RIMS Kôkyûroku Bessatsu
B25 (2011), 103–123

Introduction to algebraic and arithmetic dynamics – a survey

By

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In this survey, we discuss some topics in algebraic and arithmetic dynamics. This is based on my talk in a survey session of the RIMS workshop “Algebraic number theory and related topics 2009.” In Section 1, we see how algebraic and arithmetic dynamics are different from real and complex dynamics. Then we examine some number-theoretic aspects of the fields in Section 2, and some dynamical aspects in Section 3. In Section 4, we briefly mention some interactions between arithmetic dynamics and complex dynamics. For more complete and excellent accounts of the subjects (up to 2007), we refer the reader, for example, to Silverman [50] and Zhang [61].

§ 1. What are algebraic dynamics and arithmetic dynamics?

Let X be a set endowed with a self-map $f : X \to X$. Roughly speaking, in a discrete-time dynamical system, one is interested in asymptotic properties of points and subsets under successive iterations of f. For example, for $x \in X$, one is interested in the behavior of the f-orbit of x:

$$x,\ f(x),\ f(f(x)),\ \cdots .$$

Intuitively, X is a phase space and x is an initial state of the system (at time $t = 0$). Then $f(x)$ describes the state at $t = 1$, and $f(f(x))$ describes the state at $t = 2$, ...

Many researchers have deeply studied asymptotic properties of the system (X, f) when

Received April 01, 2010. Revised October 24, 2010.
2000 Mathematics Subject Classification(s): 11S82, 11G50, 14G40, 37Pxx
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X	f
topological space	continuous map
C^∞-manifold	C^∞-map
complex manifold	holomorphic (or sometimes meromorphic) map
measurable space	measure-preserving transformation
$\cdots$	$\cdots$

The ground field has generally been either $\mathbb{R}$ or $\mathbb{C}$. In algebraic dynamics and arithmetic dynamics, one usually works over

X	f
algebraic variety	morphism (or sometimes a rational map).

But the ground field is not $\mathbb{R}$ nor $\mathbb{C}$, but such field as a finite field, a number field, a non-Archimedean valuation field (and often on a Berkovich space) and a function field. For example, let X be a variety over $\mathbb{Q}$ and $f : X \to X$ a morphism. In arithmetic dynamics, one studies properties of *rational* points under the iterations by f.

It is rather recent that algebraic and arithmetic dynamics have attracted a number of mathematicians. Of course, there are earlier studies. For example, in 1950, Northcott [39] showed finiteness of rational periodic points for morphisms of $\mathbb{P}^N$ over a number field.

In the 2010 Mathematics Subject Classification (MSC2010), the codes

11S82 Non-Archimedean dynamical systems [See mainly 37Pxx]
37Pxx Arithmetic and non-Archimedean dynamical systems

are added. We remark that "11Sxx" represents "Algebraic number theory: local and p-adic fields," which may justify that this note be included in the proceedings of the workshop "Algebraic number theory and related topics."

Algebraic and arithmetic dynamics lie in the intersection of number theory, algebraic geometry and dynamical systems. It remains to be seen how rich they are.

Acknowledgment. We would like to thank the referee for helpful comments.

§ 2. Number-theoretic viewpoint

§ 2.1. Polarized dynamical system

Let K be a field. We denote by $\overline{K}$ the algebraic closure of K. Let A be an abelian variety over K, and $[2] : A \to A$ the twice multiplication map. An ample line bundle L

over X is said to be symmetric if $[-1]^*(L) \simeq L$. In this case, we have

$$[2]^*L \simeq L^{\otimes 4}.$$

It is easy to see that, for $x \in A(\overline{K})$, x is a torsion point if and only if $\{[2]^n(x) \mid n \geq 1\}$ is a finite set. Thus the property that x being torsion is expressed by a condition on x under the iterations by $[2]$. As we will see in Subsections 2.3–2.5, some properties of abelian varieties are expressed in the context of the dynamical system $[2] : A \to A$.

Polarized dynamical systems (cf. Zhang [61, §1.1]) are dynamical systems which include abelian varieties together with the twice multiplication map.

Definition 2.1 (Polarized dynamical system). Let K be a field, X a projective variety over K, L an ample line bundle over X, and $f : X \to X$ a morphism. The triple (X, f, L) is called a *polarized dynamical system* of degree $d \geq 2$ if

$$f^*L \simeq L^{\otimes d}.$$

In fact, Zhang's definition is more general, treating the case when X is a compact Kähler variety over $\mathbb{C}$, i.e., an analytic variety which admits a finite map to a Kähler manifold.

We give three examples of polarized dynamical systems.

Example 2.2 (Abelian variety). As we have seen, if A is an abelian variety over a field K, $[2] : A \to A$ is the twice multiplication map, and L is a symmetric ample line bundle, then the triple $(A, [2], L)$ is a polarized dynamical system of degree 4.

Example 2.3 (Projective space). Let $\mathbb{P}^N$ be projective space. Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of (algebraic) degree $d \geq 2$ over a field K. By this, we mean that there exist homogeneous polynomials $F_0, \ldots, F_N \in K[X_0, \ldots, X_N]$ of degree d with coefficients in K whose common zero is only the origin $(0, \ldots, 0)$ such that $f = (F_0 : \cdots : F_N)$. Let $\mathcal{O}_{\mathbb{P}^N}(1)$ be an ample line bundle associated to a hyperplane on $\mathbb{P}^N$. Then we have

$$f^*\mathcal{O}_{\mathbb{P}^N}(1) \simeq \mathcal{O}_{\mathbb{P}^N}(d).$$

Thus $(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1), f)$ is a polarized dynamical system of degree d.

Example 2.4 (Projective surface). Zhang [61, §2.3] classified projective surfaces X over $\mathbb{C}$ which admit polarized dynamical systems: X is either an abelian surface; a hyperelliptic surface; a toric surface; or a ruled surface $\mathbb{P}_C(\mathcal{E})$ over an elliptic curve C such that either $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{M}$ with $\mathcal{M}$ torsion of positive degree, or $\mathcal{E}$ is not decomposable and has odd degree.

We remark that Fujimoto and Nakayama studied the structure of compact complex varieties which admit non-trivial surjective endomorphisms, not necessarily polarized ones (cf. [21, 22, 23]).

Fakhruddin [17, Corollary 2.2] showed that every polarized dynamical system is embedded into projective space.

Theorem 2.5 (Fakhruddin [17]). *Let (X, L, f) be a polarized dynamical system over an infinite field K. Then there exists $m \geq 1$ such that $(X, L^{\otimes m}, f)$ can be extended to a polarized dynamical system $(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1), \widetilde{f})$ with $N = \dim H^0(X, L^{\otimes m})$:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi_{|L^{\otimes m}|}"'] & X \arrow[d, "\varphi_{|L^{\otimes m}|}"] \\ \mathbb{P}^N \arrow[r, "\widetilde{f}"] & \mathbb{P}^N. \end{tikzcd}$$

Theorem 2.5 is sometimes used to deduce some properties of (X, L, f) from those of $(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1), \widetilde{f})$.

§ 2.2. Heights, metrics and measures

This subsection is a preliminary to the following subsections. We briefly recall some basic facts on heights. Then we recall Néron–Tate's heights and cubic metrics on abelian varieties, and we see how they are generalized in polarized dynamical systems. For more details on heights, we refer the reader, for example, to [27] and [11].

Heights

Let K be a number field. We denote by M_K the set of places of K. For $v \in M_K$, let $|\cdot|_v$ be the normalized v-adic norm of K.

Definition 2.6 (Height). For $x = (x_0 : \cdots : x_N) \in \mathbb{P}^N(K)$, the (logarithmic naive) height is defined by

$$h(x) := \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} \log \max\{|x_0|_v, \ldots |x_N|_v\}.$$

The value $h(x)$ is well-defined by the product formula, and gives rise to the function $h : \mathbb{P}^N(\overline{K}) \to \mathbb{R}_{\geq 0}$.

Suppose that $x \in \mathbb{P}^N(\mathbb{Q})$. We write $x = (x_0 : \cdots : x_N)$ with $x_i \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_N) = 1$. Then $h(x) = \log \max\{|x_0|_\infty, \ldots |x_N|_\infty\}$. We see that, for any $M \in \mathbb{R}$,

$$\{x \in \mathbb{P}^N(\mathbb{Q}) \mid h(x) \leq M\}$$

is a finite set. Indeed, the cardinality is at most $(2\lfloor \exp(M) \rfloor + 1)^{N+1}$.

This property is sometimes called Northcott's finiteness property of heights, and holds in general. (The above case is when $K = \mathbb{Q}$ and $D = 1$ in Proposition 2.7.) Northcott's property is rather easy to prove but quite useful.

Proposition 2.7 (Northcott's finiteness property of heights). *For any $D \in \mathbb{Z}_{>0}$ and $M \in \mathbb{R}$,*

$$\{x \in \mathbb{P}^N(\overline{K}) \mid [K(x) : K] \leq D,\ h(x) \leq M\}$$

is a finite set.

Let X be a projective variety over K, and L a line bundle on X. We recall a height function $h_L : X(\overline{K}) \to \mathbb{R}$ associated to L.

Assume first that L is very ample. Then, choosing a basis of $H^0(X, L)$, we have an embedding $\varphi_{|L|} : X \to \mathbb{P}^N$ with $N = \dim H^0(X, L) - 1$. We define a height function $h_L : X(\overline{K}) \to \mathbb{R}$ by

$$h_L(x) = h(\varphi_{|L|}(x)).$$

The function h_L depends on the choice of a basis of $H^0(X, L)$. However, if we choose another basis of $H^0(X, L)$ and write the corresponding height function for h'_L, then the difference $h_L - h'_L$ is bounded on $X(\overline{K})$. Thus, up to bounded functions on $X(\overline{K})$, h_L does not depend on the choice of a basis of $H^0(X, L)$.

In general, we write $L = M_1 \otimes M_2^{-1}$ with very ample M_i, and define $h_L := h_{M_1} - h_{M_2}$. Then, up to bounded functions on $X(\overline{K})$, h_L is well-defined, independent of the choice of M_1 and M_2. The function h_L is called a *height function associated to L*.

Abelian varieties: Néron–Tate's heights and cubic metrics

Let A be an abelian variety over a field K, and L a symmetric ample line bundle on A.

First we let K be a number field. *Néron–Tate's height function*

$$\widehat{h}_{NT} : A(\overline{K}) \to \mathbb{R}_{\geq 0}$$

is a unique height function associated to L which satisfies $\widehat{h}_{NT}([2](x)) = 4\widehat{h}_{NT}(x)$. Note that $\widehat{h}_{NT}$ depends on L. Indeed, let h_L be any height function associated to L. Then, for $x \in A(\overline{K})$, $\widehat{h}_{NT}(x)$ is given by $\widehat{h}_{NT}(x) = \lim_{n\to\infty} \frac{1}{4^n} h_L([2]^n(x))$. We have

$$\widehat{h}_{NT}(x) = 0 \iff x \text{ is torsion.}$$

Now we let $K = \mathbb{C}$, so that A is an abelian variety over $\mathbb{C}$. We fix an isomorphism $\alpha : L^{\otimes 4} \simeq [2]^*L$. Then a *cubic metric* $\|\cdot\|_{cub}$ is a unique C^∞-hermitian metric on L over A satisfying $(\alpha^*[2]^*\|\cdot\|_{cub})^{\frac{1}{4}} = \|\cdot\|_{cub}$. Note that $\|\cdot\|_{cub}$ depends on the choice of α.

The first Chern form $c_1(L_{\mathbb{C}}, \|\cdot\|_{cub})$ is a translation-invariant $(1,1)$-form on A, and

$$\mu := \frac{1}{\deg_L(A)} c_1(L, \|\cdot\|_{cub})^{\dim A}$$

is the normalized Haar measure on A.

Example 2.8. Let $Z = X + \sqrt{-1}Y$ be a symmetric $g \times g$ matrix such that the imaginary part Y is positive definite. Then $A \simeq \mathbb{C}^g/(\mathbb{Z}^g + Z\mathbb{Z}^g)$ is a principally polarized abelian variety over $\mathbb{C}$. For $z = x + \sqrt{-1}y \in \mathbb{C}^g$, let

$$\theta(Z, z) := \sum_{n \in \mathbb{Z}^g} \exp(\pi\sqrt{-1}\,{}^t nZn + 2\pi\sqrt{-1}\,{}^t nz)$$

be the theta function, and let Θ denote the zero set of θ in A. Then the line bundle $\mathcal{O}_A(\Theta)$ associated to Θ is symmetric and ample.

A cubic metric $\|\cdot\|_{cub}$ on $\mathcal{O}_A(\Theta)$ is given by

$$\|1\|_{cub}(x + \sqrt{-1}y) = \det(Y)^{\frac{1}{4}} \cdot \exp(-\pi y Y^{-1} y) \cdot |\theta(Z, x + \sqrt{-1}y)|,$$

where 1 is the section of $\mathcal{O}_A(\Theta)$ corresponding to θ. The normalized Haar measure is given by

$$\mu = \frac{1}{\det(Y)} \left(\frac{\sqrt{-1}}{2}\right)^g dz_1 \wedge d\overline{z}_1 \cdots dz_g \wedge d\overline{z}_g,$$

where $z = {}^t(z_1, \ldots, z_g) \in \mathbb{C}^g$.

Polarized dynamical systems: Canonical heights, canonical metrics and canonical measures

Let (X, f, L) be a polarized dynamical system of degree $d \geq 2$ over a field K.

First we let K be a number field. Let h_L be a height function associated to L. Then, for $x \in X(\overline{K})$, the limit

$$\widehat{h}_{L,f}(x) = \lim_{n \to \infty} \frac{1}{d^n} h_L(f^n(x))$$

exists (cf. [12]). The function $\widehat{h}_{L,f} : X(\overline{K}) \to \mathbb{R}_{\geq 0}$ does not depend on the choice of h_L associated to L, and satisfies $\widehat{h}_{L,f}(f(x)) = d\,\widehat{h}_{L,f}(x)$. It is called a *canonical height function* for L. A point $x \in X(\overline{K})$ is said to be *preperiodic* under f if $\{f^n(x) \mid n \geq 1\}$ is a finite set. By Call–Silverman [12],

$$\widehat{h}_{L,f}(x) = 0 \Longleftrightarrow x \text{ is preperiodic under } f.$$

Now we let $K = \mathbb{C}$, and let (X, f, L) be a polarized dynamical system of degree $d \geq 2$ over $\mathbb{C}$. We fix an isomorphism $\alpha : L^{\otimes d} \simeq f^*L$. Let $\|\cdot\|$ be a C^0-hermitian metric of L. Then the limit

$$\widehat{\|\cdot\|}_f := \lim_{n\to\infty} \underbrace{\left(\alpha^* f^* \cdots (\alpha^* f^* \|\cdot\|)^{\frac{1}{d}} \cdots\right)^{\frac{1}{d}}}_{n\text{-times}}$$

exists (cf. [59]). The metric $\widehat{\|\cdot\|}_f$ is again a C^0-hermitian metric of L, does not depend on the choice of an initial metric $\|\cdot\|$ of L, and satisfies $\widehat{\|\cdot\|}_f = \left(\alpha^* f^* \widehat{\|\cdot\|}_f\right)^{\frac{1}{d}}$. Note that $\widehat{\|\cdot\|}_f$ depends on the choice of α. It is called a *canonical metric* of L for f.

Since L is assumed to be ample, we can take an initial metric $\|\cdot\|$ of L such that $\|\cdot\|$ is C^∞ and the first Chern form $c_1(L, \|\cdot\|)$ is positive. Here, $c_1(L, \|\cdot\|)$ is locally defined by $dd^c(-\log \|s\|^2)$ on $X \setminus \mathrm{Supp}(\mathrm{div}(s))$, where s is any non-zero rational section s of L. It follows that $-\log \widehat{\|s\|}_f^2$ is continuous and pluri-subharmonic on $X \setminus \mathrm{Supp}(\mathrm{div}(s))$. Then $c_1(L, \widehat{\|\cdot\|}_f)$ is positive in the sense of currents, and one can take exterior products of $c_1(L, \widehat{\|\cdot\|}_f)$ (cf. [1, Theorem 2.1]) to define

$$\mu_f := \frac{1}{\deg_L(X)} c_1(L, \widehat{\|\cdot\|}_f)^{\dim X}.$$

The normalized measure μ_f on X is called the *canonical measure* associated to (X, f, L).

Remark. For $f : \mathbb{P}^N \to \mathbb{P}^N$, the canonical measure μ_f is quite complicated in general. It is known that $\mathrm{Supp}(\mu_f)$ is equal to the N-th Julia set for f, which is a "fractal object" in $\mathbb{P}^N(\mathbb{C})$ (cf. [47, Théorème 1.6.5]).

We summarize the content of this subsection:

abelian variety	polarized dynamical system
torsion point	preperiodic point
Néron-Tate height	canonical height
cubic metric	canonical metric
normalized Haar measure	canonical measure
$\cdots$	$\cdots$

§ 2.3. Dynamical equidistribution theorem and dynamical Manin-Mumford and Bogomolov conjectures

Some arithmetic properties on abelian varieties are interpreted in the context of polarized dynamical systems. Then one can ask corresponding arithmetic properties of polarized dynamical systems. Some of them are theorems, but most of them are *conjectural.* In Subsections 2.3–2.5, let us explain some of them. This subsection is devoted to a dynamical equidistribution theorem (proved by Yuan) and dynamical Manin-Mumford and Bogomolov conjectures.

Abelian varieties

We first recall deep results over abelian varieties which serve as models in polarized dynamical systems.

Let A be an abelian variety over a number field K. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in $A(\overline{K})$. Then $\{x_n\}_{n=1}^{\infty}$ is said to be a *generic sequence of small height points* if the following conditions are satisfied:

(i) For any closed subvariety $Z \subsetneq A_{\overline{K}}$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ one has $x_n \notin Z(\overline{K})$.

(ii) $\lim_{n\to\infty} \widehat{h}_{NT}(x_n) = 0$, where $\widehat{h}_{NT}$ is the Néron–Tate height corresponding to a symmetric ample line bundle on A. (Note that the property $\lim_{n\to\infty} \widehat{h}_{NT}(x_n) = 0$ does not depend on the choice of a symmetric ample line bundle on A.)

Example 2.9. As an illustration, let E be an elliptic curve over K, and $x_1, x_2, \ldots \in E(\overline{K})$ be torsion points with $x_i \neq x_j$ $(i \neq j)$. Then $\{x_n\}_{n=1}^{\infty}$ is a generic sequence of small height points.

For $x \in A(\overline{K})$, let $G(x) := \{\sigma(x) \in A(\overline{K}) \mid \sigma \in \mathrm{Gal}(\overline{K}/K)\}$ be the Galois orbit of x. We fix an embedding $\overline{K} \hookrightarrow \mathbb{C}$. Then $G(x)$ is seen as a subset of $A(\mathbb{C})$. We define the Dirac measure $\delta_{G(x)}$ on $A(\mathbb{C})$ by

$$\delta_{G(x)} := \frac{1}{\#G(x)} \sum_{y \in G(x)} \delta_y.$$

Let μ be the normalized Haar measure on $A(\mathbb{C})$.

Szpiro–Ullmo–Zhang [53] proved the following equidistribution theorem.

Theorem 2.10 (Equidistribution on abelian varieties, Szpiro–Ullmo–Zhang [53])**.** *Let $\{x_n\}_{n=1}^{\infty} \subset A(\overline{K})$ be a generic sequence of small height points. Then $\delta_{G(x_n)}$ converges weakly to μ on $A(\mathbb{C})$. Namely, for any complex-valued continuous function f on $A(\mathbb{C})$, one has*

$$\lim_{n\to\infty} \frac{1}{\#G(x_n)} \sum_{y \in G(x_n)} f(y) = \int_{A(\mathbb{C})} f d\mu.$$

Example 2.11. As in Example 2.9, let E be an elliptic curve over a number field K and we fix an embedding $K \hookrightarrow \mathbb{C}$. Let $x_1, x_2, \ldots \in E(\overline{K})$ be torsion points with $x_i \neq x_j$ $(i \neq j)$. Then Theorem 2.10 tells us that the Galois orbit $G(x_n)$ of x_n is asymptotically equidistributed on $E(\mathbb{C}) \simeq \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ as n tends to the infinity.

The key to the proof of Theorem 2.10 is a height inequality

$$h_{\overline{L}(\varepsilon f)}(A) \leq \sup_{Z \subsetneq A_{\overline{K}}} \inf_{x \in (A \setminus Z)(\overline{K})} h_{\overline{L}(\varepsilon f)}(x)$$

due to Zhang [58, Theorem (5.2)]. Here, L is a symmetric ample line bundle on A, and $h_{\overline{L}(\varepsilon f)}$ is the height associated to the metrized line bundle $\overline{L}(\varepsilon f)$ in Arakelov geometry.

We recall the Manin–Mumford conjecture (proved by Raynaud) and the Bogomolov conjecture (proved by Ullmo and Zhang).

Theorem 2.12 (Manin–Mumford conjecture, Raynaud [42]). *Let A be an abelian variety over a number field K. Let Y be a closed subvariety of $A_{\overline{K}}$. Suppose that*

$$Y(\overline{K}) \cap A_{tors} \; (= \{x \in Y(\overline{K}) \mid \widehat{h}_{NT}(x) = 0\})$$

is Zariski dense in $Y_{\overline{K}}$. Then Y is a translate of an abelian subvariety of $A_{\overline{K}}$ by a torsion point.

If one replaces the condition "$\widehat{h}_{NT}(x) = 0$" by a weaker condition "$\widehat{h}_{NT}(x) < \varepsilon$ for any positive number $\varepsilon > 0$," then one has the Bogomolov conjecture.

Theorem 2.13 (Bogomolov conjecture, Ullmo [55] and Zhang [60]). *Let A be an abelian variety over a number field K. Let Y be a closed subvariety of $A_{\overline{K}}$. Suppose that, for any positive number $\varepsilon > 0$,*

$$\{x \in Y(\overline{K}) \mid \widehat{h}_{NT}(x) < \varepsilon\}$$

is Zariski dense in $Y_{\overline{K}}$. Then Y is a translate of an Abelian subvariety of $A_{\overline{K}}$ by a torsion point.

Ullmo proved the theorem when Y is a curve, and Zhang in general. The main ingredients of the proof were Szpiro–Ullmo–Zhang's equidistribution theorem (Theorem 2.10), and a geometric lemma for the map $A^m \to A^{m-1}, (x_1, x_2, \ldots, x_m) \mapsto (x_1 - x_2, x_2 - x_3, \ldots, x_{m-1} - x_m)$.

Polarized dynamical systems

We go back to polarized dynamical systems. Let K be a number field and (X, f, L) a polarized dynamical system over K.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in $X(\overline{K})$. Exactly as in the case of abelian varieties, the sequence is said to be a *generic sequence of small height points* if the following conditions are satisfied:

(i) For any closed subvariety $Z \subsetneq X_{\overline{K}}$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ one has $x_n \notin Z(\overline{K})$.

(ii) $\lim_{n\to\infty} \widehat{h}_L(x_n) = 0$.

For $x \in X(\overline{K})$, let $G(x) := \{\sigma(x) \in X(\overline{K}) \mid \sigma \in \mathrm{Gal}(\overline{K}/K)\}$ be the Galois orbit of x. We fix an embedding $\overline{K} \hookrightarrow \mathbb{C}$. Then $G(x)$ is seen as a subset of $X(\mathbb{C})$. We define the Dirac measure $\delta_{G(x)}$ on $X(\mathbb{C})$ by

$$\delta_{G(x)} := \frac{1}{\#G(x)} \sum_{y \in G(x)} \delta_y.$$

Let μ_f be the canonical measure on $X(\mathbb{C})$ associated to (X, f, L).

Then we can ask if a similar equidistribution property holds for polarized dynamical systems as for abelian varieties. This is Yuan's theorem.

Theorem 2.14 (Equidistribution for polarized dynamical systems, Yuan [56]). *Let $\{x_n\}_{n=1}^{\infty} \subset X(\overline{K})$ be a generic sequence of small height points. Then $\delta_{G(x_n)}$ converges weakly to μ on $X(\mathbb{C})$. Namely, for any complex-valued continuous function f on $X(\mathbb{C})$, one has*

$$\lim_{n\to\infty} \frac{1}{\#G(x_n)} \sum_{y \in G(x_n)} f(y) = \int_{X(\mathbb{C})} f d\mu.$$

Yuan proved an arithmetic version of Siu's theorem on big line bundles. Then, he proved equidistribution for polarized dynamical systems using the idea of the proof of Theorem 2.10 on abelian varieties.

Example 2.15. Let $f : \mathbb{P}^1 \to \mathbb{P}^1, (x_0 : x_1) \mapsto (x_0^2 : x_1^2)$. Then the canonical height for f is nothing but the usual height h (see Definition 2.6), and the canonical measure for f is the Haar measure on the unit circle. Let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ be a sequence of points with $x_i \neq x_j$ $(i \neq j)$ and $\lim_{n\to\infty} h(x_n) = 0$. Theorem 2.14 tells us that the Galois orbit of x_n is asymptotically equidistributed on the unit circle as n tends to the infinity.

The above example is the case of $\mathbb{G}_m$. Equidistribution for $\mathbb{G}_m^N$ is proved by Bilu [10]. Theorem 2.14 is also a generalization of Bilu's theorem.

Zhang [59, Conjecture (2.5)], [61, Conjectures 1.2.1, 4.1.7] formulated a dynamical Manin–Mumford conjecture and a dynamical Bogomolov conjecture in polarized dynamical systems, as natural generalizations from the case of abelian varieties.

Recently, Ghioca and Tucker gave a counterexample to this dynamical Manin–Mumford conjecture. Zhang [57] has reformulated dynamical Manin–Mumford conjectures.

§ 2.4. Dynamical Mordell–Lang problem

In this subsection, we explain a dynamical Mordell–Lang problem due to Ghioca and Tucker.

We first recall Faltings's celebrated theorem [18] on Mordell–Lang conjecture for abelian varieties. The following is its absolute form (cf. [26]).

Theorem 2.16 (Mordell–Lang conjecture in the absolute form). *Let A be an abelian variety over $\mathbb{C}$, V a subvariety of A, and Γ a subgroup of finite rank in $A(\mathbb{C})$. Then there are abelian subvarieties $C_1, \ldots, C_p$ of A and $\gamma_1, \ldots, \gamma_p \in \Gamma$ such that*

$$\overline{V(\mathbb{C}) \cap \Gamma} = \bigcup_{i=1}^{p} (C_i + \gamma_i) \quad \textit{and} \quad V(\mathbb{C}) \cap \Gamma = \bigcup_{i=1}^{p} (C_i(\mathbb{C}) + \gamma_i) \cap \Gamma.$$

Let A and V be as in Theorem 2.16. Let $\gamma \in A(\mathbb{C})$ be a non-torsion point, and let $f_\gamma : A \to A$ be the translation by γ. Consider the set

$$I_{f_\gamma, V}(0) = \{i \in \mathbb{Z}_{>0} \mid f_\gamma^i(0) = i\gamma \in V(\mathbb{C})\} \subset \mathbb{Z}_{>0}.$$

Let Γ be the group of $A(\mathbb{C})$ generated by γ. If $I_{f_\gamma, V}(0)$ is infinite, then Theorem 2.16 implies that V contains the translate of an abelian subvariety of dimension ≥ 1 by an element of γ. Denis [15] considers an analogue of this problem for a morphism of $\mathbb{P}^N$.

Motivated by Faltings's theorem (Mordell–Lang conjecture), Ghioca and Tucker asks the following dynamical Mordell–Lang conjecture (cf. [24, 25, 5]).

Conjecture 2.17 (Dynamical Mordell–Lang conjecture due to Ghioca and Tucker). Let X be a variety defined over $\mathbb{C}$ (not necessarily projective), V a closed subvariety of X, and $f : X \to X$ a morphism. Let $P \in X(\mathbb{C})$. Consider the set

$$I_{f,V}(P) = \{i \in \mathbb{Z}_{>0} \mid f^i(P) \in V(\mathbb{C})\} \subset \mathbb{Z}_{>0}.$$

Then $I_{f,V}(P)$ is a union of at most finitely many arithmetic progressions and finitely many other integers, i.e., there are $(k_1, \ell_1), \ldots, (k_p, \ell_p) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$ such that

$$I_{f,V}(P) = \bigcup_{j=1}^{p} \{k_j n + \ell_j \mid n \geq 1\}.$$

Bell–Ghioca–Tucker [5] show that, if f is unramified, then the dynamical Mordell–Lang conjecture is true.

There is a version of dynamical Mordell–Lang problem in which several morphisms $f_1, \ldots, f_n : X \to X$ are considered. This dynamical Mordell–Lang problem for several morphisms is not true in general, but there are some cases where this is known to be true. The following theorem for two morphisms, due to Ghioca–Tucker–Zieve [24, Theorem 1.1], is one of them.

Theorem 2.18 (Ghioca–Tucker–Zieve [24]). *Let $f, g \in \mathbb{C}[x]$ be polynomials of degree at least 2. Let $P, Q \in \mathbb{C}$. If*

$$\{f^k(P) \mid k \geq 1\} \cap \{g^k(Q) \mid k \geq 1\}$$

is infinite, then there exists positive integers m and n such that $f^m = g^n$.

§ 2.5. Uniform boundedness conjecture

Let K be a number field, and $f : \mathbb{P}^N \to \mathbb{P}^N$ a morphism of (algebraic) degree $d \geq 2$ over K. Recall that a preperiodic point is nothing but a point with zero canonical height. Northcott's finiteness property (Proposition 2.7) then implies the following.

Proposition 2.19. *The set*

$$\mathrm{PrePer}(f;K) := \{x \in \mathbb{P}^N(K) \mid x \textit{ is a preperiodic point under } f\}$$

is finite.

Morton–Silverman's uniform boundedness conjecture concerns $\#\,\mathrm{PrePer}(f;K)$.

Conjecture 2.20 (Uniform boundedness conjecture, Morton–Silverman [38])**.** For all positive integers D, N, d with $d \geq 2$, there exists a constant $\kappa(D,N,d)$ depending only on D, N, d with the following property: For any number field K with $[K : \mathbb{Q}] = D$, and any morphism $f : \mathbb{P}^N \to \mathbb{P}^N$ of degree $d \geq 2$ over K, one has

$$\#\,\mathrm{PrePer}(f;K) \leq \kappa(D,N,d).$$

Conjecture 2.20 is considered to be very difficult, because, by Theorem 2.5, it will imply the corresponding uniform boundedness of torsion points on abelian varieties. (The case of elliptic curves is celebrated results by Mazur and Merel.)

Poonen [41] has refined the uniform boundedness conjecture for quadratic polynomials over $\mathbb{Q}$. For $c \in \mathbb{Q}$, we consider a quadratic polynomial

$$f_c(X) = X^2 + c.$$

We still denote by f_c its extension to $\mathbb{P}^1$:

$$f_c : \mathbb{P}^1 \to \mathbb{P}^1, \quad (x_0 : x_1) \mapsto (x_0^2 + cx_1^2 : x_1^2).$$

Let $x \in \mathbb{P}^1(\mathbb{Q})$ be a periodic point under f_c. The *exact* period of x is the smallest positive integer m with $f_c^m(x) = x$.

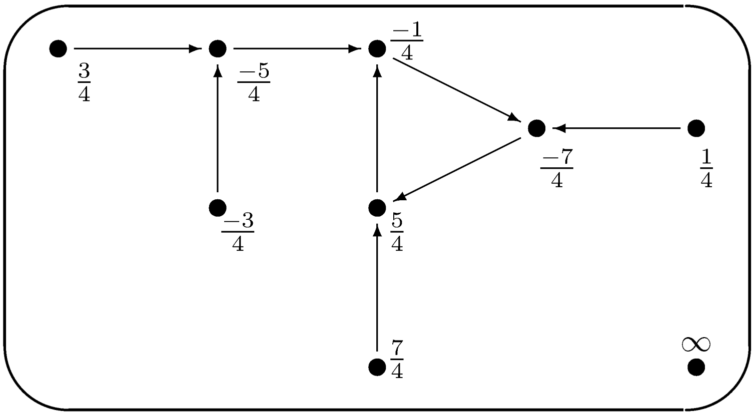
Conjecture 2.21 (Quardratic polynomials over $\mathbb{Q}$, refined by Poonen)**.** For any $c \in \mathbb{Q}$, one has the following.

(a) If $x \in \mathbb{P}^1(\mathbb{Q})$ is a periodic point under f_c, then the exact period of x is at most 3.

(b) $\#\,\mathrm{PrePer}(f_c;\mathbb{Q}) \leq 9$.

Poonen [41] showed that (a) implies (b).

Example 2.22 ([41, p. 17]). Let $c = -\frac{29}{16}$ and consider $f_{-\frac{29}{16}}(X) = X^2 - \frac{29}{16}$. Then we have the following figure. This example shows that the conditions (a) and (b) are optimal.



We list some results related to the uniform boundedness conjecture for quadratic polynomials over $\mathbb{Q}$.

Theorem 2.23 (Morton, Flynn–Poonen–Schaefer, Stoll, Benedetto, ...). *Let $c \in \mathbb{Q}$, $f_c : \mathbb{P}^1 \to \mathbb{P}^1$ be the morphism defined by $(x_0 : x_1) \mapsto (x_0^2 + cx_1^2 : x_1^2)$, and $x \in \mathbb{P}^1(\mathbb{Q})$ a periodic point under f_c.*

(a) *The exact period of x is not* 4 (Morton [37]).

(b) *The exact period of x is not* 5 (Flynn–Poonen–Schaefer [20]).

(c) *Assuming the Birch and Swinnerton-Dyer conjecture for the Jacobian of a certain curve. Then the exact period of x is not* 6 (Stoll [52]).

(d) *Let s be the number of primes of bad reduction of f_c. Then* $\#\operatorname{PrePer}(f_c; \mathbb{Q}) = O(s \log s)$ (Benedetto [8]).

Both Flynn–Poonen–Schaefer's proof and Stoll's proof use Chabauty's method. Benedetto's proof is based on careful analysis of the p-adic filled-in Julia sets.

§ 2.6. Other dynamical systems

There are interesting dynamical systems (X, f) that do not fall in polarized dynamical systems. For example, some arithmetic properties are studied for the following classes.

- Certain $K3$ surfaces with two involutions (Silverman [48]).

- X is a smooth projective surface and f is an automorphism of positive topological entropy (Kawaguchi [30]).
- Certain automorphisms of $\mathbb{A}^N$ (Silverman [49], Denis [16], Marcello [34, 35], Kawaguchi [29, 31], Lee [33]).

§ 3. Dynamical viewpoint

In this section, we would like to touch on more dynamical aspects of algebraic and arithmetic dynamics. We restrict ourselves to morphisms of $\mathbb{P}^N$. In Subsection 3.1, we recall some basic properties of complex dynamics for later comparison. In Subsections 3.2 and 3.3, we present some results on non-Archimedean dynamics over $\mathbb{C}_p$ and on Berkovich spaces. For details on complex dynamics, we refer the reader, for example, to Milnor [36] (in dimension one) and Sibony [47] (in higher dimension). For details on dynamics on Berkovich $\mathbf{P}^1$, we refer the reader, for example, to the recent book of Baker–Rumely [4].

§ 3.1. The case $\mathbb{C}$

For $x = (x_0, \dots, x_N) \in \mathbb{C}^{N+1}$, we set $\|x\| = \|x\|_\infty := \sqrt{|x_0|_\infty^2 + \cdots + |x_N|_\infty^2}$. For $x = (x_0 : \ldots : x_N), y = (y_0 : \ldots : y_N) \in \mathbb{P}^N(\mathbb{C})$, the *chordal metric* on $\mathbb{P}^N(\mathbb{C})$ is defined by

$$\rho(x, y) = \frac{\max_{i<j}\{|x_i y_j - x_j y_i|_\infty\}}{\|x\|_\infty \cdot \|y\|_\infty}.$$

Here, by slight abuse of notation, we write the same $x = (x_0, \dots, x_N)$ for a point in $\mathbb{C}^{N+1}$ satisfying $x = (x_0 : \cdots : x_N) \in \mathbb{P}^N(\mathbb{C})$.

Fatou and Julia sets

Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism over $\mathbb{C}$, and let $x = (x_0 : \ldots : x_N) \in \mathbb{P}^N(\mathbb{C})$. We say that f is *equicontinuous* at x if the family $\{f^n\}_{n\geq 1}$ is equicontinuous at x:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in \mathbb{P}^N(\mathbb{C})\ \forall n \in \mathbb{N}\ \left(\rho(x, y) < \delta \to \rho(f^n(x), f^n(y)) < \varepsilon\right).$$

Definition 3.1 (Fatou and Julia sets)**.** The *Fatou set* $\mathcal{F}(f)$ is the maximal open subset of $\mathbb{P}^N(\mathbb{C})$ on which f is equicontinuous. The *Julia set* $\mathcal{J}(f)$ is the complement of $\mathcal{F}(f)$ in $\mathbb{P}^N(\mathbb{C})$.

The Julia set of $f(X) = X^2$ is the unit circle, but the Julia sets are quite complicated in general. See Figure 1 for the Julia set for some quadratic polynomials (drawn by H. Inou).

We collect some basic facts on Julia and Fatou sets. For a morphism $f : \mathbb{P}^N \to \mathbb{P}^N$ of degree $d \geq 2$ over $\mathbb{C}$, the followings are known.

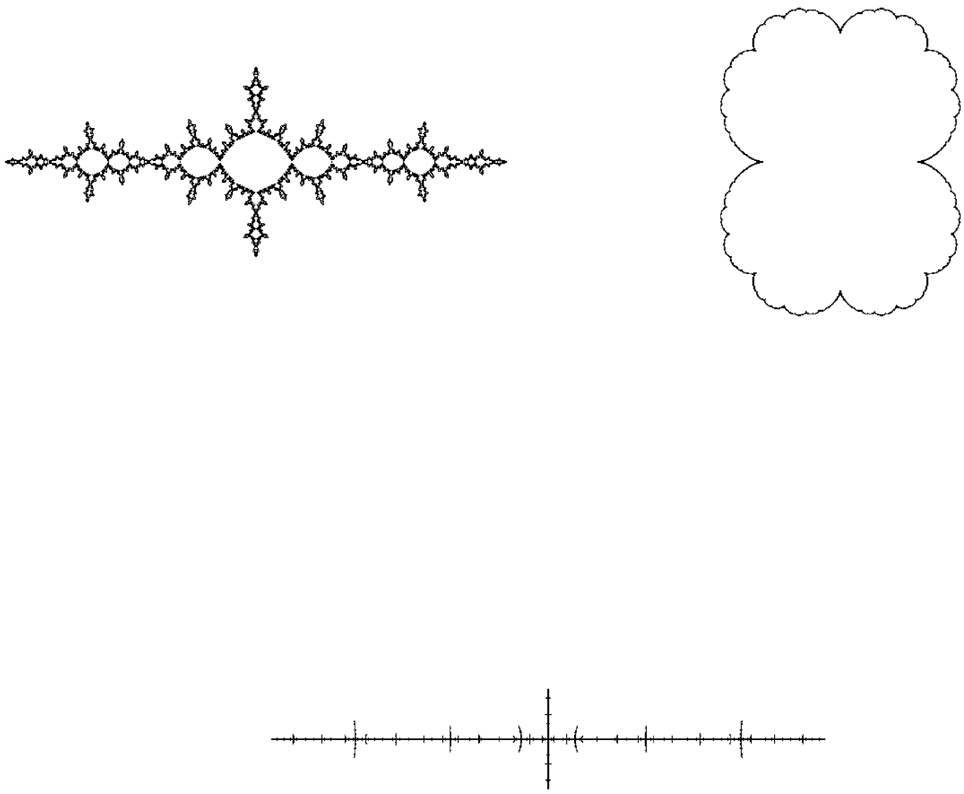


Figure 1. The Julia set of $f_c(X) = X^2 + c$. The top-left figure is when $c = \frac{-21}{16}$. The top-right figure is when $c = \frac{1}{4}$. The bottom figure is when $c = \frac{-29}{16}$.

- The Julia set is always nonempty: $\mathcal{J}(f) \neq \emptyset$.
- The Fatou set can be empty: for a certain f, $\mathcal{F}(f) = \emptyset$.

Suppose now that $N = 1$, and let $x \in \mathbb{P}^1(\mathbb{C})$ be a periodic point under $f : \mathbb{P}^1 \to \mathbb{P}^1$ with exact period n. Then x is said to be *repelling* if $|(f^n)'(x)|_\infty > 1$. The followings are known.

- Except for finitely many periodic points, periodic points are repelling.

- $\overline{\{\text{repelling periodic point}\}} = \mathcal{J}(f)$. (Here $\overline{A}$ denotes the closure of A.)

We also recall Montel's theorem, which is useful when studying $f : \mathbb{P}^1 \to \mathbb{P}^1$.

Theorem 3.2 (Montel's theorem)**.** *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism of degree $d \geq 2$ over $\mathbb{C}$. Let U be an open set of $\mathbb{P}^1(\mathbb{C})$. If $\bigcup_{n\geq 1} f^n(U)$ omits at least three points, then $U \subset \mathcal{F}(f)$.*

Green function

Let $f = (F_0 : \cdots : F_N) : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d \geq 2$ over $\mathbb{C}$. We set $F = (F_0, \cdots, F_N) : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ and call a *lift* of f. Given f, a lift F of f is determined up to a nonzero constant.

For $x = (x_0, \ldots, x_N) \in \mathbb{C}^{N+1}$, the *Green function* is defined by

$$G_F : \mathbb{C}^{N+1} \to \mathbb{R}, \quad G_F(x) = \lim_{n\to\infty} \frac{1}{d^n} \log \|F^n(x)\|_\infty.$$

We set $g_F = G_F - \log\|\cdot\|_\infty$. Then g_F is seen as a function on $\mathbb{P}^N(\mathbb{C})$.

The construction of Green functions resembles those of canonical heights and canonical metrics. Indeed, fixing an isomorphism $\alpha : \mathcal{O}(d) \simeq f^*\mathcal{O}(1)$ amounts to choosing a lift F of f. Then one has $\log \widehat{\|\cdot\|}_f(x) = -g_F(x)$.

The *Green current* T_f for f is defined by $T_f = dd^c(2g_F) = c_1(\mathcal{O}(1), \widehat{\|\cdot\|}_f)$. The canonical measure μ_f is nothing but $\mu_f := c_1(\mathcal{O}(1), \widehat{\|\cdot\|}_f)^N = \bigwedge^N T_f$.

Theorem 3.3 (cf. [54], [47, Théorème 1.6.5])**.** $\mathcal{J}(f) = \mathrm{Supp}(T)$. *In other words,*

$$\mathcal{F}(f) = \{x \in \mathbb{P}^N(\mathbb{C}) \mid g_F \textit{ is pluri-harmonic at } x\}.$$

§ 3.2. The case $\mathbb{C}_p$

Now we consider non-Archimedean dynamics over $\mathbb{C}_p$. For $x = (x_0, \ldots, x_N) \in \mathbb{C}_p^{N+1}$, we set $\|x\|_p := \max\{|x_0|_p, \ldots, |x_N|_p\}$. For $x = (x_0 : \ldots : x_N), y = (y_0 : \ldots : y_N) \in \mathbb{P}^N(\mathbb{C}_p)$, the *chordal metric* on $\mathbb{P}^N(\mathbb{C}_p)$ is defined by

$$\rho(x, y) = \frac{\max_{i<j}\{|x_i y_j - x_j y_i|_p\}}{\|x\|_p \cdot \|y\|_p}.$$

Here, by slight abuse of notation, we write the same $x = (x_0, \ldots, x_N)$ for a point in $\mathbb{C}_p^{N+1}$ satisfying $x = (x_0 : \cdots : x_N) \in \mathbb{P}^N(\mathbb{C}_p)$.

Fatou and Julia sets

One is interested in how similar (or how different) are dynamics over $\mathbb{C}$ and dynamics over $\mathbb{C}_p$.

Definition 3.4. The *equicontinuous locus* $\mathcal{F}(f)$ is the maximal open subset of $\mathbb{P}^N(\mathbb{C}_p)$ on which f is equicontinuous (with respect to the chordal metric). The *non-equicontinuous locus* $\mathcal{J}(f)$ is the complement of $\mathcal{F}(f)$ in $\mathbb{P}^N(\mathbb{C}_p)$.

The loci $\mathcal{F}(f)$ and $\mathcal{J}(f)$ correspond to Fatou and Julia sets over $\mathbb{C}$. On Berkovich $\mathbf{P}^1$, there are other definitions of Fatou and Julia sets (cf. [4]). Here, for clarity, we call $\mathcal{F}(f)$ the equicontinuous locus instead of the Fatou set, and similarly for $\mathcal{J}(f)$.

Let us consider corresponding properties of $\mathcal{F}(f)$ and $\mathcal{J}(f)$ over $\mathbb{C}_p$. For a morphism $f : \mathbb{P}^N \to \mathbb{P}^N$ of degree $d \geq 2$ over $\mathbb{C}_p$, the followings are known.

- $\mathcal{F}(f) \neq \emptyset$.
- For a certain f, $\mathcal{J}(f) = \emptyset$.

Compared with $\mathbb{C}$, these are exactly the opposite. However, if we work on Berkovich $\mathbf{P}^1$, $\mathcal{J}(f)$ is always nonempty.

Suppose now that $N = 1$, and let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism of degree $d \geq 2$ over $\mathbb{C}_p$. let $x \in \mathbb{P}^1(\mathbb{C}_p)$ be a periodic point with exact period n. Then x is said to be *repelling* if $|(f^n)'(x)|_p > 1$.

Theorem 3.5 (Bézivin [9]). *If there is at least one repelling periodic point, then there are infinitely many repelling periodic points, and*

$$\overline{\{\textit{repelling periodic point}\}} = \mathcal{J}(f).$$

Conjecture 3.6 (Hsia [28, Conjecture 4.3]). Does $\overline{\{\text{repelling periodic point}\}} = \mathcal{J}(f)$ hold without any assumption? By Theorem 3.5, this means that, if $\mathcal{J}(f) \neq \emptyset$, then does there exist at least one repelling periodic point?

A Montel type theorem over $\mathbb{C}_p$ is obtained by Hsia.

Theorem 3.7 (Hsia [28, Theorem 2.4]). *Let $\overline{D}(x,r) = \{y \in \mathbb{P}^1(\mathbb{C}_p) \mid \rho(x,y) \leq r\}$. If $\bigcup_{n\geq 1} f^n(\overline{D}(x,r))$ omits at least two points, then $\overline{D}(x,r) \subset \mathcal{F}(f)$.*

Wandering domains in $\mathbb{P}^1(\mathbb{C}_p)$

For a polynomial map f over $\mathbb{C}$, a deep result of Sullivan [51] is that there is no wandering domain, i.e., every Fatou component is preperiodic under f.

Benedetto [7] showed that there *does* exist a wandering domain if a polynomial map f is defined over $\mathbb{C}_p$. He also showed that, if f is defined over $\mathbb{Q}_p$, then with some conditions on f there is no wandering domain (see [6]).

Green function

Let $f = (F_0 : \cdots : F_N) : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d \geq 2$ over $\mathbb{C}_p$. Let $F = (F_0, \cdots, F_N) : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ be a lift of f.

For $x = (x_0, \ldots, x_N) \in \mathbb{C}_p^{N+1}$, the *p-adic Green function* is defined by

$$G_F : \mathbb{C}_p^{N+1} \to \mathbb{R}, \quad G_F(x) = \lim_{n\to\infty} \frac{1}{d^n} \log \|F^n(x)\|_p.$$

We set $g_F = G_F - \log \|\cdot\|_p$. Then g_F is seen as a function on $\mathbb{P}^N(\mathbb{C}_p)$.

Theorem 3.8 ([32]). *$\mathcal{F}(f) = \{x \in \mathbb{P}^N(\mathbb{C}_p) \mid g_F$ is locally constant at $x\}$.*

§ 3.3. Berkovich space

The field $\mathbb{C}_p$ is complete and algebraically closed, but is totally disconnected and not locally compact. People now prefer to work over Berkovich spaces (especially in dimension 1). Dynamical properties on Berkovich $\mathbf{P}^1$ are studied deeply by Rivera-Letelier in a series of papers (cf. [43, 44, 45, 46]). See also [4].

In this note, let us only mention equidistribution of small height points on Berkovich spaces. Let (X, f, L) be a polarized dynamical system over a number field K. Let v be a finite place of K, and let $X^{an}_{\mathbb{C}_v}$ denote the Berkovich space associated to X.

Chambert-Loir [13] has defined the canonical measure for f on $X^{an}_{\mathbb{C}_v}$. Yuan [56] proved equidistribution of small height points in this context.

Let $X = \mathbb{P}^1$. Prior to [56], equidistribution of small height points on Berkovich $\mathbf{P}^1$ is obtained by Chambert-Loir [13], Baker–Rumely [3] and Favre–Rivera-Letelier [19].

§ 4. Interactions between arithmetic dynamics and complex dynamics

Some problems in complex dynamics are solved by passing to $\mathbb{C}_p$ and Berkovich spaces. Let us just mention a few topics. Each topic is about a statement over $\mathbb{C}$, but its proof uses non-Archimedean dynamics.

- DeMarco–Rumely [14] obtain the transfinite diameter of the filled-in Julia set of a regular polynomial map $F : \mathbb{C}^N \to \mathbb{C}^N$ in terms of the resultant of F.
- Let (X, f, L) and (X, g, L) be polarized dynamical systems over $\mathbb{C}$ (or any field of characteristic zero). Yuan and Zhang [57] showed that

  (4.1) $\mathrm{PrePer}(f) = \mathrm{PrePer}(g) \iff \mathrm{PrePer}(f) \cap \mathrm{PrePer}(g)$ is Zariski dense in X.

  When $X = \mathbb{P}^1$ and f, g are defined over a number field, Petsche–Szpiro–Tucker [40] showed among other things that the above conditions are also equivalent to the condition that the canonical heights of f and g are the same.
- Baker–DeMarco [2] showed that for any fixed $a, b \in \mathbb{C}$ and $d \geq 2$, the set of complex numbers c for which both a and b are preperiodic for $z^d + c$ is infinite if and only if $a^d = b^d$, answering a question of Zannier in the affirmative. They also showed (4.1) independently when $X = \mathbb{P}^1$.

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