On a Bilinear Estimate of Schrödinger Waves

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Abstract

In this paper we want to consider a bilinear space-time estimate for homogeneous Schrödinger equations. We give an elementary proof for the estimates in Bourgain space, which is in a form of scaling invariance.

1 Introduction

Consider the homogeneous Schrödinger equations

\[
\begin{align*}
\left\{ \begin{array}{l}
u_t - \Delta u &= 0, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\
u(0) &= f;
\end{array} \right. \\
u_t - \Delta v &= 0, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\
v(0) &= g.
\end{align*}
\]

(1.1)

Via Fourier transform, the solution \(u\) and \(v\) can be written as

\[
u(t) = e^{-it\Delta} f \quad \text{and} \quad v(t) = e^{-it\Delta} g.
\]

Thus to study the estimates of the product of Schrödinger waves, \(uv\), is to study the estimates of the product

\[
 e^{-it\Delta} f e^{-it\Delta} g.
\]

There are many literature investigating on the topic of bilinear estimates for Schrödinger waves. In '98, Ozawa and Tsutsumi [OT] proved an \(L^2\) estimate for \(uv\) with \(\frac{1}{2}\) derivative for \(n = 1\),

\[
\left\| (-\Delta)^{\frac{1}{2}} (e^{it\Delta} f)(e^{-it\Delta} g) \right\|_{L^2_{t,x}} = \frac{1}{\sqrt{2}} \| f \|_{L^2} \| g \|_{L^2}.
\]

(1.4)

In '98, Bourgain [Bo] showed a refinements of Strichartz’ inequality for \(n = 2\). If \(\hat{f}\) is supported on \(|\xi| \sim N\), \(\hat{g}\) is supported on \(|\xi| \sim M\), and \(M << N\), then

\[
\left\| (e^{it\Delta} f)(e^{\pm it\Delta} g) \right\|_{L^2_{t,x}} \lesssim \left( \frac{M}{N} \right)^{\frac{1}{2}} \| f \|_{L^2} \| g \|_{L^2}.
\]

(1.5)

In '01, Kenig etc. [CDKS] obtained a bilinear estimate in Bourgain space for nonlinear Schrödinger equation in two dimension. Let \(b = \frac{1}{2} + \). If \(-\frac{1}{4} - (1 - b) < s\) and \(\sigma < \min \left( s + \frac{1}{2}, 2s + 2(1 - b) \right)\), then

\[
\| uv \|_{X^{\sigma, b-1}} \lesssim \| u \|_{X^{s, b}} \| v \|_{X^{s, b}}.
\]

(1.6)

In '03, Tao [T1] obtained a sharp bilinear restriction estimate for paraboloids. Let \(q > \frac{n + 3}{n + 1}\), \(n \geq 2\), \(N > 0\), and \(f\) and \(g\) have Fourier transform supported in the region \(|\xi| \leq N\). Suppose
that \( \text{dist}(\text{supp} \hat{f}, \text{supp} \hat{g}) \geq cN \). Then we have
\[
\left\| e^{-it\Delta} f e^{-it\Delta} g \right\|_{L^2_t L^{n+2}_x (\mathbb{R}^{n+1})} \lesssim N^{n-\frac{n+2}{q}} \| f \|_{L^2} \| g \|_{L^2}.
\] (1.7)

In ’05, Burq, Jérad, and Tzvetkov [BGT] derived bilinear eigenfunction estimates on sphere and on Zoll surfaces. In ’09, Keraani and Vargas [KV] showed a bilinear estimate of \( uv \) in \( L^{\frac{n+2}{n}} \) norm, where \( n \geq 2 \). If \( b \in (0, \frac{2}{n+2}) \), then
\[
\left\| e^{-it\Delta} f e^{-it\Delta} g \right\|_{L^{\frac{n+2}{n}}(R^{n+1})} \lesssim C \| f \|_{\dot{H}^{b}} \| g \|_{\dot{H}^{1-b}}.
\] (1.8)

In ’09, Kishimoto [K] derived an improved bilinear estimate for quadratic Schrödinger equation in one and two dimensions. The estimate is in a variant of Bourgain space with weighted norm. In ’10, Chae, Cho, and Lee [CCL] proved an interactive estimate of \( uv \) in a mixed norm. Let \( n \geq 2 \). If \( \frac{2}{q} = n(1 - \frac{1}{r}) \), \( 1 < r \leq 2 \), \( \frac{q}{2} > 1 \), \( |s| < 1 - \frac{1}{r} \), then
\[
\left\| e^{-it\Delta} f e^{-it\Delta} g \right\|_{L^{q}_t L^{r}_x} \leq C \| f \|_{\dot{H}^{s}} \| g \|_{\dot{H}^{-s}}.
\] (1.9)

Let \( D, S_+, \) and \( S_- \) be the operators with the symbols
\[
\hat{D} \overset{\text{def}}{=} |\xi|, \quad \hat{S}_+ \overset{\text{def}}{=} ||\tau| + |\xi|^2|, \quad \text{and} \quad \hat{S}_- \overset{\text{def}}{=} ||\tau| - |\xi|^2|,
\] (1.10)
respectively.

**Theorem 1.** Let \( n \geq 2 \). If for \( j = 1, 2, \)
\[
\beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2,
\]
\[
\beta_- \geq 0, \quad \beta_- - \alpha_j + \frac{n-1}{2} \geq 0,
\]
\[
(\beta_-, \alpha_j) \neq (0, \frac{n-1}{2}), \text{ and } \beta_0 > -\frac{n-1}{2},
\] (1.11)
then
\[
\left\| D^{\beta_0} S_+^{\beta_+} S_-^{\beta_-} \left( e^{-it\Delta} f e^{-it\Delta} g \right) \right\|_{L^2_t(R^{n+1})} \leq C \| f \|_{\dot{H}^{\alpha_1}(R^n)} \| g \|_{\dot{H}^{\alpha_2}(R^n)}.
\] (1.12)

Notice that Strichartz Estimate for Homogeneous Schrödinger equation for \( n = 2 \) reads
\[
\| u \|_{L^4} \lesssim \| f \|_{L^2},
\]
which coincides with the bilinear estimate
\[
\| u \|_{L^4}^2 = \| uu \|_{L^2} \lesssim \| f \|_{L^2} \| f \|_{L^2},
\]
when
\[
\beta_0 = \beta_+ = \beta_- = \alpha_1 = \alpha_2 = 0.
\]
The estimate is given in the form of scaling invariance. The conditions stated in the theorem come from the scaling invariance and interactions between frequencies.
The proof of Theorem 1 is based on the ideas of the work of Foschi and Klainerman [FK], and the work of Klainerman and Machedon [KM], however some modifications for adapting the case of Schrödinger are required. The purpose of this work is to derive a new estimate with an elementary proof.

The paper is organized as follows: In Section 2, we prove Theorem 1. In Section 3, we state and prove some properties which are the technical parts left in the proof of Theorem 1.

2 Bilinear Estimates for Schrödinger waves

We denote the Fourier transform of the function \( u(t, x) \) with respect to the space variable and by \( \hat{u}(\tau, \xi) \) with respect to the space-time variables. For simplicity, we call

\[
\hat{A} = \hat{D}^{2\beta_{0}} \hat{S}_{+}^{2(\beta_{+})} \hat{S}_{-}^{2(\beta_{-})}.
\]

We now prove Theorem 1.

**Proof.** First we compute the Fourier transform of the product \( e^{-it\Delta}fe^{-it\Delta}g \) with respect to space variables,

\[
\hat{e^{-it\Delta}f} \ast \hat{e^{-it\Delta}g}(\xi) = \int e^{it|\xi-\eta|^{2}}\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta = \int e^{it(|\xi-\eta|^{2}+|\eta|^{2})}\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta.
\]

Thus its Fourier transform with respect to space-time variables is

\[
\int \delta(\tau - |\xi-\eta|^{2} - |\eta|^{2})\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta,
\]

where \( \delta(\tau - |\xi-\eta|^{2} - |\eta|^{2})d\eta \) is viewed as a measure supported on surfaces \( \{ \eta : \tau = |\xi-\eta|^{2} + |\eta|^{2} \} \).

We split the integral into three parts in the following way. First we define a function

\[
h(\gamma) \equiv \frac{\sqrt{2\gamma-1}}{\gamma}
\]

which will appear in the proof later, see figure 1. Since the equation \( h(\gamma) = 1/3 \) has two roots \( 9 \pm 6\sqrt{2} \), we denote the two roots by \( \gamma_1 = 9 - 6\sqrt{2} \) and \( \gamma_2 = 9 + 6\sqrt{2} \). Then we decompose the \( \eta \)-space into \( S_a \cup S_b \cup S_c \), see figure 2, where

\[
S_a \equiv \{ \eta : \frac{1}{2}|\xi|^{2} \leq |\xi-\eta|^{2} + |\eta|^{2} \leq \gamma_1|\xi|^{2} \},
S_b \equiv \{ \eta : \gamma_1|\xi|^{2} \leq |\xi-\eta|^{2} + |\eta|^{2} \leq \gamma_2|\xi|^{2} \}, \text{ and }
S_c \equiv \{ \eta : \gamma_2|\xi|^{2} \leq |\xi-\eta|^{2} + |\eta|^{2} \}.
\]

Thus we have
\[ \left\| D^{\beta_0} S^+_{\beta_+} S^-_{\beta-} (e^{-it\Delta} f e^{-it\Delta} g) \right\|_{L^2} = \int \int \hat{\int} \delta(\tau - |\xi - \eta|^2 - |\eta|^2) \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta \right|^{2} d\tau d\xi \]

(2.3)

Hence it is sufficient to bound each of the above integrals. For simplicity, we denote \( \Phi(\eta) \equiv \tau - |\xi - \eta|^2 - |\eta|^2 \). Using Hölder inequality, we can bound the first integral in (2.3) as follows.

\[
\int \int \hat{A} \left| \int_{S_a} \delta(\tau - |\xi - \eta|^2 - |\eta|^2) \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta \right|^{2} d\tau d\xi \leq \int \int \hat{A} \left| \int_{S_a} \frac{\delta(\Phi(\eta))}{|\xi - \eta|^{2\alpha_1} |\eta|^{2\alpha_2}} d\eta \right| \int \hat{\int} \delta(\Phi(\eta)) \left| |\xi - \eta|^{\alpha_1} \hat{f}(\xi - \eta) |\eta|^{\alpha_2} \hat{g}(\eta) \right|^{2} d\eta d\tau d\xi
\]

\[
\leq \int \int \left\{ \int \delta(\Phi(\eta)) d\tau \right\} \left| |\xi - \eta|^{\alpha_1} \hat{f}(\xi - \eta) |\eta|^{\alpha_2} \hat{g}(\eta) \right|^{2} d\eta d\xi \leq C \| f \|_{\dot{H}^{\alpha_1}} \| g \|_{\dot{H}^{\alpha_2}},
\]

provided that

\[
\hat{A} \int_{S_a} \frac{\delta(\Phi(\eta))}{|\xi - \eta|^{2\alpha_1} |\eta|^{2\alpha_2}} d\eta \leq C, \quad \text{for all } \tau, \xi.
\] (2.4)
For the second integral we can get the desired bound in the same vain,
\[
\iint \hat{A} \left| \int_{S_b} \delta(\Phi(\eta)) \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta \right|^2 d\tau d\xi \leq C \| f \|_{\dot{H}^{\alpha_1}} \| g \|_{\dot{H}^{\alpha_2}},
\] (2.5)
provided that
\[
\hat{A} \int_{S_b} \frac{\delta(\Phi(\eta))}{|\xi - \eta|^{2\alpha_1} |\eta|^{2\alpha_2}} d\eta \leq C, \quad \text{for all } \tau, \xi.
\] (2.6)

Notice that we have \(|\xi - \eta| \sim |\eta|\) and \(|\xi - \varphi| \sim |\varphi|\) on the set \(S_c\). Using the fact that \(|z|^2 = z \overline{z}\), the Fubini theorem, and change of variables, we can bound the third integral in (2.3) as follows.
\[
\iint \hat{A} \left| \int_{S_c} \delta(\Phi(\eta)) \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta \right|^2 d\tau d\xi = \iint \hat{A} \left| \int_{S_c} \frac{\delta(\Phi(\eta) - \Phi(\xi - \varphi))}{|\varphi||\eta|^{\alpha_1 + \alpha_2}} \left| \frac{\hat{f}(\xi - \varphi)}{\hat{g}(\eta)} \right|^2 d\varphi d\eta d\xi \right|^2 d\varphi d\eta d\xi \leq C \| f \|_{\dot{H}^{\alpha_1}} \| g \|_{\dot{H}^{\alpha_2}}^2,
\] (2.7)
where \(T_c^\mathrm{d} = \{ \xi : |\xi - \varphi|^2 + |\varphi|^2, |\xi - \eta|^2 + |\eta|^2 \geq \gamma_2 |\xi|^2 \}\) and \(\tau = |\xi - \varphi|^2 + |\varphi|^2 = |\xi - \eta|^2 + |\eta|^2\).

Therefore the proof of the Theorem is complete once the claims, (2.4), (2.6), and (2.7) are proved.

\[\square\]

Remark 1. What left to be done are the following estimates. Claims: There is a constant \(C\) which is independent of \(\tau, \xi, \varphi, \) and \(\eta\) such that the following inequalities hold.
\[
\hat{D}^{2\beta_0} \hat{S}_+^{2\beta_+} \hat{S}_-^{2\beta_-} \int_{S_a} \frac{\delta(\tau - |\xi - \eta|^2 - |\eta|^2)}{|\xi - \eta|^{2\alpha_1} |\eta|^{2\alpha_2}} d\eta \leq C, \quad \text{for all } \tau, \xi.
\] (2.8)
\[
\hat{D}^{2\beta_0} \hat{S}_+^{2\beta_+} \hat{S}_-^{2\beta_-} \int_{S_b} \frac{\delta(\tau - |\xi - \eta|^2 - |\eta|^2)}{|\xi - \eta|^{2\alpha_1} |\eta|^{2\alpha_2}} d\eta \leq C, \quad \text{for all } \tau, \xi.
\] (2.9)
\[ \int_{T_{c}} \hat{D}^{2\beta_{0}} \hat{S}_{+}^{2 \beta_{+}} \hat{S}_{-}^{2 \beta_{-}} \frac{\delta(|\xi-\varphi|^{2} + |\varphi|^{2} - |\xi-\eta|^{2} - |\eta|^{2})}{(|\varphi||\eta|)^{\alpha_{1}+\alpha_{2}}} d\xi \leq C, \text{ for all } \varphi \text{ and } \eta, \]  

(2.10)

where \( \tau = |\xi-\varphi|^{2} + |\varphi|^{2} = |\xi-\eta|^{2} + |\eta|^{2} \). The proofs of the above claims will be given in the next section.

### 3 Proofs of Claims

Now we are ready to prove the claims. First we prove the claims which come from the proof of bilinear estimates for \( uv \).

**Lemma 1** (Claim (2.8)). Let \( S_{a} = \{ \eta : \frac{1}{2}|\xi|^{2} \leq |\eta|^{2} + |\xi-\eta|^{2} \leq \gamma_{1}|\xi|^{2} \} \). If \( \beta_{0} + 2\beta_{+} + 2\beta_{-} + \frac{n-2}{2} = \alpha_{1} + \alpha_{2} \) and \( n \geq 2 \), then

\[ \hat{D}^{2\beta_{0}} \hat{S}_{+}^{2 \beta_{+}} \hat{S}_{-}^{2 \beta_{-}} \int_{S_{a}} \frac{\delta(\tau - |\xi-\eta|^{2} - |\eta|^{2})}{|\xi-\eta|^{2\alpha_{1}}|\eta|^{2\alpha_{2}}} d\eta \leq C, \]  

(3.1)

for all \( \tau \) and \( \xi \).

*Proof.* We set \( \zeta = R(\eta - \frac{\xi}{2}) \), where \( R \) is the rotation such that \( R\xi = |\xi|e_{1} \), then we have the identities

\[ |\xi-\eta| = |\zeta - \frac{1}{2}|\xi|e_{1}| \quad \text{and} \quad |\eta| = |\zeta + \frac{1}{2}|\xi|e_{1}|. \]  

(3.2)

Then we use spherical coordinates

\[ \zeta = (X_{1}, \ldots, X_{n}) \overset{\text{def}}{=} \rho(\cos \phi, \sin \phi \omega') \overset{\text{def}}{=} \rho \omega, \]  

(3.3)

where \( \omega \in S^{n} \) and \( \omega' \in S^{n-1} \), so that we can rewrite the integral in (3.1) as

\[ \int_{S_{a}} \frac{\delta(\tau - |\xi-\eta|^{2} - |\eta|^{2})}{|\xi-\eta|^{2\alpha_{1}}|\eta|^{2\alpha_{2}}} d\eta = \int \frac{1}{|\xi-\eta|^{2\alpha_{1}}|\eta|^{2\alpha_{2}} |\xi|^{2\alpha_{2}} - 4\rho_{0}^{n-1}} d\omega, \]  

(3.4)

where \( \rho_{0}^{2} = \frac{1}{4}(2\tau - |\xi|^{2}) \). Using the identity \( d\omega = (\sin \phi)^{n-2}d\phi d\omega' \), the identities for \( |\xi-\eta| \) and \( |\eta| \) in (3.2), and the change of variables \( p = \cos \phi \), we can simplify the integral further.

\[ \int \frac{1}{|\xi-\eta|^{2\alpha_{1}}|\eta|^{2\alpha_{2}}} \rho_{0}^{n-2} \sin^{n-2} \phi d\phi d\omega \sim \rho_{0}^{n-2} \int_{-1}^{1} \frac{(1-p^{2})^{(n-3)/2}}{(1+\lambda p)^{\alpha_{1}}(1-\lambda p)^{\alpha_{2}}} dp, \]  

(3.5)

where \( \lambda = \frac{2|\xi|\rho_{0}}{\tau} = \frac{|\xi|\sqrt{2\tau - |\xi|^{2}}}{\tau} \). Notice that \( 0 \leq \lambda \leq \frac{1}{3} \) under the restriction \( \tau = |\xi-\eta|^{2} + |\eta|^{2} \) for \( \eta \in S_{a} \).
We set $\tau \overset{\text{def}}{=} \gamma |\xi|^2$ which implies that $\lambda = \frac{\sqrt{2\gamma - 1}}{\gamma} = h(\gamma)$. Then we have $\frac{1}{2} \leq \gamma \leq \gamma_1$ which implies that $\tau \sim |\xi|^2$, and $\rho_0 = \frac{1}{2} \sqrt{2\gamma - 1} |\xi| \leq |\xi|$. Thus we can estimate the quantity in (3.1) as follows.

\[
\hat{D}^{2\beta_0} \hat{S}_+^{2\beta_+} \hat{S}_-^{2\beta_-} \int_{S_b} \frac{\delta(\tau - |\xi - \eta|^2 - |\eta|^2)}{|\xi - \eta|^{2\alpha_1} |\eta|^{2\alpha_2}} d\eta
\]

\[
\leq \left|\xi|^{2\beta_0 + 4\beta_+ + 4\beta_- - 2\alpha_1 - 2\alpha_2}\right| |\gamma| - 1|^{2\beta_-} \left(\frac{\sqrt{2\gamma - 1}|\xi|}{2}\right)^{n-2} \int_{-1}^{1} \frac{(1 - p^2)^{(n-3)/2}}{(1 + \lambda p)^{\alpha_1}(1 - \lambda p)^{\alpha_2}} dp.
\]

The above quantity is bounded if we require that $n \geq 2$ and $\beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2$.

**Lemma 2** (Claim (2.9)). Let $S_b = \{ \eta : \gamma_1|\xi|^2 \leq |\eta|^2 + |\xi - \eta|^2 \leq \gamma_2|\xi|^2 \}$. If

\[
\beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2,
\]

\[
n \geq 2, \quad \beta_- \geq \alpha_j - \frac{n-1}{2}, \quad \beta_- \geq 0, \quad \text{and} \quad (\beta_-, \alpha_j) \neq (0, \frac{n-1}{2})
\]

for $j = 1, 2$, then

\[
\hat{D}^{2\beta_0} \hat{S}_+^{2\beta_+} \hat{S}_-^{2\beta_-} \int_{S_b} \frac{\delta(\tau - |\xi - \eta|^2 - |\eta|^2)}{|\xi - \eta|^{2\alpha_1} |\eta|^{2\alpha_2}} d\eta \leq C,
\]

for all $\tau$ and $\xi$.

**Proof.** As in the proof of Lemma 1, we set $\zeta \overset{\text{def}}{=} R\left(\eta - \frac{\xi}{2}\right)$, where the rotation $R\xi = |\xi| e_1$, and

\[
\zeta = (X_1, \ldots, X_n) \overset{\text{def}}{=} \rho(\cos \phi, \sin \phi \omega') \overset{\text{def}}{=} \rho \omega.
\]

Using the above, the identity $d\omega = (\sin \phi)^{n-2} d\phi d\omega'$, and the identities for $|\xi - \eta|$ and $|\eta|$, we can rewrite the integral in (3.6) as

\[
\int_{S_b} \frac{\delta(\tau - |\xi - \eta|^2 - |\eta|^2)}{|\xi - \eta|^{2\alpha_1} |\eta|^{2\alpha_2}} d\eta \sim \rho_0^{-n-2} \int_{-1}^{1} \frac{(1 - p^2)^{(n-3)/2}}{(1 + \lambda p)^{\alpha_1}(1 - \lambda p)^{\alpha_2}} dp,
\]

where $\rho_0^2 = \frac{1}{4} (2\tau - |\xi|^2)$, the change of variables $p = \cos \phi$, and $\lambda = \frac{2|\xi|\rho_0}{\tau} = \frac{|\xi|\sqrt{2\tau - |\xi|^2}}{\tau}$.

Again we set $\tau \overset{\text{def}}{=} \gamma|\xi|^2$ and then we have $\gamma_1 \leq \gamma \leq \gamma_2$ and $\frac{1}{3} \leq \lambda \leq 1$ under the restriction $\tau = |\xi - \eta|^2 + |\eta|^2$ for $\eta \in S_b$. These imply that

\[
\tau \sim |\xi|^2, \quad (\gamma - 1)^2 \sim (1 - \lambda), \quad \text{and} \quad \rho_0 = \frac{1}{2} \sqrt{2\gamma - 1} |\xi| \sim |\xi|.
\]
Now we can combine the above observations to simplify the quantity in (3.1) as follows.

\[ \hat{D}^{2\beta_0} \hat{S}_+^{2\beta_+} \hat{S}_-^{2\beta_-} \int_{S_b} \frac{\delta(\tau - |\xi - \eta|^2 - |\eta|^2)}{|\xi - \eta|^{2\alpha_1} |\eta|^{2\alpha_2}} \, d\eta \]

\[ \sim |\xi|^{2\beta_0 + 4\beta_+ + 4\beta_- + n - 2 - 2\alpha_1 - 2\alpha_2} \int_{-1}^{1} \frac{(1 - p^2)^{(n-3)/2}}{(1 + \lambda p)^{\alpha_1} (1 - \lambda p)^{\alpha_2}} \, dp. \]

To bound the above integral we split it into two parts, one is over \([-1, 0]\) while the other is over \([0, 1]\). Since the estimates for the two parts are the same, thus we only prove the second part. First we have

\[ \int_{0}^{1} \frac{(1 - p^2)^{(n-3)/2}}{(1 + \lambda p)^{\alpha_1} (1 - \lambda p)^{\alpha_2}} \, dp \sim \int_{0}^{1} \frac{(1 - p)^{(n-3)/2}}{(1 - \lambda p)^{\alpha_2}} \, dp \]  

(3.10)

Using the change of variables \( p = -(1 - \lambda)q + \lambda \), the integral is changed into

\[ (1 - \lambda)^{n-3/2 + 1 - \alpha_2} \int_{-1}^{1} \frac{(1 + q)^{(n-3)/2}}{(1 + \lambda + \lambda q)^{\alpha_2}} \, dq. \]

Again we split the above integral into two parts. For the first part, we have

\[ \int_{-1}^{1} \frac{(1 + q)^{(n-3)/2}}{(1 + \lambda + \lambda q)^{\alpha_2}} \, dq \leq C, \]

provided that \( n \geq 2 \). For the second part, we have

\[ \int_{1}^{\frac{\lambda}{1-\lambda}} \frac{(1 + q)^{(n-3)/2}}{(1 + \lambda + \lambda q)^{\alpha_2}} \, dq \sim \begin{cases} 
(1 - \lambda)^{-\frac{n-1}{2} + \alpha_2} & \text{for } \alpha_2 < \frac{n-1}{2}, \\
|\log(1 - \lambda)| & \text{for } \alpha_2 = \frac{n-1}{2}, \\
C & \text{for } \alpha_2 > \frac{n-1}{2},
\end{cases} \]

Thus we get

\[ (1 - \lambda)^{\beta_-} \int_{0}^{1} \frac{(1 - p^2)^{(n-3)/2}}{(1 + \lambda p)^{\alpha_1} (1 - \lambda p)^{\alpha_2}} \, dp \]

\[ \sim C(1 - \lambda)^{\beta_- + \frac{n-1}{2} - \alpha_2} + \begin{cases} 
(1 - \lambda)^{\beta_- + \frac{n-1}{2} - \alpha_2} (1 - \lambda)^{-\frac{n-1}{2} + \alpha_2} & \text{for } \alpha_2 < \frac{n-1}{2}, \\
(1 - \lambda)^{\beta_- + \frac{n-1}{2} - \alpha_2} |\log(1 - \lambda)| & \text{for } \alpha_2 = \frac{n-1}{2}, \\
(1 - \lambda)^{\beta_- + \frac{n-1}{2} - \alpha_2} C & \text{for } \alpha_2 > \frac{n-1}{2},
\end{cases} \]

Hence the above integral is bounded if \((\beta_-, \alpha_2)\) is in the set \( S_1 \cap (S_2 \cup S_3 \cup S_4) \), where

\[ S_1 \overset{\text{def}}{=} \{(\beta_-, \alpha_2) : \beta_- \geq \alpha_2 - \frac{n-1}{2}\}, \quad S_2 \overset{\text{def}}{=} \{(\beta_-, \alpha_2) : \beta_- \geq 0, \alpha_2 < \frac{n-1}{2}\}, \]

\[ S_3 \overset{\text{def}}{=} \{(\beta_-, \alpha_2) : \beta_- > \alpha_2 - \frac{n-1}{2}, \alpha_2 = \frac{n-1}{2}\}, \quad \text{and} \quad S_4 \overset{\text{def}}{=} \{(\beta_-, \alpha_2) : \beta_- \geq \alpha_2 - \frac{n-1}{2}, \alpha_2 > \frac{n-1}{2}\}. \]
Therefore the quantity in (3.6) is bounded if we require that

\[ \beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2, \]

\[ n \geq 2, \quad \beta_- \geq \alpha_2 - \frac{n-1}{2}, \quad \beta_- \geq 0, \quad \text{and} \quad (\beta_-, \alpha_2) \neq (0, \frac{n-1}{2}). \]

The conditions for \( \alpha_1 \) is the same as that of \( \alpha_2 \). This completes the proof. \( \square \)

**Remark 2.** Notice that for the integral over \( S_c \) we have \( 0 \leq \lambda \leq \frac{1}{3}, \gamma_2 \leq \gamma, \tau = \gamma|\xi|^2 \), and \( \rho_0 = \sqrt{2\gamma - 1}|\xi|/2 \sim \sqrt{\gamma}|\xi| \). If we follow the same path to estimates it, then we obtain

\[
\hat{D}^{2\beta_0} \hat{S}_+^{2\beta_+} \hat{S}_-^{2\beta_-} \int_{S_b} \frac{\delta(|\tau - |\xi - \eta|^2 - |\eta|^2)}{|\xi - \eta|^{2\alpha_1}|\eta|^{2\alpha_2}} d\eta \\
\sim |\xi|^{2\beta_0+4\beta_++4\beta_-+n-2-2\alpha_1-2\alpha_2} \gamma^{\beta_0+2\beta_++2\beta_-+\frac{n-2}{2}-\alpha_1-\alpha_2} \int_{-1}^{1} \frac{(1-u^2)^{\frac{n-3}{2}}}{(1+\lambda p)^{\alpha_1}(1-\lambda p)^{\alpha_2}} dp \leq C,
\]

(3.11)

provided that \( \beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2, n \geq 2 \), and \( 2\beta_+ + 2\beta_- + n-2 - \alpha_1 - \alpha_2 \leq 0 \).

The last condition implies that \( \beta_0 \geq 0 \) which we shall see that this is not good enough.

**Lemma 3** (Claim (2.10)). Let \( T_c(\eta, \varphi) \stackrel{\text{def}}{=} \{ \xi : |\xi - \eta|^2 + |\eta|^2, |\xi - \varphi|^2 \geq \gamma_2|\xi|^2 \} \). If \( \beta_0 + 2\beta_+ + 2\beta_- + \frac{n-2}{2} = \alpha_1 + \alpha_2 \) and \( \beta_0 > -\frac{n-1}{2} \), then

\[
\int_{T_c} \hat{D}^{2\beta_0} \hat{S}_+^{2\beta_+} \hat{S}_-^{2\beta_-} \frac{\delta(|\xi - \varphi|^2 + |\varphi|^2 - |\xi - \eta|^2 - |\eta|^2)}{|(\varphi||\eta|)^{\alpha_1+\alpha_2}} d\xi \leq C,
\]

(3.12)

where \( \tau = |\xi - \varphi|^2 + |\varphi|^2 = |\xi - \eta|^2 + |\eta|^2 \).

**Proof.** Let \( \Phi(\xi) \stackrel{\text{def}}{=} |\xi - \varphi|^2 + |\varphi|^2 - |\xi - \eta|^2 - |\eta|^2 \) and \( P(\varphi, \eta) \stackrel{\text{def}}{=} \{ \xi : \Phi(\xi) = 0 \} \). Since \( |\xi - \varphi|^2 + |\varphi|^2 \geq \gamma_2|\xi|^2 \) and \( \gamma_2 > 16 \), thus we have \( |\varphi| + |\xi - \varphi| \geq 4|\xi| \) and analogously we have \( |\eta| + |\xi - \eta| \geq 4|\xi| \).

Using triangle inequality, we get

\[
\frac{3}{5} |\eta| \leq |\xi - \eta| \leq \frac{5}{3} |\eta| \quad \text{and} \quad \frac{3}{5} |\varphi| \leq |\xi - \varphi| \leq \frac{5}{3} |\varphi|,
\]

and then

\[
|\xi| \leq \frac{2}{3} \min\{|\eta|, |\xi - \eta|, |\varphi|, |\xi - \varphi|\}.
\]

On the plane \( P \) we have \( |\xi - \varphi|^2 + |\varphi|^2 = |\xi - \eta|^2 + |\eta|^2 \) which implies that

\[
|\xi - \varphi| \sim |\varphi| \sim |\xi - \eta| \sim |\eta|.
\]

Set \( (\xi - \eta) \cdot (-\eta) \stackrel{\text{def}}{=} |\xi - \eta||\eta| \cos \theta \), through some calculations we can show that

\[
\cos \theta \geq \cos \theta_0 = \frac{\sqrt{5}}{3} > \frac{\sqrt{2}}{2} = \cos \frac{\pi}{4}.
\]

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Hence the angle between $-\eta$ and $\xi - \eta$ is restricted on $0 \leq \theta \leq \theta_0 \leq \frac{\pi}{4}$.

Without loss of generality, we assume $|\varphi| > |\eta|$. Follow the idea used in [FK], we decompose $S^{n-2} = \bigcup_{j=1}^{N} \Omega_j$, where $\Omega_j$ are disjoint and the angle between any two unit vectors lie in the same $\Omega_j$ is less than $\theta_0$, and $N$ is a finite integer. Denote

$$
\Gamma_j \overset{\text{def}}{=} \left\{ \xi \in \mathbb{R}^n / \{0\} : \frac{\xi}{|\xi|} \in \Omega_j \right\}, \chi_j \overset{\text{def}}{=} \text{characteristic function of } \Gamma_j, f_j \overset{\text{def}}{=} \chi_j f, \text{ and } g_j \overset{\text{def}}{=} \chi_j g.
$$

Thus we have $f = \sum_{j=1}^{N} f_j$ and $g = \sum_{j=1}^{N} g_j$. Then we can split the integral into finitely many pieces,

$$
\left\| \int D^{\beta_0} S_+^{\beta_+} S_-^{\beta_-} \delta(\tau - |\xi - \eta|^2 - |\eta|^2) \hat{f_j}(\xi - \eta) \hat{g_k}(\eta) d\eta \right\|_{L^2(\tau \geq 16|\xi|^2)} \leq \sum_{j,k} \left\| \int D^{\beta_0} S_+^{\beta_+} S_-^{\beta_-} \delta(\tau - |\xi - \eta|^2 - |\eta|^2) \hat{f_j}(\xi - \eta) \hat{g_k}(\eta) d\eta \right\|_{L^2(\tau \geq 16|\xi|^2, \theta_0 < \pi/4)}.
$$

There exists a cone $\Gamma$ with an aperture $2\theta_0$ such that $\eta \in \Gamma_k \subset \Gamma$ and $\xi - \eta \in \Gamma_j \subset -\Gamma$.

$$
\left\| \int D^{\beta_0} S_+^{\beta_+} S_-^{\beta_-} \delta(\tau - |\xi - \eta|^2 - |\eta|^2) \hat{f}_j(\xi - \eta) \hat{g}(\eta) d\eta \right\|_{L^2(\tau \geq 16|\xi|^2)}^2 \leq \int \int \int D^{\beta_0} \frac{\delta(\Phi(\xi))}{(|\varphi||\eta|)^{\alpha_1 + \alpha_2}} d\mu \frac{1}{|\nabla \Phi(\xi)|},
$$

where $\eta \in \Gamma_k \subset \Gamma$, $\xi - \eta \in \Gamma_j \subset -\Gamma$, $\varphi \in \Gamma_j \subset -\Gamma$, and $\xi - \varphi \in \Gamma_k \subset \Gamma$. Through elementary argument, we have the following identity, see [H],

$$
\int D^{\beta_0} S_+^{\beta_+} S_-^{\beta_-} \frac{\delta(\Phi(\xi))}{(|\varphi||\eta|)^{\alpha_1 + \alpha_2}} d\xi = \int D^{\beta_0} S_+^{\beta_+} S_-^{\beta_-} \frac{d\mu}{(|\varphi||\eta|)^{\alpha_1 + \alpha_2} |\nabla \Phi(\xi)|},
$$

(3.13)
where $d\mu$ is the surface measure on the surface $\{\xi : \Phi(\xi) = 0\}$. The facts $\xi - \varphi \in \Gamma$ and $\xi - \eta \in -\Gamma$ imply that $|\nabla \Phi(\xi)| \sim |\varphi|$ since

$$|\nabla \Phi(\xi)| = |-2(\varphi - \eta)| = 2\sqrt{|\varphi|^2 + |\eta|^2 - 2\varphi \cdot \eta} \sim |\varphi|.$$ 

Let $\xi'$ be the projection of $\xi$ onto the plane $P$, see figure 5,

and $P_{\frac{\varphi + \eta}{2}}$ the projection of $\frac{\varphi + \eta}{2}$ onto the plane $P$,

$$P_{\frac{\varphi + \eta}{2}} \overset{\text{def}}{=} \frac{\varphi + \eta}{2} - \frac{\varphi + \eta}{2} \frac{\varphi - \eta}{|\varphi - \eta|} \frac{\varphi - \eta}{|\varphi - \eta|}.$$ 

Denote the rotation taking $\varphi - \eta$ to $|\varphi - \eta|e_1$ by $R$ and the change of coordinates

$$\nu \overset{\text{def}}{=} R\left(\xi - \frac{\varphi + \eta}{2}\right) \overset{\text{def}}{=} (X_1, X_2, \ldots, X_n).$$
Thus we get $|\xi'| \leq |\xi|$ and

$$
R\left(\xi' - P_{\frac{\varphi + \eta}{2}}\right) = R\left(\xi - \frac{\varphi + \eta}{2}\right) - R\left(\xi - \frac{\varphi - \eta}{2}\right) \cdot R\left(\frac{\varphi - \eta}{|\varphi - \eta|}\right)
$$

$$
= \nu - \nu \cdot e_1 e_1 = (X_1, X_2, \ldots, X_n) - (X_1, 0, \ldots, 0) = (0, X_2, \ldots, X_n),
$$

and then $d\xi' = dX_2 \cdots dX_n = d\mu$. Hence we obtain $\delta(\Phi(\xi)) d\xi = \frac{d\mu}{|\nabla \Phi(\xi)|} \sim \frac{d\xi'}{|\varphi|}$.

Therefore we can now bound the integral (3.13) as follows.

$$
\int_{P(\eta, \varphi), |\xi| \leq \frac{2}{3} |\eta|} \left|\xi\right|^{2\beta_0} \left|\xi - \varphi\right|^2 + \left|\varphi\right|^2 + \left|\xi\right|^2 \left|\xi - \varphi\right|^2 + \left|\varphi\right|^2 - \left|\xi\right|^2 \left|\xi - \varphi\right|^2 + \frac{\delta(\Phi(\xi))}{(|\varphi||\eta|)^{\alpha_1+\alpha_2}} d\xi
$$

$$
\sim |\varphi|^{4\beta_+ + 4\beta_- - 2\alpha_1 - 2\alpha_2} \int_{|\xi| < |\varphi|} |\xi|^{2\beta_0} \frac{d\xi'}{|\varphi|}.
$$

If $\beta_0 \geq 0$, then we get the bound

$$
\int_{|\xi| < |\varphi|} |\xi|^{2\beta_0} d\xi' \leq \int_{|\xi| < |\varphi|} |\varphi|^{2\beta_0} d\xi' \sim |\varphi|^{2\beta_0 + n - 1}.
$$

If $\beta_0 < 0$, then we get the bound

$$
\int_{|\xi'| \leq |\varphi|} |\xi'|^{2\beta_0} d\xi' \leq \int_{|\xi'| \leq |\varphi|} |\xi'|^{2\beta_0} d\xi' \sim |\varphi|^{2\beta_0 + n - 1},
$$

provided that $2\beta_0 + n - 1 > 0$. Finally combining the above results we have

$$
(3.13) \lesssim |\varphi|^{4\beta_+ + 4\beta_- - 2\alpha_1 - 2\alpha_2 + 2\beta_0 + n - 2} \leq C,
$$

provided that $2\beta_+ + 2\beta_- + \beta_0 + \frac{n-2}{2} = \alpha_1 + \alpha_2$ and $2\beta_0 + n - 1 > 0$. This completes the proof. \(\square\)

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