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Non-existence of certain Galois representations with a uniform tame inertia weight: A resume

By

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Abstract

In this paper, we announce some results on the non-existence of certain semistable Galois representations. We apply them to a conjecture of Rasmussen and Tamagawa.

§ 1. Main results

Our main concern in this paper is the non-existence of certain semistable Galois representations of a number field. Let ℓ be a prime number and K a number field of degree d and discriminant d_K. Choose an algebraic closure $\bar{K}$ of K. Fix non-negative integers n, r and w, and a prime number $\ell_0 \neq \ell$. Put $\bullet := (n, \ell_0, r, w)$. Let $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet$ be the set of isomorphism classes of n-dimensional ℓ-adic representations V of the absolute Galois group $G_K = \mathrm{Gal}(\bar{K}/K)$ of K which satisfy the following four conditions:

(A) For any place λ of K above ℓ, the restriction of V to the decomposition group of (an extension to $\bar{K}$ of) λ is semistable and has Hodge-Tate weights in $[0, r]$.

(B) For some place λ_0 of K above ℓ_0, the representation V is unramified at λ_0 and the characteristic polynomial $\det(T - \mathrm{Fr}_{\lambda_0} | V)$ has rational integer coefficients. Furthermore, the roots of the above characteristic polynomial have complex absolute value $q_{\lambda_0}^{w/2}$ for every embedding $\bar{\mathbb{Q}}_\ell$ into $\mathbb{C}$. Here Fr_{λ_0} and q_{λ_0} are the arithmetic Frobenius and the order of the residue field of λ_0, respectively.

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(C) For any finite place v of K not above ℓ, the action of the inertia group at v on $\bar{V}$ is unipotent. Here $\bar{V}$ is a residual representation of V[1].

(D) The representation $\bar{V}$ has a filtration of G_K-modules

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{n-1} \subset \bar{V}_n = \bar{V}$$

such that $\bar{V}_k$ has dimension k for each $0 \leq k \leq n$.

Furthermore, we denote by $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}_{\mathrm{cycl}}$ the subset of $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}$ whose elements V satisfy the additional property (E) below:

(E) For each $1 \leq k \leq n$, the G_K-action on the quotient $\bar{V}_k/\bar{V}_{k-1}$ is given by a power of the mod ℓ cyclotomic character χ_ℓ.

Example 1.1. Let X be a proper smooth scheme over K which has semistable reduction everywhere and has good reduction at some place of K above ℓ_0. Let n be the w-th Betti number of $X(\mathbb{C})$ and $w \leq r$. Then the dual of $H^w_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ satisfies the conditions (A), (B) and (C).

Our main results are the following:

Theorem 1.2 ([O], Theorem 3.10)**.** *Suppose that w is odd or $w > 2r$. Then there exists an explicit constant C depending only on d_K, n, ℓ_0, r and w such that* $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}_{\mathrm{cycl}}$ *is empty for any prime number $\ell > C$.*

Theorem 1.3 ([O], Theorem 3.11)**.** *Suppose that w is odd or $w > 2r$. Then there exists an explicit constant C' depending only on K, n, ℓ_0, r and w such that* $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}$ *is empty for any prime number $\ell > C'$ which does not split in K.*

The key of the proofs of the above Theorems is a relation between tame inertia weights and Frobenius weights. This relation is obtained by a result of Caruso [Ca] which gives an upper bound of tame inertia weights of semistable Galois representations.

§ 2. Rasmussen-Tamagawa Conjecture

We describe an application, which is a special case of the Rasmussen-Tamagawa conjecture ([RT]) related with the finiteness of the set of isomorphism classes of abelian varieties with constrained prime power torsion. Our work is motivated by this conjecture. We denote by $\tilde{K}_\ell$ the maximal pro-ℓ extension of $K(\mu_\ell)$ which is unramified away from ℓ.

[1] A residual representation $\bar{V}$ is not uniquely defined (it depends on the choice of a G_K-stable lattice), but the validity of conditions (C), (D) or (E) does not depend on the choice of $\bar{V}$.

Definition 2.1. Let $g \geq 0$ be an integer. We denote by $\mathcal{A}(K, g, \ell)$ the set of K-isomorphism classes of abelian varieties A over K, of dimension g, which satisfy the following equivalent conditions:
(1) $K(A[\ell^\infty]) \subset \tilde{K}_\ell$;
(2) The abelian variety A has good reduction outside ℓ and $A[\ell]$ admits a filtration

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{2g-1} \subset \bar{V}_{2g} = A[\ell]$$

such that $\bar{V}_k$ has dimension k for each $0 \leq k \leq 2g$. Furthermore, for each $1 \leq k \leq 2g$, the G_K-action on the space $\bar{V}_k/\bar{V}_{k-1}$ is given by a power of the mod ℓ cyclotomic character χ_ℓ.

The equivalence of (1) and (2) follows from the criterion of Néron-Ogg-Shafarevich and Lemma 3 of [RT][2]. The set $\mathcal{A}(K, g, \ell)$ is a finite set because of the Shafarevich conjecture proved by Faltings. Rasmussen and Tamagawa conjectured that this set is in fact empty for any ℓ large enough:

Conjecture 2.2 ([RT], Conjecture 1). *The set $\mathcal{A}(K, g, \ell)$ is empty for any prime ℓ large enough.*

It is known that this conjecture holds under the following conditions:

(i) $K = \mathbb{Q}$ and $g = 1$ ([RT], Theorem 2);

(ii) K is a quadratic number field other than the imaginary quadratic fields of class number one and $g = 1$ ([RT], Theorem 4).

We consider the semistable reduction case of Conjecture 2.2.

Definition 2.3. (1) We denote by $\mathcal{A}(K, g, \ell)_{\rm st}$ the set of K-isomorphism classes of abelian varieties in $\mathcal{A}(K, g, \ell)$ with everywhere semistable reduction.
(2) We denote by $\mathcal{A}(K, g, \ell_0, \ell)_{\rm st}$ the set of K-isomorphism classes of abelian varieties A over K with everywhere semistable reduction, of dimension g, which satisfy the following condition: The abelian variety A has good reduction at some place of K above ℓ_0 and $A[\ell]$ admits a filtration of G_K-modules

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{2g-1} \subset \bar{V}_{2g} = A[\ell]$$

such that $\bar{V}_k$ has dimension k for each $0 \leq k \leq 2g$.

We have $\mathcal{A}(K, g, \ell)_{\rm st} \subset \mathcal{A}(K, g, \ell_0, \ell)_{\rm st}$. We can show the following easily as corollaries of Theorems 1 and 2[3]:

[2]Lemma 3 of [RT] is stated in the setting $K = \mathbb{Q}$. However, an easy argument allows us to extend this setting to any number field K.
[3]Rasmussen and Tamagawa have shown the emptiness of the set $\mathcal{A}(K, g, \ell)_{\rm st}$ for ℓ large enough by using the result of [Ra] instead of [Ca] (unpublished).

Corollary 2.4 ([O], Corollary 4.5). *There exists an explicit constant D depending only on d_K and g such that the set $\mathcal{A}(K,g,\ell)_{\mathrm{st}}$ is empty for any prime number $\ell > D$.*

Corollary 2.5 ([O], Corollary 4.6). *There exists an explicit constant D' depending only on K, g and ℓ_0 such that the set $\mathcal{A}(K,g,\ell_0,\ell)_{\mathrm{st}}$ is empty for any prime number $\ell > D'$ which does not split in K.*

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