Ramification of local fields and Fontaine’s property $(P_m)$: a résumé

By

MANABU YOSHIDA*

Abstract

This is a résumé of our results ([7], [9]) on Fontaine’s property $(P)$ which is an effective tool for estimating the ramification of torsion Galois representations.

§1. Fontaine’s property $(P_m)$

Let $K$ be a complete discrete valuation field with perfect residue field $k$ of characteristic $p > 0$, $K^{\text{alg}}$ a fixed algebraic closure of $K$, $\overline{K}$ the separable closure of $K$ in $K^{\text{alg}}$ and $v_K$ the valuation on $K^{\text{alg}}$ normalized by $v_K(K^\times) = \mathbb{Z}$. We denote by $G_K$ the absolute Galois group of $K$. Let $G_K^{(m)}$ be the $m$th upper numbering ramification group in the sense of [3]. Namely, we put $G_K^{(m)} = G_K^{m-1}$, where the latter is the upper numbering ramification group defined in [6].

One of the classical problems in ramification theory is to obtain a ramification bound of torsion geometric Galois representations. Assume char$(K)=0$ for the moment. Let $e$ be the absolute ramification index of $K$. Consider a proper smooth variety $X_K$ over $K$ and put $X_{\overline{K}} = X_K \times K \overline{K}$. In [3], Fontaine conjectured the upper numbering ramification group $G_K^{(m)}$ acts trivially on the $r$th étale cohomology group $V = H^r_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z})$ for $m > e(n + r/(p-1))$ if $X_K$ has good reduction. This is equivalent to the inequality $u_{L/K} \leq e(n + r/(p-1))$, where $L$ is defined by $G_L = \text{Ker}(G_K \to \text{Aut}(V))$ and $u_{L/K}$ is the infimum of the real numbers $m$ such that $G_K^{(m)} \subset G_L$. For $e = 1$ and $r < p-1$, this conjecture was proved independently by himself ([4], for $n = 1$) and Abrashkin ([1], for any $n$). There are also similar ramification bounds if $X_K$ has semi-stable reduction by...
Caruso-Liu ([2]) and Hattori ([5]). All of them used Fontaine’s property \((P_m)\) studied in [3] to obtain each ramification bound.

Now, let us go back to the general case where \(K\) is of either characteristic. For an algebraic extension \(E/K\), we denote by \(\mathcal{O}_E\) the integral closure of \(\mathcal{O}_K\) in \(E\). Fontaine’s property \((P_m)\) is the following condition for a finite Galois extension \(L\) of \(K\) and a real number \(m\):

\[
(P_m) \quad \text{For any algebraic extension } E/K, \text{ if there exists an } \mathcal{O}_K\text{-algebra homomorphism } \mathcal{O}_L \to \mathcal{O}_E/\mathfrak{a}_{E/K}^m, \text{ then there exists a } K\text{-embedding } L \hookrightarrow E,
\]

where \(\mathfrak{a}_{E/K}^m = \{ x \in \mathcal{O}_E \mid v_K(x) \geq m \}\). We define the greatest upper numbering ramification break \(u_{L/K}\) as above. Fontaine proved the following:

**Proposition 1.1** ([3], Prop. 1.5). Let \(L\) be a finite Galois extension of \(K\), \(m\) a real number and \(e_{L/K}\) the ramification index of \(L/K\). Then there are following relations:

(i) If we have \(m > u_{L/K}\), then \((P_m)\) is true.

(ii) If \((P_m)\) is true, then we have \(m > u_{L/K} - e_{L/K}^{-1}\).

**Remark.** The above (i) can be generalized to the imperfect residue field case by Abbes-Saito’s ramification theory ([9], Prop. 4.3).

Given a torsion Galois representation \(V\) of \(G_K\), we can use the above proposition to bound its ramification as follows: Let \(L/K\) be the finite Galois extension defined by \(G_L = \text{Ker}(G_K \to \text{Aut}(V))\). If \(V\) is of some geometric origin, it is often possible to verify \((P_m)\) for \(L/K\) and a suitable \(m\). Then the above inequality of (ii) gives the upper bound \(u_{L/K} < m + e_{L/K}^{-1}\). Thus it will be useful to sharpen the inequality. Indeed, our first main theorem below shows that we can improve the bound to \(u_{L/K} \leq m\). This result is actually used in [5], Section 5, Proposition 5.6.

For a finite Galois extension \(L\) of \(K\), we put

\[
m_{L/K} = \inf \{ m \in \mathbb{R} \mid (P_m) \text{ is true for } L/K \}
\]

By the above proposition, we have the inequalities

\[
u_{L/K} - e_{L/K}^{-1} \leq m_{L/K} \leq u_{L/K}.
\]

More precisely, we have the following equality:

**Theorem 1.2** ([9], Prop. 3.3). We have \(u_{L/K} = m_{L/K}\).

An outline of the proof: We can prove easily that \((P_m)\) is not true if \(m = u_{L/K}\) (hence \(u_{L/K} = m_{L/K}\)) in the case where \(L/K\) is at most tamely ramified. Hence we may assume \(L/K\) is wildly ramified. It suffices to show the inequality \(u_{L/K} - (e')^{-1} \leq m_{L/K}\) with an
arbitrarily large integer $e'$. Take an arbitrary finite tamely ramified Galois extension $K'$ of $K$ and put $L' = LK'$. Then we have $u_{L'/K} = u_{L/K}$ since $L/K$ is wildly ramified. If we apply (ii) of Proposition 1.1 to $L'/K$, and (i) of Proposition 1.1 to $K'/K$ respectively, then we can prove the inequality $u_{L/K} - e_{L/K}^{-1} \leq m_{L/K}$.

§2. ($P_m$) at the ramification break

Let $L$ be a finite Galois extension of $K$ and $e$ its ramification index. Assume $L/K$ to be totally and wildly ramified for simplicity. In this section, we completely determine the truth of ($P_m$)\(^1\). The equality $u_{L/K} = m_{L/K}$ in Theorem 1.2 gives no information about the truth of ($P_m$) at the break $m = u_{L/K}$. The behavior of ($P_m$) at the break depends on the residue field:

**Theorem 2.1** ([7], Thm. 1.1). The property ($P_m$) is true for $m = u_{L/K}$ if and only if the residue field $k$ has no Galois extension whose degree is divisible by $p$.

This theorem can be proved by using the local class field theory of Serre and Hazewinkel. On the other hand, we consider a weaker property ($P_m^e$) as follows:

($P_m^e$) For any totally ramified extension $E/K$ of degree $e$, if there exists an $\mathcal{O}_K$-algebra homomorphism $\mathcal{O}_L \to \mathcal{O}_E / a_{E/K}^m$, then there exists a $K$-embedding $L \hookrightarrow E$.

Then we have the following theorem, which is a similar result as Theorem 2.1. The proof employs the notion of a non-Archimedean metric on the set of all Eisenstein polynomials over $K$.

**Theorem 2.2** ([8], Thm. A, Prop. 5.1). The property ($P_m^e$) is true for $m = u_{L/K}$ if and only if the residue field $k$ has no Galois extension of degree $p$.

**Remark.** Both (i) and (ii) of Proposition 1.1 remain true for a finite totally and wildly ramified Galois extension $L$ of $K$ if we consider ($P_m^e$) instead of ($P_m$). However, ($P_m^e$) does not satisfy the equality in Proposition 1.2 in general. In fact, we can check that ($P_m^e$) is true for $m = u_{L/K}$ if and only if ($P_m^e$) is true for $m > u_{L/K} - e_{L/K}^{-1}$.

\(^1\)The results in this section were obtained by joint work with Takashi Suzuki after the talk in "Algebraic Number Theory and Related Topics 2009".
References

[7] Suzuki, T. and Yoshida, M., Fontaine’s property \((P_m)\) at the maximal ramification break, *preprint*