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Ramification of local fields and Fontaine's property (P_m): a résumé

By

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Abstract

This is a résumé of our results ([7], [9]) on Fontaine's property (P_m) which is an effective tool for estimating the ramification of torsion Galois representations.

§ 1. Fontaine's property (P_m)

Let K be a complete discrete valuation field with perfect residue field k of characteristic $p > 0$, K^{alg} a fixed algebraic closure of K, $\bar{K}$ the separable closure of K in K^{alg} and v_K the valuation on K^{alg} normalized by $v_K(K^\times) = \mathbb{Z}$. We denote by G_K the absolute Galois group of K. Let $G_K^{(m)}$ be the mth upper numbering ramification group in the sense of [3]. Namely, we put $G_K^{(m)} = G_K^{m-1}$, where the latter is the upper numbering ramification group defined in [6].

One of the classical problems in ramification theory is to obtain a ramification bound of torsion geometric Galois representations. Assume char(K)=0 for the moment. Let e be the absolute ramification index of K. Consider a proper smooth variety X_K over K and put $X_{\bar{K}} = X_K \times_K \bar{K}$. In [3], Fontaine conjectured the upper numbering ramification group $G_K^{(m)}$ acts trivially on the rth étale cohomology group $V = H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z})$ for $m > e(n + r/(p-1))$ if X_K has good reduction. This is equivalent to the inequality $u_{L/K} \leq e(n + r/(p-1))$, where L is defined by $G_L = \mathrm{Ker}(G_K \to \mathrm{Aut}(V))$ and $u_{L/K}$ is the infimum of the real numbers m such that $G_K^{(m)} \subset G_L$. For $e = 1$ and $r < p-1$, this conjecture was proved independently by himself ([4], for $n = 1$) and Abrashkin ([1], for any n). There are also similar ramification bounds if X_K has semi-stable reduction by

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Caruso-Liu ([2]) and Hattori ([5]). All of them used Fontaine's property (P_m) studied in [3] to obtain each ramification bound.

Now, let us go back to the general case where K is of either characteristic. For an algebraic extension E/K, we denote by $\mathcal{O}_E$ the integral closure of $\mathcal{O}_K$ in E. Fontaine's property (P_m) is the following condition for a finite Galois extension L of K and a real number m:

(P_m) *For any algebraic extension E/K, if there exists an $\mathcal{O}_K$-algebra homomorphism $\mathcal{O}_L \to \mathcal{O}_E/\mathfrak{a}^m_{E/K}$, then there exists a K-embedding $L \hookrightarrow E$,*

where $\mathfrak{a}^m_{E/K} = \{x \in \mathcal{O}_E \mid v_K(x) \geq m\}$. We define the greatest upper numbering ramification break $u_{L/K}$ as above. Fontaine proved the following:

Proposition 1.1 ([3], Prop. 1.5). *Let L be a finite Galois extension of K, m a real number and $e_{L/K}$ the ramification index of L/K. Then there are following relations:*
(i) *If we have $m > u_{L/K}$, then (P_m) is true.*
(ii) *If (P_m) is true, then we have $m > u_{L/K} - e^{-1}_{L/K}$.*

Remark. The above (i) can be generalized to the imperfect residue field case by Abbes-Saito's ramification theory ([9], Prop. 4.3).

Given a torsion Galois representation V of G_K, we can use the above proposition to bound its ramification as follows: Let L/K be the finite Galois extension defined by $G_L = \mathrm{Ker}(G_K \to \mathrm{Aut}(V))$. If V is of some geometric origin, it is often possible to verify (P_m) for L/K and a suitable m. Then the above inequality of (ii) gives the upper bound $u_{L/K} < m + e^{-1}_{L/K}$. Thus it will be useful to sharpen the inequality. Indeed, our first main theorem below shows that we can improve the bound to $u_{L/K} \leq m$. This result is actually used in [5], Section 5, Proposition 5.6.

For a finite Galois extension L of K, we put

$$m_{L/K} = \inf\{m \in \mathbb{R} \mid (\mathrm{P}_m) \text{ is true for } L/K\ \}$$

By the above proposition, we have the inequalities

$$u_{L/K} - e^{-1}_{L/K} \leq m_{L/K} \leq u_{L/K}.$$

More precisely, we have the following equality:

Theorem 1.2 ([9], Prop. 3.3). *We have $u_{L/K} = m_{L/K}$.*

An outline of the proof: We can prove easily that (P_m) is not true if $m = u_{L/K}$ (hence $u_{L/K} = m_{L/K}$) in the case where L/K is at most tamely ramified. Hence we may assume L/K is wildly ramified. It suffices to show the inequality $u_{L/K} - (e')^{-1} \leq m_{L/K}$ with an arbitrarily large integer e'. Take an arbitrary finite tamely ramified Galois extension K' of K and put $L' = LK'$. Then we have $u_{L'/K} = u_{L/K}$ since L/K is wildly ramified. If we apply (ii) of Proposition 1.1 to L'/K, and (i) of Proposition 1.1 to K'/K respectively, then we can prove the inequality $u_{L/K} - e_{L'/K}^{-1} \leq m_{L/K}$.

§ 2. $(\mathbf{P}_m)$ at the ramification break

Let L be a finite Galois extension of K and e its ramification index. Assume L/K to be totally and wildly ramified for simplicity. In this section, we completely determine the truth of (P_m)[1]. The equality $u_{L/K} = m_{L/K}$ in Theorem 1.2 gives no information about the truth of (P_m) at the break $m = u_{L/K}$. The behavior of (P_m) at the break depends on the residue field:

Theorem 2.1 ([7], Thm. 1.1). *The property* (P_m) *is true for* $m = u_{L/K}$ *if and only if the residue field* k *has no Galois extension whose degree is divisible by* p.

This theorem can be proved by using the local class field theory of Serre and Hazewinkel. On the other hand, we consider a weaker property (P^e_m) as follows:

(P^e_m) *For any totally ramified extension* E/K *of degree* e, *if there exists an* $\mathcal{O}_K$-*algebra homomorphism* $\mathcal{O}_L \to \mathcal{O}_E/\mathfrak{a}^m_{E/K}$, *then there exists a* K-*embedding* $L \hookrightarrow E$.

Then we have the following theorem, which is a similar result as Theorem 2.1. The proof employs the notion of a non-Archimedean metric on the set of all Eisenstein polynomials over K.

Theorem 2.2 ([8], Thm. A, Prop. 5.1). *The property* (P^e_m) *is true for* $m = u_{L/K}$ *if and only if the residue field* k *has no Galois extension of degree* p.

Remark. Both (i) and (ii) of Proposition 1.1 remain true for a finite totally and wildly ramified Galois extension L of K if we consider (P^e_m) instead of (P_m). However, (P^e_m) does not satisfy the equality in Proposition 1.2 in general. In fact, we can check that (P^e_m) is true for $m = u_{L/K}$ if and only if (P^e_m) is true for $m > u_{L/K} - e_{L/K}^{-1}$.

[1]The results in this section were obtained by joint work with Takashi Suzuki after the talk in "Algebraic Number Theory and Related Topics 2009".

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