \mathrm{Gal}(F/K)} K_a^*(0, \rho_a^\sigma, s; \Gamma) = w_{F/F_a} \sum_{\sigma' \in \mathrm{Gal}(F_a/K)} K_a^*(0, \rho_a^{\sigma'}, s; \Gamma)$$

$$= w_{F/F_a} \frac{w_a}{w_0} \sum_{v \in (\mathcal{O}_K/\mathfrak{f}_a)^\times} \sum_{\gamma \in \Gamma} \frac{\overline{\gamma}^a}{|\gamma|^{2s}} \langle \gamma, \psi(v)\rho_a \rangle$$

$$= w_{F/F_a} \frac{w_a}{w_0} \sum_{v \in (\mathcal{O}_K/\mathfrak{f}_a)^\times} \sum_{\gamma \in \Gamma} \frac{\overline{\varepsilon}^a(v)\overline{\gamma}^a}{|\gamma|^{2s}} \langle \gamma, v\rho_a \rangle.$$

Our assertion follows from the fact that $\Gamma = \Omega\mathcal{O}_K$, $A = |\Omega|^2\sqrt{d_K}/2\pi$ and the definition (4.2) of the Eisenstein-Kronecker-Lerch series. $\qquad\square$

The right hand side of Proposition 5.2 may be used to express the Hecke L-function on the other side of the functional equation as follows.

Lemma 5.3.   *We have*

$$(5.2) \quad \frac{1}{w_a w_{F/F_a}} \sum_{\sigma \in \mathrm{Gal}(F/K)} E_{n,a-n,\rho_a^\sigma}^\infty(\mathbb{C}/\Gamma, dz, \nu)$$

$$= \frac{A^{1-n} N(\mathfrak{f}_a)^{a+1-n}\overline{\Omega}^a}{\overline{f}_a^a |\Omega|^{2(a+1-n)}} \Gamma(s) L(\overline{\psi}^a, s)\Big|_{s=a+1-n}.$$

Proof.   We have by definition

$$\sum_{v \in \mathcal{O}_K/\mathfrak{f}_a} K_a^*(\psi(v)\rho_a, 0, a+1-s; \Gamma) = \sum_{v \in \mathcal{O}_K/\mathfrak{f}_a} \sideset{}{^*}\sum_{\gamma \in \Gamma} \frac{(\overline{\psi(v)\rho_a + \gamma})^a}{|\psi(v)\rho_a + \gamma|^{2(a+1-s)}}$$

$$= \frac{N(\mathfrak{f}_a)^{a+1-s}\overline{\Omega}^a}{\overline{f}_a^a |\Omega|^{2(a+1-s)}} L(\overline{\psi}^a, a+1-s)$$

for $\mathrm{Re}(s) < a/2$, hence for any $s \in \mathbb{C}$ by analytic continuation. Our assertion follows from the functional equation

$$\Gamma(s) K_a^*(0, \psi(v)\rho_a, s; \Gamma) = A^{a+1-2s}\Gamma(a+1-s) K_a^*(\psi(v)\rho_a, 0, a+1-s; \Gamma)$$

and the definition (4.2) of the Eisenstein-Kronecker-Lerch series. $\qquad\square$

The case when $\mathfrak{f}_a = (1)$ is given as follows.

Proposition 5.4. *Suppose $\mathfrak{f}_a = (1)$. Then we have*

$$\frac{1}{w_a} \sum_{\rho \in E[N] \setminus \{0\}} E_{n,a-n,\rho}^\infty(\mathbb{C}/\Gamma, dz, \nu) = \left(\frac{N^{a+2}}{N^{2n}} - 1\right) \frac{\overline{\Omega}^a}{A^{a-n} |\Omega|^{2n}} \Gamma(s) L(\psi^a, s)|_{s=n}.$$

Proof. By definition, we have

$$\sum_{\rho \in \frac{1}{N}\Gamma/\Gamma} K_a^*(0, \rho, s; \Gamma) = \sum_{\gamma \in \Gamma} \sum_{\rho \in \frac{1}{N}\Gamma/\Gamma} \frac{\overline{\gamma}^a}{|\gamma|^{2s}} \langle \gamma, \rho \rangle = \frac{N^{a+2}}{N^{2s}} \sum_{\gamma \in \Gamma} \frac{\gamma^a}{|\gamma|^{2s}},$$

where the last equality follows from the equality

$$\sum_{\rho \in \frac{1}{N}\Gamma/\Gamma} \langle \gamma, \rho \rangle = \begin{cases} N^2 & \gamma \in N\Gamma \\ 0 & \text{otherwise} \end{cases}$$

and the fact that complex conjugation acts bijectively on Γ. Our assertion follows from the definition (4.2) of the Eisenstein-Kronecker-Lerch series. $\qquad\square$

Similarly to Lemma 5.3, we have the following.

Lemma 5.5. *We have*

$$(5.3) \quad \frac{1}{w_a} \sum_{\rho \in E[N] \setminus \{0\}} E_{n,a-n,\rho}^\infty(\mathbb{C}/\Gamma, dz, \nu))$$

$$= \left(\frac{N^{a+2}}{N^{2n}} - 1\right) \frac{A^{1-n}\overline{\Omega}^a}{|\Omega|^{2(a+1-n)}} \Gamma(s) L(\overline{\psi}^a, s)\Big|_{s=a+1-n}.$$

§6. The Main Result

In this section, we give an outline of the proof of our main theorem. We will mainly deal with the case when $\mathfrak{f}_a \neq (1)$, as the case for $\mathfrak{f}_a = (1)$ is essentialy the same except for the factor $(N^{a+2}/N^{2n} - 1)$. We first calculate the p-adic and complex periods $\Omega_p(n)$ and $\Omega(n)$. From the definition of c_n^a and from the compatibility of the syntomic regulator with respect to pull-back morphisms, the restriction of the syntomic Eisenstein class through the decomposition of Lemma 3.1 gives the image by the syntomic regulator of the element c_n^a in $H_{\text{mot}}^1(K, M^a(n))$.

Let the notations be as in the previous section. We denote by ω^* the class in $H_{\text{dR}}^1(E/\mathbb{C})$ corresponding to $d\overline{z}/A$, which is in fact a class in $H_{\text{dR}}^1(E/K)$. Let $k = 2n -$

$a - 2$. Then $\omega^{k-j+1,j+1} := \omega^{\vee k-j}\omega^{*\vee j}(1)$ for $0 \le j \le k$ form a basis of $\mathrm{Sym}^k \iota^* \mathscr{H}_{\mathrm{dR}}(1)$. The relation between the basis $\widetilde{\omega}^{m,n}$ and $\omega^{m,n}$ is given by $\widetilde{\omega}^{m,n} = \Omega_{\mathfrak{p}}^{n-m}\omega^{m,n}$. In what follows, let φ_a be as in Definition 3.2. By Theorem 4.3, the pull-back of the syntomic Eisenstein class $\mathrm{Eis}_{\mathrm{syn}}^{k+2}(\varphi_a)$ to $H_{\mathrm{syn}}^1(K_{\mathfrak{p}}, \mathrm{Sym}^k \iota^* \mathscr{H}(1))$ is expressed by the element

$$\iota^* \widetilde{\alpha}_{\mathrm{Eis}}^{k+2}(\varphi_a) = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{j!} \Omega_{\mathfrak{p}}^{2j-k} E_{j+1,j-k-1,\varphi_a}^{(p)}(E,\omega,\nu) \omega^{k-j+1,j+1}.$$

Hence the element $r_{\mathrm{syn}}(c_n^a)$ in $H_{\mathrm{syn}}^1(K_{\mathfrak{p}}, M_{\mathrm{cris}}^a(n))$ corresponding by definition to the direct factor $j = n-1$ and $j = n-a-1$ is represented by

$$\frac{(-1)^{n-a+1}}{\Gamma(n)} \Omega_{\mathfrak{p}}^a E_{n,a-n,\varphi_a}^{(p)}(E,\omega,\nu) \omega^{n-a,n} + \frac{(-1)^{n-1}}{\Gamma(n-a)} \Omega_{\mathfrak{p}}^{-a} E_{n-a,-n,\varphi_a}^{(p)}(E,\omega,\nu) \omega^{n,n-a}.$$

By definition of the exponential map, the element in $t_n^a \otimes \mathbb{Q}_p$ corresponding to $r_{\mathrm{syn}}(c_n^a)$ through the isomorphism (3.3) is

$$\frac{(-1)^{n-a+1}}{\Gamma(n)} \Omega_{\mathfrak{p}}^a E_{n,a-n,\varphi_a}(E,\omega,\nu) \omega^{n-a,n} + \frac{(-1)^{n-1}}{\Gamma(n-a)} \Omega_{\mathfrak{p}}^{-a} E_{n-a,-n,\varphi_a}(E,\omega,\nu) \omega^{n,n-a},$$

where $E_{n,a-n,\varphi_a}(E,\omega,\nu)$ is the element in $\widehat{K}_{\mathfrak{p}}^{\mathrm{ur}}$ satisfying

$$(6.1) \qquad \left(1 - p^{a-n}\sigma^*\right) E_{n,a-n,\varphi_a}(E,\omega,\nu) = E_{n,a-n,\varphi_a}^{(p)}(E,\omega,\nu).$$

From the definition of c_n^a and the discussion at the end of §3, this shows that we have

$$\log_p \circ r_p(c_n^a) = \frac{(-1)^{n-a+1}}{\Gamma(n)} \Omega_{\mathfrak{p}}^a E_{n,a-n,\varphi_a}(E,\omega,\nu) \omega^{n-a,n}$$
$$+ \frac{(-1)^{n-1}}{\Gamma(n-a)} \Omega_{\mathfrak{p}}^{-a} E_{n-a,-n,\varphi_a}(E,\omega,\nu) \omega^{n,n-a}.$$

This gives the following.

Theorem 6.1. *Let n be an integer $> a$ and assume that the p-adic regulator r_p is injective. Then the p-adic period $\Omega_p(n) \in K_{\mathfrak{p}} \bigoplus K_{\mathfrak{p}^*}$ of the motive $M^a(n)$ is given by*

$$\Omega_p(n) = \frac{(-1)^{n-a+1}}{\Gamma(n)} \Omega_{\mathfrak{p}}^a E_{n,a-n,\varphi_a}(E,\omega,\nu)$$

Proof. The theorem follows from the definition given in (2.5) of the p-adic period, noting that we have an isomorphism

$$(K \otimes_{\mathbb{Q}} \mathbb{Q}_p) \omega_{\mathrm{tg},n}^a \cong \mathbb{Q}_p \, \omega^{n,n-a} \bigoplus \mathbb{Q}_p \, \omega^{n-a,n}$$

induced from the canonical splitting $K \otimes \mathbb{Q}_p = K_{\mathfrak{p}} \bigoplus K_{\mathfrak{p}^*} \cong \mathbb{Q}_p \bigoplus \mathbb{Q}_p$. $\qquad \square$

The calculation of the complex period, originally due to Deninger [De1] may be done in a similar fashion. If we let $\Gamma = \Omega \mathcal{O}_K$ as in the previous section, then the Betti homology of E is given by $H_1^B(E(\mathbb{C}), \mathbb{Z}) = \Gamma$. We let $\gamma_1 := \Omega \in \Gamma$, which is a generator of Γ as a $\mathcal{O}_K$-module. If we fix a $v \in \mathcal{O}_K$ such that $\mathcal{O}_K := \mathbb{Z} \oplus \mathbb{Z}v$ and if we let $\gamma_2 := v(\gamma_1)$ where v acts through the complex multiplication of E, then we have $\Gamma := \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$ as a $\mathbb{Z}$-module. The period relation gives the equality

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \Omega & \overline{\Omega}/A \\ \tau\Omega & \overline{\tau}\overline{\Omega}/A \end{pmatrix} \begin{pmatrix} \omega^\vee \\ \omega^{*\vee} \end{pmatrix}.$$

The K-basis γ_1 induces a K-basis of $M_B^a(n) \subset \mathrm{Sym}^k H_B^1(E(\mathbb{C}), \mathbb{Q}(1))$, which we denote by ω_B. Then the inclusion

$$M_B^a(n) \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow t_n^a \otimes_{\mathbb{Q}} \mathbb{R}$$

maps ω_B to

$$\Omega^{n-a} \frac{\overline{\Omega}^n}{A^n} \omega^{n-a,n} + \Omega^n \frac{\overline{\Omega}^{n-a}}{A^{n-a}} \omega^{n,n-a}.$$

Furthermore, one may prove the following.

Theorem 6.2 (Deninger [De1]).   *The image $r_\infty(c_n^a)$ in*

$$H_{\mathscr{D}}^1(K \otimes_{\mathbb{Q}} \mathbb{R}, V_\infty^a(n)) \cong (t_n^a \otimes_{\mathbb{Q}} \mathbb{R})/(M_B^a(n) \otimes_{\mathbb{Q}} \mathbb{R})$$

of c_n^a by the Beilinson regulator (2.1) is represented by the element

$$\frac{(-1)^{n-1}}{\Gamma(n)} E_{n,a-n,\varphi_a}^\infty(\mathbb{C}/\Gamma, dz, \nu)\, \omega^{n-a,n} + \frac{(-1)^{n-a+1}}{\Gamma(n-a)} E_{n-a,-n,\varphi_a}^\infty(\mathbb{C}/\Gamma, dz, \nu)\, \omega^{n,n-a}$$

in $t_n^a \otimes_{\mathbb{Q}} \mathbb{C}$.

Proof.   The Eisenstein class in this paper defined using the elliptic polylogarithm is related to the Eisenstein class defined by Beilinson and Deninger. The theorem is then a special case of the weak Beilinson conjecture for Hecke character associated to imaginary quadratic fields proved by Deninger [De1] (see also [DW] for the case of an elliptic curve defined over $\mathbb{Q}$ with complex multiplication.) The theorem may also be proved by explicitly calculating the Hodge realization of the elliptic polylogarithm [BL] (see also [BKT] Theorem A 29.)   □

By taking the determinant of the complex (2.2) with respect to the basis $r_\infty(c_n^a)$, ω_B, $\omega^{n-a,n}$ and $\omega^{n,n-a}$, the above calculation and the definition of the complex period give the following.

Proposition 6.3. *The complex period $\Omega_\infty(n)$ of $M^a(n)$ in $K \otimes_\mathbb{Q} \mathbb{R}$ is given by*

$$\Omega_\infty(n) = (-1)^{a+n-1} G(\varepsilon^a) L(\psi^a, n) \bigoplus (-1)^{n+1} G(\overline{\varepsilon}^a) L(\overline{\psi}^a, n)$$

in $K \otimes_\mathbb{Q} \mathbb{C} \cong \mathbb{C} \bigoplus \mathbb{C}$ if $\mathfrak{f}_a \neq (1)$. A similar formula holds for the case when $\mathfrak{f}_a = (1)$, but with a factor $(N^{a+2}/N^{2n} - 1)$ multiplied to the L-value.

Proof. The assertion follows from Theorem 6.2 by explicit calcuation, using the definition of φ_a (Definition 3.2), the calculation of the complex period above, and the relation between Eisenstein-Kronecker-Lerch series and special values of L-functions (Proposition 5.2 if $\mathfrak{f}_a \neq (1)$, or Proposition 5.4 if $\mathfrak{f}_a = (1)$). $\square$

This gives the following corollary, which we stated in Theorem 2.1.

Corollary 6.4. *If $\mathfrak{f}_a \neq (1)$, then we have*

$$\frac{L(\psi^a, n)}{\Omega_\infty(n)} = \frac{(-1)^{a+n-1}}{G(\varepsilon^a)} \in K^\times \subset K \otimes_\mathbb{Q} \mathbb{C}.$$

A similar formula holds when $\mathfrak{f}_a = (1)$, but with multiplication by $(N^{a+2}/N^{2n} - 1)^{-1}$ on the right hand side.

Proof. The equalitiy follow from the calculation of the complex period in Proposition 6.3. Since ε^a is a primitive Hecke character with values in K, we see that this value is in K. $\square$

We next construct the p-adic measure μ^a which appears in the formulation of Conjecture 2.2. Since E has good ordinary reduction at p, the prime p splits as $p = \mathfrak{p}\mathfrak{p}^*$ in K. In what follows, we fix once and for all complex and p-adic embeddings of our coefficient K as follows. We let $\tau : K \hookrightarrow \mathbb{C}$ as in §5 and an embedding $K \hookrightarrow \mathbb{C}_p$ mapping $\mathfrak{p}$ to a prime in $\mathbb{C}_p$. With this convention, we may regard the complex and p-adic periods as elements respectively in $\mathbb{C}$ and $\mathbb{C}_p$, by taking the first components in Proposition 6.3 and Theorem 6.1.

Let $\widehat{E}$ be the formal group of E over $\mathcal{O}_K$, and let $\widehat{K}_\mathfrak{p}^{\mathrm{ur}}$ be the p-adic completion of the maximal unramified extension $K_\mathfrak{p}^{\mathrm{ur}}$ of $K_\mathfrak{p}$, which we regard as a subfield of $\mathbb{C}_p$ through our fixed embedding. Since p is an ordinary prime, there exists an isomorphism of formal groups η over $\mathcal{O}_{\widehat{K}_\mathfrak{p}^{\mathrm{ur}}}$

$$\eta : \widehat{E} \xrightarrow{\cong} \widehat{\mathbb{G}}_m$$

given by a power series $\eta(t) = \exp(\lambda(t)/\Omega_\mathfrak{p}) - 1$, where $\Omega_\mathfrak{p}$ is a p-adic period of E which is an element in $\mathcal{O}_{\widehat{K}_\mathfrak{p}^{\mathrm{ur}}}$ satisfying

$$(6.2) \qquad\qquad \Omega_\mathfrak{p}^{\sigma-1} = \psi(\mathfrak{p}) p^{-1}.$$

The above isomorphism gives the equality

$$(6.3) \qquad\qquad \eta^* \left(d \log(1 + T) \right) = \omega / \Omega_{\mathfrak{p}}.$$

Again let $N \geq 3$ be an integer as in §3 divisible by $\mathfrak{f}$ and prime to p. By [Katz] 5.10.1, the value of the p-adic Eisenstein series E_{k+2,l,φ_a} at the test object (E, ω, ν) is defined by

$$(6.4) \qquad\qquad E_{k+2,l,\varphi_a}(E, \omega, \nu) := \Omega_{\mathfrak{p}}^{k+l+2} E_{k+2,l,\varphi_a}(E, \eta, \nu).$$

In addition, the comparison theorem [Katz] 8.0.9 states that this value for integers $k > 0, l \geq 0$ is an element in $F := K(\mathfrak{f})$ satisfying the equality

$$E_{k+2,l,\varphi_a}(E, \omega, \nu) = E_{k+2,l,\varphi_a}^{\infty}(E, \omega, \nu).$$

Then the calculation above and Theorem 4.1 gives the following.

Proposition 6.5. *We let φ_a be as in Definition 3.2, and we denote again by μ_{φ_a} the p-adic measure on $\mathbb{Z}_p \times \mathbb{Z}_p^\times$ obtained as the value of μ_{φ_a} of Theorem 4.1 at (E, ω, ν). If $\mathfrak{f}_a \neq (1)$, then we have*

$$\frac{(-1)^a}{\Omega_{\mathfrak{p}}^a} \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} x^{n-1} y^{a-n} \mu_{\varphi_a}(x, y) = G(\varepsilon^a) \left(1 - \frac{\psi(\mathfrak{p})^a}{p^n} \right) \frac{\overline{\Omega}^a \Gamma(n) L(\psi^a, n)}{A^{a-n} |\Omega|^{2n}}$$

for integers $a \geq n > 0$. A similar formula holds for the case when $\mathfrak{f}_a = (1)$, but with multiplication by $(N^{a+2}/N^{2n} - 1)$ on the right hand side.

Proof. The relation between the action of the Frobenius on $V_p(\mathbb{Q}_p, \Gamma(N))$ and its specialization is given by

$$\phi^*(E_{k+2,l,\varphi})(E, \eta, \nu) = E_{k+2,l,\varphi}(E, \eta, \nu^{\sigma_{\mathfrak{p}}}),$$

since (E, ω) is define over K and hence $E^{\sigma_{\mathfrak{p}}} = E$ and $\omega^{\sigma_{\mathfrak{p}}} = \omega$. Then from the definition of the specialization of p-adic modular forms (6.4) and the action of the Frobenius on the p-adic period (6.2), we have

$$p^l \phi^*(E_{k+2,l,\varphi})(E, \omega, \nu) = p^l \left(\Omega_{\mathfrak{p}}^{\sigma_{\mathfrak{p}}} \right)^{k+l+2} E_{k+2,l,\varphi}(E, \eta, \nu^{\sigma_{\mathfrak{p}}})$$
$$= \psi(\mathfrak{p})^{k+l+2} p^{-k-2} \Omega_{\mathfrak{p}}^{k+l+2} E_{k+2,l,\varphi^{\sigma_{\mathfrak{p}}}}(E, \eta, \nu).$$

Applying the above calculation to the case $a = k + l + 2$ and $n = k + 2$, our assertion now follows from Theorem 4.1, noting the definition of the p-adic Eisenstein series (4.5), the definition of φ_a, and the fact that the sum over all $\sigma \in G_{F/K}$ of φ_a^σ is invariant by the action of $\sigma_{\mathfrak{p}}$. $\qquad\square$

Proposition 6.6. *Let $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ be the surjection defined by $\rho : (x, y) \mapsto x/y$. We define the measure μ'^a on $\mathbb{Z}_p^\times$ by*

$$\int_{\mathbb{Z}_p^\times} f(w)\mu'^a(w) = \frac{(-1)^a}{G(\varepsilon^a)} \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} x^{-1}y^a \rho^*(f)(x, y)\mu_{\varphi_a}(x, y).$$

If $\mathfrak{f}_a \neq (1)$, then this measure satisfies the interpolation property

$$\frac{1}{\Omega_{\mathfrak{p}}^a} \int_{\mathbb{Z}_p^\times} w^n \mu'^a(w) = \left(1 - \frac{\psi(\mathfrak{p})^a}{p^n}\right)\left(1 - \frac{\overline{\psi}(\mathfrak{p}^*)^a}{p^{a+1-n}}\right) \frac{\overline{\Omega}^a \Gamma(n) L(\psi^a, n)}{A^{a-n}|\Omega|^{2n}}.$$

for any integers a and n such that $a \geq n > 0$. A similar formula holds for the case when $\mathfrak{f}_a = (1)$, but with multiplication by $(N^{a+2}/N^{2n} - 1)$ on the right hand side.

Proof. The equality is obtained from the definition of μ'^a and in calculating the restriction of the measure μ_{φ_a} to $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. The calculation directly follows from Katz [Katz] 8.7.6, using the functional equation (see Remark 6 below.) One may also do the calculation using an alternative construction of Katz p-adic measure ([BKo] Proposition 3.5 and Theorem 3.7), again after using the functional equation. $\square$

Remark. Combining Proposition 5.2 and (5.2) (or if $\mathfrak{f}_a = (1)$, then Proposition 5.4 and (5.3)), we obtain the functional equation

$$\frac{\Gamma(n) L(\psi^a, n)}{A^{a-n}|\Omega|^{2n}} = \frac{N(\mathfrak{f}_a)^{a+1-n}\Gamma(a+1-n)L(\overline{\psi}^a, a+1-n)}{(-1)^a G(\varepsilon^a)\overline{f}_a^a A^{n-1}|\Omega|^{2(a+1-n)}}.$$

We regard f_a and $\overline{f}_a$ in $\mathcal{O}_K$ as elements in $\mathbb{Z}_p^\times$ through the canonical isomorphism $\mathcal{O}_{K_{\mathfrak{p}}} \cong \mathbb{Z}_p$. Denote by $\widetilde{\mu}_{\varphi_a}$ the measure on $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ obtained as the pull-back of $f_a^{-1}\mu_{\varphi_a}$ through the isomorphism

$$\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \xrightarrow{\cong} \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$$

given by $(x, y) \mapsto (\overline{f}_a x, f_a^{-1} y)$. Since $A = |\Omega|^2 \sqrt{d_K}/2\pi$, if we let $k_1 = a+1-n$ and $k_2 = 1-n$, then the interpolation property of $\widetilde{\mu}_{\varphi_a}$ at (E, ω, ν) becomes

$$\frac{1}{\Omega_{\mathfrak{p}}^{k_1-k_2}} \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} x^{-k_2} y^{k_1-1} \widetilde{\mu}_{\varphi_a}(x, y)$$

$$= \left(1 - \frac{\psi(\mathfrak{p})^{k_1-k_2}}{p^{1-k_2}}\right)\left(1 - \frac{\overline{\psi}(\mathfrak{p}^*)^{k_1-k_2}}{p^{k_1}}\right)\left(\frac{\sqrt{d_K}}{2\pi}\right)^{k_2} \frac{\Gamma(k_1) L(\overline{\psi}^{k_1-k_2}, k_1)}{\Omega^{k_1-k_2}}$$

for $k_1 > -k_2 \geq 0$. This coincides with the interpolation property of the two-variable p-adic measure constructed by Katz and Yager (see [Yag] §1.)

If $\mathfrak{f}_a \neq (1)$, then the measure μ'^a defined in Proposition 6.6 satisfies the condition of Conjecture 2.2. If $\mathfrak{f}_a = (1)$, then we need to cancel the factor $(N^{a+2}/N^{2n} - 1)$ which appears in the interpolation formula.

Definition 6.7. We define the pseudo-measure μ_a on $\mathbb{Z}_p^\times$ as follows.

1. If $\mathfrak{f}_a \neq (1)$, then we let $\mu^a := \mu'^a$.

2. If $\mathfrak{f}_a = (1)$, then we let μ_N^a be the measure on $\mathbb{Z}_p^\times$ defined by

$$\int_{\mathbb{Z}_p^\times} x^n \mu_N^a = \left(\frac{N^{a+2}}{N^{2n}} - 1 \right)$$

for any integer n. We define μ^a to be the pseudo-measure on $\mathbb{Z}_p^\times$ obtained as the quotient of μ'^a by μ_N^a (see for example [Col] §1.2 for the definition of a pseudo-measure).

When $\mathfrak{f}_a \neq (1)$, then μ_a is by definition a p-adic measure on $\mathbb{Z}_p^\times$.

We now have the following.

Theorem 6.8. *Let $a > 0$ be an integer and let μ^a be the pseudo-measure on $\mathbb{Z}_p^\times$ defined in Definition 6.7. If we let*

$$L_p(\psi^a \otimes \chi_{\mathrm{cyc}}^n) := \int_{\mathbb{Z}_p^\times} w^n \mu^a(w),$$

then we have

$$\frac{L_p(\psi^a \otimes \chi_{\mathrm{cyc}}^n)}{\Omega_p(n)} = \left(1 - \frac{\psi(\mathfrak{p})^a}{p^n} \right) \left(1 - \frac{\overline{\psi}(\mathfrak{p}^*)^a}{p^{a+1-n}} \right) \frac{\Gamma(n) L(\psi^a, n)}{\Omega_\infty(n)}$$

for any integer $n > a$ such that the p-adic period $\Omega_p(n)$ is non-zero.

Proof. We first consider the case when $\mathfrak{f}_a \neq (1)$. By definition of the p-adic Eisenstein series in Definition 4.2, the moments of the measure μ_{φ_a} constructed in Theorem 4.1 is given by p-adic Eisenstein series. As in (6.1), let $E_{n,a-n,\varphi_a}(E, \omega, \nu)$ be the element in $\widehat{K}_{\mathfrak{p}}^{\mathrm{ur}}$ satisfying

$$\left(1 - p^{a-n} \sigma^* \right) E_{n,a-n,\varphi_a}(E, \omega, \nu) = E_{n,a-n,\varphi_a}^{(p)}(E, \omega, \nu).$$

Then the compatibility between the Frobenius on the modular curve and a point, as well as the restriction of the measure on $\mathbb{Z}_p \times \mathbb{Z}_p^\times$ to $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ shows that we have the relation

$$\frac{1}{\Omega_{\mathfrak{p}}^a} \int_{\mathbb{Z}_p^\times} w^n \mu^a(w) = \frac{1}{G(\varepsilon^a)} \left(1 - \frac{\psi(\mathfrak{p})^a}{p^n} \right) \left(1 - \frac{\overline{\psi}(\mathfrak{p}^*)^a}{p^{a+1-n}} \right) E_{n,a-n,\varphi_a}(E, \omega, \nu).$$

The calculation of the p-adic period in Theorem 6.1 shows that

$$\Omega_p(n) = \frac{(-1)^{a-n+1}}{\Gamma(n)} \Omega_{\mathfrak{p}}^a E_{n,a-n,\varphi_a}(E, \omega, \nu)$$

for our fixed embedding $K \hookrightarrow \mathbb{C}_p$. This proves in particular that

$$\frac{(-1)^{a-n+1}}{\Omega_p(n)} \int_{\mathbb{Z}_p^\times} w^n \mu^a(w) = \left(1 - \frac{\psi(\mathfrak{p})^a}{p^n}\right) \left(1 - \frac{\overline{\psi}(\mathfrak{p}^*)^a}{p^{a+1-n}}\right) \frac{\Gamma(n)}{G(\varepsilon^a)}.$$

Our assertion now follows from Corollary 6.4. The case for $\mathfrak{f}_a = (1)$ follows in a similar fashion, noting the interpolation property of μ'^a and μ_N^a. $\qquad\square$

Remark. If there were a case such that $\Omega_p(n)$ is zero for $n > a$, our calculation would imply simply that $L_p(\psi^a \otimes \chi_{\mathrm{cyc}}^n) = 0$.

References

[BK] K. Bannai and G. Kings, p-adic elliptic polylogarithm, p-adic Eisenstein series and Katz measure, *American J. Math.* **132** no. 6 (2010), 1609-1654.

[BKo] K. Bannai and S. Kobayashi, Algebraic theta functions and p-adic interpolation of Eisenstein-Kronecker numbers, *Duke Math. J.* **153** no. 2 (2010), 229-295.

[BKT] K. Bannai, S. Kobayashi and T. Tsuji, On the de Rham and p-adic realizations of the elliptic polylogarithm for CM elliptic curves, *Annales scientifiques de l'ENS* **43**, fascicule 2 (2010), 185-234.

[Bes] A. Besser, Syntomic regulators and p-adic integration I: Rigid syntomic regulators, *Israel Journal of Math.* **120** (2000), 291-334.

[BL] A. Beilinson and A. Levin, The Elliptic Polylogarithm, *Motives*, (Seattle, WA, 1991), 123-192.

[Col] P. Colmez, Fonctions L p-adiques (Exposé 851), *Astérisque* **266** (2000), 21-58.

[De1] C. Deninger, Higher regulators and Hecke L-series of imaginary quadratic fields I. *Invent. math.* **96**, 1–69 (1989).

[De2] _____, Extensions of motives associated to symmetric powers of elliptic curves and to Hecke characters of imaginary quadratic fields, In: F. Catanese (ed.) *Proc. Arithmetic Geometry, Cortona 1994*, Symp. Math. **37**, 99-137, Cambridge Univ. Press (1997).

[DW] C. Deninger and K. Wingberg, On the Beilinson conjectures for elliptic curves with complex multiplication, In: Rapoport, M., Schappacher, N., Schneider, P. (ed.) *Beilinson's conjectures on special values of L-functions*, Perspect. Math. vol. **4**, 249–272, Academic Press, (1988).

[GS] C. Goldstein and N. Schappacher, Séries d'Eisenstein et fonctions L des courbes elliptiques á multiplication complexe, *J. Reine Angew. Math.* **327** (1981), 184-218.

[HK] A. Huber and G. Kings, Degeneration of l-adic Eisenstein classes and of the elliptic polylog, *Invent. Math.* **135** (1999), no. 3, 545–594.

[Jan] U. Jannsen, On the l-adic cohomology of varieties over number fields and its Galois cohomology, In: *Galois groups over* $\mathbb{Q}$, Springer (1989), 315–360.

[Kato]   K. Kato, *p*-adic Hodge theory and values of zeta functions of modular forms, In: *Cohomologies p-adiques et applications arithmétiques (III)*, *Astérisque* **295** (2004), 117–290.

[Katz]   N. Katz, *p*-adic interpolation of real analytic Eisenstein series, *Ann. of Math.* (2) **104** (1976), no. 3, 459–571.

[Lan]   S. Lang, *Elliptic Functions*, Second Edition, *GTM* **112**, Springer 1987.

[Ni]   W. Nizioł, Cohomology of crystalline smooth sheaves, *Comp. Math.* **129** (2001), no. 2, 123–147.

[PR]   B. Perrin-Riou, *Fonctions L p-adiques des représentations p-adiques*, *Astérisque* **229**, (1995).

[VM]   S. Vishik and Ju. I. Manin, *p*-adic Hecke series for quadratic imaginary fields, *Math. Sbornik*, V, **95** (137), No. 3 (11), 1974.

[Wei]   A. Weil, *Elliptic Functions according to Eisenstein and Kronecker*, Springer-Verlag, 1976.

[Yag]   R. Yager, On two variable *p*-adic *L*-functions, *Ann. of Math.* **115** (1982), 411-449.