RIMS Kôkyûroku Bessatsu
B26 (2011), 51–80

Olsen's inequality and its applications to Schrödinger equations

By

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Abstract

Morrey spaces turned out to be useful in that it grasps the subtle property of the fractional integral operators.

In 1995 Olsen obtained a bilinear estimate. Olsen applied his estimate called the Olsen inequality to the Schrödiger equation. We improve this estimate.

This Olsen inequality is a part of trace inequality that has kin relation with potential theory. Here we present some applications of our results to PDEs and the potential theory.

§ 1. Introduction

The aim of this note is to establish the following inequality.

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r} \le C\,\|g\|_{\mathcal{M}^{p_0}_p}\|f\|_{\mathcal{M}^{q_0}_q}, \tag{1.1}$$

where $\|\cdot\|_{\mathcal{M}^{p_0}_p}$ denotes the Morrey (quasi-)norm given by

$$\|f\|_{\mathcal{M}^{p_0}_p} = \sup_{Q \in \mathcal{D}} |Q|^{1/p_0 - 1/p}\left(\int_Q |f(y)|^p\,dy\right)^{1/p},\ 0 < p \le p_0 < \infty \tag{1.2}$$

Received August 31, 2010. Revised December 18, 2010.
2000 Mathematics Subject Classification(s): 2000 Mathematics Subject Classification(s):42B35, 42B25,26A33,35J10
Key Words: *Key Words*: Fractional integral operators, Schrödinger equations, multiplication operators.
The first author is supported financially by Grant-in-Aid for Young Scientists (B) (No. 21740104) Japan Society for the Promotion of Science. The third author is supported by the Global COE program at Graduate School of Mathematical Sciences the University of Tokyo, and was supported by Fūjyukai foundation.
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and I_α denotes the fractional integral operator defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,\ 0<\alpha<n. \tag{1.3}$$

Here $\mathcal{D}$ denotes the set of all dyadic cubes in $\mathbb{R}^n$. It is well known that I_2 is the inverse of $-\Delta$ modulo multiplicative constants. In this note we shall prove the following theorem on the fractional integral operator I_α.

Here is a series of equivalent definitions of Morrey norms.

$$\|f\|^{(1)}_{\mathcal{M}^p_q} = \sup_{Q\in\mathcal{Q}} |Q|^{1/p_0-1/p} \left(\int_Q |f(y)|^p\, dy\right)^{1/p}$$
$$\|f\|^{(2)}_{\mathcal{M}^p_q} = \sup_{Q\in\mathcal{Q}^*} |Q|^{1/p_0-1/p} \left(\int_Q |f(y)|^p\, dy\right)^{1/p},$$

where $\mathcal{Q}$ denotes the set of all cubes whose edges are parallel to the coordinate axis and $\mathcal{Q}^*$ denotes the set of all cubes whose edges are not always parallel to the coordinate axis. In the present paper we identify $\|f\|^{(1)}_{\mathcal{M}^p_q}$ with $\|f\|_{\mathcal{M}^p_q}$ but we do not use $\|f\|^{(2)}_{\mathcal{M}^p_q}$. If we formally let $p_0=\infty$, then we obtain the L^∞ space.

Theorem 1.1. *Let $0<\alpha<n$, $1<p\le p_0<\infty$, $1<q\le q_0<\infty$ and $1<r\le r_0<\infty$. Suppose that*

$$q>r,\ \frac{1}{p_0}>\frac{\alpha}{n},\ \frac{1}{q_0}\le\frac{\alpha}{n}, \tag{1.4}$$

and that

$$\frac{1}{r_0}=\frac{1}{q_0}+\frac{1}{p_0}-\frac{\alpha}{n},\ \frac{r}{r_0}=\frac{p}{p_0}. \tag{1.5}$$

Then

$$\|g\cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r} \le C\|g\|_{\mathcal{M}^{q_0}_q}\cdot\|f\|_{\mathcal{M}^{p_0}_p},$$

where the constant C is independent of f and g.

Remark. Hölder's inequality yields

$$L^{p_0}=\mathcal{M}^{p_0}_{p_0}\hookrightarrow\mathcal{M}^{p_0}_{p_1}\hookrightarrow\mathcal{M}^{p_0}_{p_2} \tag{1.6}$$

for all $p_0\ge p_1\ge p_2\ge 1$.

Here is a precise result by Olsen.

Theorem 1.2 ([10, Theorem 2])**.** *Let $0<\alpha<n$, $1<p\le p_0<\infty$, $1<q\le q_0<\infty$ and $1<r\le r_0<\infty$. Suppose that*

$$q>r,\ \frac{1}{p_0}>\frac{\alpha}{n},\ \frac{1}{q_0}\le\frac{\alpha}{n}, \tag{1.7}$$

and that

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n}, \ \frac{1}{r} = \frac{1}{q_0} + \frac{1}{p} - \frac{\alpha}{n}. \tag{1.8}$$

Then

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r} \le C \|g\|_{\mathcal{M}^{q_0}_q} \cdot \|f\|_{\mathcal{M}^{p_0}_p},$$

where the constant C is independent of f and g.

However, this result is not sharp as the following calculation shows.

Remark. Using naively the Adams theorem [1] and Hölder's inequality, one can prove a minor part of q in Theorem 1.1. That is, the proof of Theorem 1.1 is fundamental provided $\frac{p}{p_0} q_0 \le q \le q_0$. Indeed, by virtue of the Adams theorem we have, for any cube $Q \in \mathcal{Q}$,

$$|Q|^{1/s_0} \left(\frac{1}{|Q|} \int_Q |I_\alpha f(x)|^s \, dx \right)^{1/s} \le C \|f\|_{\mathcal{M}^{p_0}_p}, \quad \frac{1}{s} = \frac{p_0}{p} \frac{1}{s_0}, \ \frac{1}{s_0} = \frac{1}{p_0} - \frac{\alpha}{n}. \tag{1.9}$$

The condition $\frac{r}{r_0} = \frac{p}{p_0}, \ \frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n}$ reads

$$\frac{1}{r} = \frac{p_0}{p} \left(\frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n} \right) = \frac{p_0}{p} \frac{1}{q_0} + \frac{1}{s}.$$

These yield

$$|Q|^{1/q_0 + 1/s_0} \left(\frac{1}{|Q|} \int_Q |g(x) I_\alpha f(x)|^r \, dx \right)^{1/r} \le C \|g\|_{\mathcal{M}^{q_0}_q} \|f\|_{\mathcal{M}^{p_0}_p}$$

if $\frac{r}{r_0} = \frac{p}{p_0} = \frac{q}{q_0}$. In view of the inclusion (1.6), the same can be said when

$$\frac{p}{p_0} q_0 \le q \le q_0.$$

Also observe that $\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n} > \frac{1}{q_0}$. Hence we have $q_0 > r_0$. Thus, since the condition $q > r$, Theorem 1.1 is significant only when $\frac{p}{p_0} r_0 < q < \frac{p}{p_0} q_0$.

We generalized Theorem 1.1 in our subsequent papers [12, 13]. The motivation stemed from the earlier works due to Sugano and Tanaka [17, 18]. To prove them we used some auxiliary results of maximal operators, which strenghten those by Nakai [8, 9] (see Lemmas 8.1 and 8.2).

As we have established in [14], the fractional maximal operater I_α is not surjective from $\mathcal{M}^p_q$ to $\mathcal{M}^s_t$ if

$$1 < q \le p < \infty,\ 1 < t \le s < \infty,\ \frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n},\ \frac{q}{p} = \frac{t}{s}. \tag{1.10}$$

Therefore, one can expect such a strange phenomenon like Theorem 1.1.

§ 2. Fundamentals on Morrey spaces

Here we shall present several examples of the functions in Morrey spaces. Here and below given a cube Q and $\kappa > 0$, we denote by κQ the cube concentric to Q and having volume κ^n.

Example Let $1 < q < q_0 < \infty$. Then $|x|^{-n/q_0} \in \mathcal{M}^{q_0}_q$.

The following proposition is due to Pérez [11]. It seems that we can construct counterexamples by using the following proposition.

Proposition 2.1. *Let $p_0 > p > 0$ and $R > 1$ be fixed so that*

$$(R+1)^{-1/p_0} = 2^{1/p}(1+R)^{-1/p}. \tag{2.1}$$

For $\varepsilon = \{\varepsilon_a\}_{a=1}^n \in \{0,1\}^n$, we define an Affine transformation T_ε by

$$T_\varepsilon(x) = \frac{1}{R+1}x + \frac{R}{R+1}\varepsilon \quad (x \in \mathbb{R}^n). \tag{2.2}$$

Let $E_0 = [0,1]^n$. Suppose that we have defined $E_0, E_1, E_2, \cdots, E_j$. Define

$$E_{j+1} = \bigcup_{\varepsilon \in \{0,1\}^n} T_\varepsilon(E_j). \tag{2.3}$$

Then we have

$$\|\chi_{E_j}\|_{\mathcal{M}^{p_0}_p} \sim (1+R)^{-jn/p_0}, \tag{2.4}$$

where the implicit constants in $\sim$ does not depend on j but can depend on p, p_0.

Before the proof we make a preparatory observation. Note that

$$\|E_{j,0}\|_{L^{p_0}} = \|E_j\|_{L^p}(1+R)^{-jn/p_0}, \tag{2.5}$$

where $E_{j,0} = [0,(1+R)^{-j}]^n$.

Proof. Let us calculate

$$\|\chi_{E_j}\|_{\mathcal{M}^{p_0}_p} \sim \sup_{S \in \mathcal{Q}} |S|^{1/p_0 - 1/p}|S \cap E_j|^{1/p}, \tag{2.6}$$

where $\mathcal{Q}$ denotes the set of all cubes.

Let us temporally say that $Q \in \mathcal{Q}$ is wasteful, if there exists a cube $S \in \mathcal{Q}$ such that

$$|Q|^{1/p_0-1/p}|Q \cap E_j|^{1/p} < |S|^{1/p_0-1/p}|S \cap E_j|^{1/p}. \tag{2.7}$$

Thus, by definition, any cube is clearly wasteful unless it is contained in $[0,1]^n$. Also, if the side-length of a cube Q is less than $(R+1)^{-j}$, then it is wasteful. Indeed, then the equality

$$\begin{aligned}\sup_{Q\,:\,cube\,:\,|Q|\le(R+1)^{-jn}} |Q|^{1/p_0-1/p}|Q \cap E_j|^{1/p} &= \sup_{Q\,:\,cube\,:\,Q\subset E_j} |Q|^{1/p_0-1/p}|Q \cap E_j|^{1/p} \\ &= \sup_{Q\,:\,cube\,:\,Q\subset E_j} |Q|^{1/p_0} \\ &= |E_{j,0}|^{1/p_0}\end{aligned}$$

holds. Thus, if the cube Q is not wasteful and $(R+1)^{-kn} \le |Q| \le (R+1)^{-(k-1)n}$, then $\dfrac{R+1}{R-1}Q$ contains a connected component of E_k. Hence it follows that

$$\begin{aligned}&\|\chi_{E_j}\|_{\mathcal{M}^{p_0}_p} \\ &\sim \sup\{|Q|^{1/p_0-1/p}|Q \cap E_j|^{1/p} \,:\, Q \text{ contains a connected component of } E_k\ \}.\end{aligned}$$

Let us write

$$\alpha = \sup\left\{\frac{|S \cap E_j|^{1/p}}{|S|^{1/p-1/p_0}} \,:\, S \text{ contains a connected component of } E_k\right\}. \tag{2.8}$$

Let S be a cube which contains a connected component of E_j. We define

$$S^* = \operatorname{co}\left(\bigcup\{W \,:\, W \text{ is a connected component of } E_j \text{ intersecting } S\}\right), \tag{2.9}$$

where $\operatorname{co}(A)$ denotes the smallest convex set containing a set A. Then a geometric observation shows that S^* engulfs k^n connected component of E_j for some $1 \le k \le 2^j$. Take an integer l so that $2^{l-1} \le k \le 2^l$. Then we have

$$|S^* \cap E_j| = k^n(1+R)^{-jn},\ |S^*| = (1+R)^{-jn+ln}. \tag{2.10}$$

Consequently, we obtain

$$\begin{aligned}|S^*|^{1/p_0-1/p}|S^* \cap E_j|^{1/p} &\sim 2^{ln/p}(1+R)^{-jn/p}(1+R)^{(-j+l)(n/p_0-n/p)} \\ &\sim 2^{ln/p}(1+R)^{-jn/p_0+l(n/p_0-n/p)} \\ &\sim (1+R)^{-jn/p_0}.\end{aligned}$$

Thus, we obtain (2.4). □

We remark that this example is essentially employed in our paper [12] to prove the sharpness of our result.

Proposition 2.2. *Let $0 < \alpha < n$, $1 < q \le p < \infty$ and $1 < t \le s < \infty$. If there exists a constant $C > 0$*

$$\|I_\alpha f\|_{\mathcal{M}^s_t} \le C\|f\|_{\mathcal{M}^p_q} \tag{2.11}$$

for all positive measurable functions f, it is necessary and sufficient that

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \ \frac{t}{s} \le \frac{q}{p}. \tag{2.12}$$

Proof. The sufficiency is well known as Adam's theorem and we will use in this lecture. For the necessity, we first obtain

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n} \tag{2.13}$$

by the dilation

$$\|f(\lambda\cdot)\|_{\mathcal{M}^p_q} = \lambda^{-\frac{n}{p}}\|f\|_{\mathcal{M}^p_q}, \ \|f(\lambda\cdot)\|_{\mathcal{M}^s_t} = \lambda^{-\frac{n}{s}}\|f\|_{\mathcal{M}^s_t}, \ I_\alpha[f(\lambda\cdot)] = \lambda^{-\alpha} I_\alpha f(\lambda\cdot) \tag{2.14}$$

for $\lambda > 0$. Consequently, we obtain

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}. \tag{2.15}$$

We may assume that $q < p$ for the purpose of establishing $\dfrac{t}{s} = \dfrac{q}{p}$. Let $R > 1$ be the solution of $(1+R)^{-\frac{1}{p}} = 2^{\frac{1}{q}}(1+R)^{-\frac{1}{q}}$. Then we have

$$\|\chi_{E_j}\|_{\mathcal{M}^p_q} \sim (1+R)^{-jn/p}. \tag{2.16}$$

If we use E_j above, then we have

$$I_\alpha \chi_{E_j}(x) \ge c\,(R+1)^{-j\alpha}\chi_{E_j}(x). \tag{2.17}$$

Indeed, as the equality

$$I_\alpha \chi_{E_j}(x) = \int_{x-E_j} \frac{dy}{|y|^{n-\alpha}}, \quad x - E_j := \{x - y \,:\, y \in E_j\} \tag{2.18}$$

shows, $I_\alpha \chi_{E_j}$ is continuous. So we can take $c = \min_{x\in[0,1]^n} I_\alpha \chi_{E_j}(x)$.

Also, from the definition of $\mathcal{M}^s_t$, we have

$$(R+1)^{-\frac{jnq}{pt}} = \|\chi_{E_j}\|_{L^t} \le \|\chi_{E_j}\|_{\mathcal{M}^s_t} \tag{2.19}$$

from the defintion of the Morrey norm. Consequently, it follows that

$$(R+1)^{-\frac{jnq}{pt}} \le C\,(R+1)^{j\alpha}\|\chi_{E_j}\|_{\mathcal{M}^p_q} = C\,(R+1)^{j\left(\alpha - \frac{n}{p}\right)} = C\,(R+1)^{-\frac{jn}{s}}.$$

Since $j \in \mathbb{N}$ is arbitrary, it follows that $t \le \dfrac{qs}{p}$. □

§ 3. Boundedness of the operators on Morrey spaces

Now we present a typical argument proving the Morrey-boundedness. We accept

$$\|Mf\|_q \le C\,\|f\|_q,\ 1 < q \le \infty, \tag{3.1}$$

where M denotes the Hardy-Littlewood maximal operator given by

$$Mf(x) = \sup_{x \in Q; cube} \frac{1}{|Q|}\int_Q |f(y)|\,dy. \tag{3.2}$$

Here and below given a cube Q, we define

$$\ell(Q) = |Q|^{1/n}. \tag{3.3}$$

Theorem 3.1. *Let $1 < q \le p < \infty$. Then there exists $C_{p,q}$ such that*

$$\|Mf\|_{\mathcal{M}^p_q} \le C_{p,q}\|f\|_{\mathcal{M}^p_q} \tag{3.4}$$

for all $f \in \mathcal{M}^p_q$.

Proof. For the proof we shall show, from the definition, that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q Mf(y)^q\,dy\right)^{\frac{1}{q}} \le C_{p,q}\|f\|_{\mathcal{M}^p_q}, Q \in \mathcal{Q}. \tag{3.5}$$

Once this is achieve, we have only to consider the supremum over all Q.

Write $f = f_1 + f_2$, where $f_1 = f$ on $5Q$ and $f_2 = f$ outside $5Q$. The estimate of (3.5) can be decomposed into

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q Mf_1(y)^q\,dy\right)^{\frac{1}{q}} \le C_{p,q}\|f\|_{\mathcal{M}^p_q}, \tag{3.6}$$

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q Mf_2(y)^q\,dy\right)^{\frac{1}{q}} \le C_{p,q}\|f\|_{\mathcal{M}^p_q} \tag{3.7}$$

by virtue of the triangle inequality.

As we have seen in (3.1), M is L^q-bounded. Thus (3.6) can be shown easily;

$$\begin{aligned}
|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q Mf_1(y)^q\,dy\right)^{\frac{1}{q}} &\le |Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{\mathbb{R}^n} Mf_1(y)^q\,dy\right)^{\frac{1}{q}}\\
&\le C_{p,q}|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{5Q} |f(y)|^q\,dy\right)^{\frac{1}{q}}\\
&\le C_{p,q}|5Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{5Q} |f(y)|^q\,dy\right)^{\frac{1}{q}}\\
&\le C_{p,q}\|f\|_{\mathcal{M}^p_q}.
\end{aligned}$$

Thus, we obtain (3.6).

To prove (3.7) we have to keep in mind the following fundamental geometric observation.

If $R \in \mathcal{Q}$ is a cube that meets both Q and $\mathbb{R}^n \setminus 5Q$, then $\ell(R) \ge 2\ell(Q)$ and $2R \supset Q$.

This geometric observation yields

$$Mf_2(y) \le 2^n \sup_{Q \subset R \in \mathcal{Q}} \frac{1}{|R|} \int_R |f(y)|\, dy. \tag{3.8}$$

If we insert this inequality to (3.7), then we obtain

$$|Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q Mf_2(y)^q\, dy \right)^{\frac{1}{q}} \le 2^n |Q|^{\frac{1}{p}} \sup_{Q \subset R \in \mathcal{Q}} \frac{1}{|R|} \int_R |f(y)|\, dy.$$

Taking into account $\mathcal{M}^p_q \subset \mathcal{M}^p_1$, we see that

$$|Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q Mf_2(y)^q\, dy \right)^{\frac{1}{q}} \le 2^n \sup_{R \in \mathcal{Q}} |R|^{\frac{1}{p}-1} \int_R |f(y)|\, dy = 2^n \|f\|_{\mathcal{M}^p_1} \le 2^n \|f\|_{\mathcal{M}^p_q}. \tag{3.9}$$

Estimate (3.7) is therefore proved. With (3.6) and (3.7) proved, we obtain (3.4). □

We now turn to the boundedness of the fractional integral operators. The following lemma is named the comparison lemma by Volberg.

Lemma 3.2 (Comparison lemma). *Let* $0 < \alpha < n$.

$$\int_0^\infty \chi_{B(x,l)}(y) \frac{dl}{l^{n+1-\alpha}} = \frac{|x-y|^{-n+\alpha}}{n-\alpha}. \tag{3.10}$$

Proof. The proof is simple. Indeed, we have

$$\int_0^\infty \chi_{B(x,l)}(y) \frac{dl}{l^{n+1-\alpha}} = \int_0^\infty \chi_{(|x-y|,\infty)}(l) \frac{dl}{l^{n+1-\alpha}} \tag{3.11}$$

$$= \int_{|x-y|}^\infty \frac{dl}{l^{n+1-\alpha}} \tag{3.12}$$

$$= \frac{|x-y|^{-n+\alpha}}{n-\alpha}, \tag{3.13}$$

which is the desired result. □

We use a notation; if we are given a cube Q and a locally integrable function f, then we write

$$m_Q(f) = \frac{1}{|Q|} \int_Q f(x)\, dx,$$

the average of f over a cube Q.

Morrey spaces, the BMO space and Hölder spaces lie in a line. More precisely we have the following.

Theorem 3.3 (I_α and the modified fractional maximal operators $\tilde{I}_\alpha$). *Suppose that $0<\alpha<n$. We define*

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \tag{3.14}$$

$$\tilde{I}_\alpha f(x) := \int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{\chi_{{Q_0}^c}(y)}{|x_0-y|^{n-\alpha}} \right) f(y)\,dy, \tag{3.15}$$

where Q_0 is a fixed cube centered at x_0.

1. (Subcritical case) *Let $p<\dfrac{n}{\alpha}$. Assume that the parameters s,t satisfy*

$$1<q\le p<\infty,\ 1<t\le s<\infty,\ \frac{1}{s}=\frac{1}{p}-\frac{\alpha}{n},\ \frac{t}{s}=\frac{q}{p}. \tag{3.16}$$

Then

$$\|I_\alpha f\|_{\mathcal{M}^s_t} \le C_{p,q,\alpha}\|f\|_{\mathcal{M}^p_q} \tag{3.17}$$

for every (positive) $f\in\mathcal{M}^p_q$.

2. (Critical case) *Assume that $1\le q\le p=\dfrac{n}{\alpha}$. Then*

$$\|\tilde{I}_\alpha f\|_* \le C_{p,q,\alpha}\|f\|_{\mathcal{M}^p_q} \tag{3.18}$$

for every $f\in\mathcal{M}^p_q$, where $\|\cdot\|_*$ denotes the BMO *norm given by*

$$\|g\|_* = \sup_{Q\in\mathcal{Q}} m_Q(|g-m_Q(g)|). \tag{3.19}$$

3. (Supercritical case) *Assume that $1\le q\le p<\infty$ and that $0<\alpha-\dfrac{n}{p}<1$. Then*

$$\|\tilde{I}_\alpha f\|_{\mathrm{Lip}\,(\alpha-\frac{n}{p})} \le C_{p,q,\alpha}\|f\|_{\mathcal{M}^p_q} \tag{3.20}$$

for every $f\in\mathcal{M}^p_q$, where we denoted

$$\|g\|_{\mathrm{Lip}(\theta)} = \sup\left\{ \frac{|g(x)-g(y)|}{|x-y|^\theta} \,:\, x,y\in\mathbb{R}^n,\, x\ne y \right\}, \quad 0<\theta<1 \tag{3.21}$$

for a function g.

Before we come to the proof of this theorem, let us remark that the integral kernel of $\tilde{I}_\alpha$ is better than that of I_α. With the better kernel, we can apply $\tilde{I}_\alpha$ for $\mathcal{M}_q^{n/\alpha}$ functions.

Here we content ourselves with proving (2), other results being proved similarly.

Proof. We have to prove

$$m_Q(|\tilde{I}_\alpha f - m_Q(\tilde{I}_\alpha f)|) \le c\,\|f\|_{\mathcal{M}_q^{n/\alpha}} \tag{3.22}$$

for all cubes Q. For the proof we may assume $q < \dfrac{n}{\alpha}$ because we always have

$$\|f\|_{\mathcal{M}_q^{n/\alpha}} \le \|f\|_{\mathcal{M}_{n/\alpha}^{n/\alpha}} = \|f\|_{L^{n/\alpha}}. \tag{3.23}$$

We decompose f according to $2Q$ as usual. That is, we split $f = f_1 + f_2$ with $f_1 = \chi_{2Q} \cdot f$ and $f_2 = f - f_1$. By virtue of the triangle inequality our present task is partitioned into proving

$$m_Q(|\tilde{I}_\alpha f_1 - m_Q(\tilde{I}_\alpha f_1)|) + m_Q(|\tilde{I}_\alpha f_2 - m_Q(\tilde{I}_\alpha f_2)|) \le c\,\|f\|_{\mathcal{M}_q^{n/\alpha}}. \tag{3.24}$$

Then the estimate for f_1 is simple. Indeed, to estimate f_1, we define an auxiliary index $w \in (q, \infty)$ by $\dfrac{1}{w} = \dfrac{1}{q} - \dfrac{\alpha}{n}$. By the triangle inequality and the Hölder inequality, we have

$$m_Q(|\tilde{I}_\alpha f_1 - m_Q(\tilde{I}_\alpha f_1)|) \le 2m_Q(|\tilde{I}_\alpha f_1|) \le 2m_Q^{(w)}(|\tilde{I}_\alpha f_1|), \tag{3.25}$$

where we wrote

$$m_Q^{(w)}(F) := \left(\frac{1}{|Q|}\int_Q F(x)^w\,dx\right)^{\frac{1}{w}} \tag{3.26}$$

for positive measurable functions F. By using the L^q-L^w boundedness of the fractional integral operator we obtain

$$\begin{aligned} m_Q(|\tilde{I}_\alpha f_1 - m_Q(\tilde{I}_\alpha f_1)|) &\le c\left(\frac{1}{|Q|}\int_{\mathbb{R}^n} |\tilde{I}_\alpha f_1(x)|^w\,dx\right)^{1/w} \\ &\le c\,m_{2Q}(|f|^q)^{1/w} \\ &\le c\,\|f\|_{\mathcal{M}_q^{n/\alpha}}. \end{aligned}$$

For the proof of the second inequality, we write the left-side out in full.

$$\begin{aligned} &m_Q(|\tilde{I}_\alpha f_2 - m_Q(\tilde{I}_\alpha f_2)|) \\ &= \frac{1}{|Q|^2}\int_Q \left|\iint_{Q\times(\mathbb{R}^n\setminus 2Q)} \left(\frac{f(z)}{|x-z|^{n-\alpha}} - \frac{f(z)}{|y-z|^{n-\alpha}}\right) dy\,dz\right| dx. \end{aligned}$$

First, we bound the right-hand side with the triangle inequality and arrange it. Then the right-hand side is majorized by

$$\frac{1}{|Q|^2}\iiint_{Q\times Q\times(\mathbb{R}^n\setminus 2Q)}|f(z)|\cdot\left|\frac{1}{|x-z|^{n-\alpha}}-\frac{1}{|y-z|^{n-\alpha}}\right|dx\,dy\,dz. \tag{3.27}$$

Denote by $c(Q)$ the center of the cube Q. By virtue of the mean value theorem, we have

$$\left|\frac{1}{|x-z|^{n-\alpha}}-\frac{1}{|y-z|^{n-\alpha}}\right|\le c\,\frac{|x-y|}{|z-c(Q)|^{n-\alpha+1}}\le c\,\frac{\ell(Q)}{|z-c(Q)|^{n-\alpha+1}}. \tag{3.28}$$

Thus, inserting this inequality gives us

$$m_Q(|\tilde{I}_\alpha f_2-m_Q(\tilde{I}_\alpha f_2)|)\le c\,\ell(Q)\int_{\mathbb{R}^n\setminus 2Q}\frac{|f(z)|}{|z-c(Q)|^{n-\alpha+1}}\,dz. \tag{3.29}$$

By the comparison lemma, the integral of the right-hand side is bounded by

$$\int_{2\ell(Q)}^{\infty}\left(\int_{B(c(Q),\ell)}|f(z)|\,dz\right)\frac{d\ell}{\ell^{n-\alpha+2}}\le c\,\|f\|_{\mathcal{M}_1^{n/\alpha}}\cdot\int_{\ell(Q)}^{\infty}\frac{d\ell}{\ell^2}=c\frac{\|f\|_{\mathcal{M}_1^{n/\alpha}}}{\ell(Q)}\le c\frac{\|f\|_{\mathcal{M}_q^{n/\alpha}}}{\ell(Q)}. \tag{3.30}$$

Thus, the estimate of the second inequality is complete and the proof is concluded. □

§ 4. Proof of Theorem 1.1

In [10] the proof was rather lengthy. But our proof is a little shorter than that by Olsen.

We need the following observation below. This lemma dates back to [18] for example.

Lemma 4.1. *For a nonnegative function h in $L^\infty(Q_0)$ we let $\gamma_0=m_{Q_0}(h)$ and $c=2^{n+1}$. For $k=1,2,\dots$ let*

$$D_k:=\bigcup\left\{Q\,:\,Q\in\mathcal{D}_1(Q_0):\,m_Q(h)>\gamma_0c^k\right\}\subset\mathbb{R}^n.$$

(0) *Define*

$$\mathcal{Z}_k=\left\{Q\,:\,Q\in\mathcal{D}_1(Q_0):\,m_Q(h)>\gamma_0c^k\right\}. \tag{4.1}$$

Considering the maximal cubes in $\mathcal{Z}_k$ with respect to inclusion, we can write

$$D_k=\bigcup_j Q_{k,j},$$

where the cubes $\{Q_{k,j}\}\subset\mathcal{D}_1(Q_0)$ are nonoverlapping.

(1) *By virtue of the maximality of $Q_{k,j}$ one has that*

$$\gamma_0 c^k < m_{Q_{k,j}}(h) \le 2^n \gamma_0 c^k.$$

(2) *Let*

$$E_0 = Q_0 \setminus D_1 \text{ and } E_{k,j} = Q_{k,j} \setminus D_{k+1}.$$

Then $\{E_0\} \cup \{E_{k,j}\}$ is a disjoint family of sets which decomposes Q_0 and satisfies

$$|Q_0| \le 2|E_0| \text{ and } |Q_{k,j}| \le 2|E_{k,j}|. \tag{4.2}$$

(3) *Also, we set*

$$\mathcal{D}_0 := \{Q \in \mathcal{D}_1(Q_0) : m_Q(h) \le \gamma_0 c\} \subset \mathcal{D}$$
$$\mathcal{D}_{k,j} := \left\{Q \in \mathcal{D}_1(Q_0) : Q \subset Q_{k,j}, \gamma_0 c^k < m_Q(h) \le \gamma_0 c^{k+1}\right\} \subset \mathcal{D}.$$

Then $\mathcal{D}_1(Q_0)$ is partitioned as follows:

$$\mathcal{D}_1(Q_0) = \mathcal{D}_0 \cup \bigcup_{k,j} \mathcal{D}_{k,j}. \tag{4.3}$$

Proof. We choose $Q_{k,j}$ as is indicated in (0).

(1) The left inequality is a consequence of the fact that we chose $Q_{k,j}$ from $\mathcal{Z}_k$. If we consider the dyadic parent R of $Q_{k,j}$, then we have

$$m_R(h) \le \gamma_0 c^k. \tag{4.4}$$

Otherwise, instead of $Q_{k,j}$ R would have been chosen as an element of $\mathcal{Z}_k$. Since

$$m_{Q_{j,k}}(h) \le 2^n m_R(h), \tag{4.5}$$

we obtain the right inequality.

(2) A geometric observation shows that two dyadic cube never intersect unless one is not included in the other. Let j and k be freezed. We write

$$D_{k+1} = \bigcup_{j^*} Q_{k+1,j^*}, \quad Q_{k+1,j^*} \in \mathcal{Z}_{k+1}. \tag{4.6}$$

¿From the observation above, we have

$$D_{k+1} \cap Q_{k,j} = \bigcup \{Q_{k+1,j^*} : Q_{k+1,j^*} \subset Q_{k,j}\} \tag{4.7}$$

because $Q_{k+1,j^*} \supsetneq Q_{k,j}$ never happens thanks to the maximality of $Q_{k,j}$. Note that

$$|Q_{k+1,j^*}|\gamma_0 c^{k+1} \le \int_{Q_{k+1,j^*}} h(x)\,dx \tag{4.8}$$

since $Q_{k+1,j^*} \in \mathcal{Z}_{k+1}$. If we add this estimate for all j^* such that $Q_{k+1,j^*} \subset Q_{k,j}$, we obtain

$$|D_{k+1} \cap Q_{k,j}|\gamma_0 c^{k+1} \le \int_{Q_{k,j}} h(x)\,dx \le |Q_{k,j}|2^n\gamma_0. \tag{4.9}$$

So, if we consider the complement set of D_{k+1}, then we obtain the desired result.

(3) By maximality if $R \subset Q_0$ is a dyadic cube such that $m_R(h) > \gamma_0$, then R is contained in some $Q_{k,j}$. So this assertion follows.

□

We begin by discretizing the operator $I_\alpha f$ following the idea of C. Pérez (see [11]):

$$I_\alpha f(x) = \sum_{\nu\in\mathbb{Z}} \int_{2^{-\nu-1}<|x-y|\le 2^{-\nu}} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \le C\sum_{\nu\in\mathbb{Z}} 2^{\nu(n-\alpha)} \int_{B(x,2^{-\nu})} f(y)\,dy.$$

Denote by $\mathcal{D}_\nu$ the set of all dyadic cubes of volume $2^{-\nu n}$;

$$\mathcal{D}_\nu = \{2^{-\nu}m + 2^{-\nu}[0,1)^n \,:\, m \in \mathbb{Z}^n\}. \tag{4.10}$$

Then we have

$$\begin{aligned} I_\alpha f(x) &\le C\sum_{\nu\in\mathbb{Z}}\sum_{x\in Q\in\mathcal{D}_\nu} \frac{\ell(Q)^\alpha}{|Q|}\int_{3Q} f(y)\,dy \\ &= C\sum_{Q\in\mathcal{D}} \frac{\ell(Q)^\alpha}{|Q|}\int_{3Q} f(y)\,dy \cdot \chi_Q(x) \\ &= C\sum_{Q\in\mathcal{D}} \ell(Q)^\alpha m_{3Q}(f)\cdot\chi_Q(x). \end{aligned}$$

It suffices, from the definition of the Morrey norm, to show that

$$\left(\int_{Q_0} (g(x)I_\alpha f(x))^r\,dx\right)^{1/r} \le C\|g\|_{\mathcal{M}^{q_0}_q}\cdot\|M_\alpha f\|_{\mathcal{M}^{r_0}_r}\cdot|Q_0|^{1/r-1/r_0},$$

for a given dyadic cube Q_0, where M_α denotes the fractional maximal operator given by

$$M_\alpha f(x) = \sup_{x\in Q\in\mathcal{Q}} |Q|^{\alpha/n}\left(\frac{1}{|Q|}\int_Q |f(y)|\,dy\right). \tag{4.11}$$

Indeed, a geometric observation shows

$$|Q|^{\alpha/n-1}\chi_Q(y) \le c|x-y|^{-n+\alpha}, \quad x \in Q. \tag{4.12}$$

So the pointwise estimate $M_\alpha f(x) \le cI_\alpha f(x)$ follows. If we invoke the Adams theorem, then we obtain the desired result. Furthermore, by a simple limiting argument, we can assume that $g \in L^\infty$. Hereafter, we write

$$\begin{cases} \mathcal{D}_1(Q_0) = \{Q \in \mathcal{D} :\, Q \subset Q_0\}, \\ \mathcal{D}_2(Q_0) = \{Q \in \mathcal{D} :\, Q \supsetneq Q_0\}. \end{cases}$$

Let us define for $i = 1, 2$

$$F_i(x) = \sum_{Q \in \mathcal{D}_i(Q_0)} \ell(Q)^\alpha m_{3Q}(f)\chi_Q(x)$$

and we shall estimate

$$\left(\int_{Q_0} (g(x)F_i(x))^r\, dx\right)^{1/r}.$$

First, we establish

$$\left(\int_{Q_0} (g(x)F_1(x))^r\, dx\right)^{1/r} \le C\|g\|_{\mathcal{M}^{q_0}_q} \left(\int_{Q_0} M_{\alpha-n/q_0}f(x)^r\, dx\right)^{1/r}, \tag{4.13}$$

by the duality argument. Take a nonnegative function $w \in L^{r'}$, $1/r+1/r' = 1$, satisfying that w is supported on $\overline{Q_0}$, that $\|w\|_{L^{r'}(Q_0)} = 1$ and that

$$\left(\int_{Q_0} (g(x)F_1(x))^r\, dx\right)^{1/r} = \int_{Q_0} g(x)F_1(x)w(x)\, dx.$$

Letting $h = g \cdot w$, we shall apply Lemma 4.1 to estimate this quantity. It follows that

$$\begin{aligned} \int_{Q_0} g(x)F_1(x)w(x)\, dx &= \sum_{Q \in \mathcal{D}_1(Q_0)} \ell(Q)^\alpha m_{3Q}(f) \int_Q g(x)w(x)\, dx \\ &= \sum_{Q \in \mathcal{D}_0} \ell(Q)^\alpha m_{3Q}(f) \int_Q g(x)w(x)\, dx \\ &\quad + \sum_{j=1}^\infty \sum_k \sum_{Q \in \mathcal{D}_{j,k}} \ell(Q)^\alpha m_{3Q}(f) \int_Q g(x)w(x)\, dx. \end{aligned}$$

First, we evaluate

$$\sum_{Q \in \mathcal{D}_{k,j}} \ell(Q)^\alpha m_{3Q}(f) \int_Q g(x)w(x)\, dx. \tag{4.14}$$

In view of the definition of $\mathcal{D}_{k,j}$, we have

$$\text{R.H.S. of (4.14)} = \sum_{l=0}^{\infty} \sum_{\substack{Q\in\mathcal{D}_{k,j} \\ \ell(Q)=2^{-l}\ell(Q_{k,j})}} \ell(Q)^\alpha m_{3Q}(f) \int_Q g(x)w(x)\,dx$$
$$= 3^n \sum_{l=0}^{\infty} \sum_{\substack{Q\in\mathcal{D}_{k,j} \\ \ell(Q)=2^{-l}\ell(Q_{k,j})}} \ell(Q)^\alpha m_Q(gw) \int_{3Q} f(x)\,dx$$

Let $Q \in \mathcal{D}_{k,j}$. Now that we have

$$\gamma_0 c^k \le m_Q(gw) \le \gamma_0 c^{k+1},\ \gamma_0 c^k \le m_{Q_{k,j}}(gw) \le 2^n \gamma_0 c^k,\ Q \subset Q_{k,j}, \tag{4.15}$$

we obtain $m_Q(gw) \sim m_{Q_{k,j}}(gw)$. Hence it follows that

$$\text{R.H.S. of (4.14)} \le C_{p,q,\alpha} m_{Q_{k,j}}(gw) \sum_{l=0}^{\infty} \sum_{\substack{Q\in\mathcal{D}_{k,j} \\ \ell(Q)=2^{-l}\ell(Q_{k,j})}} \ell(Q)^\alpha \int_{3Q} f(x)\,dx$$
$$\le C_{p,q,\alpha} \sum_{l=0}^{\infty} 2^{-l\alpha} \ell(Q_{k,j})^\alpha m_{Q_{k,j}}(gw) \int_{3Q_{k,j}} f(x)\,dx$$
$$\le C_{p,q,\alpha} \ell(Q_{k,j})^\alpha m_{Q_{k,j}}(gw) \int_{3Q_{k,j}} f(x)\,dx.$$

In summary, we have obtained

$$\sum_{Q\in\mathcal{D}_{k,j}} \ell(Q)^\alpha m_{3Q}(f) \int_Q g(x)w(x)\,dx \le C_{p,q,\alpha} \ell(Q_{k,j})^\alpha m_{3Q_{k,j}}(f)\, m_{Q_{k,j}}(g\,w)\,|E_{k,j}|. \tag{4.16}$$

Using Hölder's inequality, we have

$$m_{Q_{k,j}}(g\,w) \le m^{(q)}_{Q_{k,j}}(g) m^{(q')}_{Q_{k,j}}(w), \tag{4.17}$$

and

$$m^{(q)}_{Q_{k,j}}(g) \le \|g\|_{\mathcal{M}^{q_0}_q} \ell(Q_{k,j})^{-\frac{n}{q_0}} \tag{4.18}$$

We denote

$$M^{(q')}w(x) = \sup_{x\in Q\in\mathcal{Q}} \left(\frac{1}{|Q|}\int_Q w(x)^{q'}\,dx\right)^{1/q'}. \tag{4.19}$$

Estimates (4.17) and (4.18) yield

$$(4.16) \le C\|g\|_{\mathcal{M}^{q_0}_q} \ell(Q_{k,j})^{\alpha-\frac{n}{q_0}} m_{3Q_{k,j}}(f) m^{(q')}_{Q_{k,j}}(w)|E_{k,j}|$$
$$\le C\|g\|_{\mathcal{M}^{q_0}_q} \ell(Q_{k,j})^{\alpha-\frac{n}{q_0}} m_{3Q_{k,j}}(f)|E_{k,j}| \inf_{x\in E_{k,j}} M^{(q')}w(x)$$
$$\le C\|g\|_{\mathcal{M}^{q_0}_q} \int_{E_{k,j}} M_{\alpha-n/q_0}f(x)\, M^{(q')}w(x)\,dx.$$

Similarly, we have

$$\sum_{Q\in\mathcal{D}_0} \ell(Q)^\alpha m_{3Q}(f) \int_Q g(x)w(x)\,dx \le C\|g\|_{\mathcal{M}^{q_0}_q} \int_{E_0} M_{\alpha-n/q_0}f(x)\, M^{(q')}w(x)\,dx.$$

Summing up all factors we obtain

$$(4.14) \le C\|g\|_{\mathcal{M}^{q_0}_{q_0}} \int_{Q_0} M_{\alpha-n/q_0}f(x)\, M^{({q_0}')}w(x)\,dx.$$

Another application of Hölder's inequality gives us that

$$(4.14) \le C\|g\|_{\mathcal{M}^{q_0}_q} \left(\int_{Q_0} M_{\alpha-n/q_0}f(x)^r\,dx\right)^{1/r} \left(\int_{Q_0} M^{(q_0')}w(x)^{r'}\,dx\right)^{1/r'}.$$

The fact $r' > q'$ and the $L^{r'/q'}$-boundedness of maximal operator M yield

$$\begin{aligned}(4.14) &\le C\|g\|_{\mathcal{M}^{q_0}_q} \left(\int_{Q_0} M_{\alpha-n/q_0}f(x)^r\,dx\right)^{1/r} \left(\int_{Q_0} w(x)^{r'}\,dx\right)^{1/r'}\\ &= C\|g\|_{\mathcal{M}^{q_0}_q} \left(\int_{Q_0} M_{\alpha-n/q_0}f(x)^r\,dx\right)^{1/r}.\end{aligned}$$

This is our desired inequality.

The case $i = 2$ A cruder estimate suffices in this case. By a property of the dyadic cubes, for all $x \in Q_0$ we have

$$F_2(x) = \sum_{Q\in\mathcal{D}_2(Q_0)} \ell(Q)^\alpha m_{3Q}(f) \le \sum_{Q\in\mathcal{D}_2(Q_0)} \ell(Q)^{\alpha-\frac{n}{p_0}} \|f\|_{\mathcal{M}^{p_0}_p}. \tag{4.20}$$

In view of the definition of $\mathcal{D}_2(Q_0)$, we have

$$F_2(x) \le \mu\ell(Q_0)^{\alpha-\frac{n}{p_0}} \|f\|_{\mathcal{M}^{p_0}_p} \quad \left(\mu = \sum_{j=0}^{\infty} 2^{j(\alpha-n/p_0)}\right) \tag{4.21}$$

Thus, for all $x \in Q_0$ we obtain

$$F_2(x) \le C\|f\|_{\mathcal{M}^{p_0}_p} \ell(Q_0)^{\alpha-\frac{n}{p_0}} \tag{4.22}$$

and

$$\left(\int_{Q_0} (g(x)F_2(x))^r\,dx\right)^{1/r} \le C\, m^{(q)}_{Q_0}(g)\|f\|_{\mathcal{M}^{p_0}_p} \ell(Q_0)^{\alpha-\frac{n}{p_0}+\frac{n}{r}}. \tag{4.23}$$

This is our desired inequality.

Remark. In the course of the proof we have proved

$$I_\alpha f(x) \le C \sum_{Q\in\mathcal{D}} \ell(Q)^\alpha m_{3Q}(f)\chi_Q(x). \tag{4.24}$$

Here is a converse inequality:

$$C^{-1}\sum_{Q\in\mathcal{D}} \ell(Q)^\alpha m_{3Q}(f)\chi_Q \le \sum_{Q\in\mathcal{D}} \ell(Q)^\alpha \left(\inf_Q Mf\right)\chi_Q \le C\, I_\alpha(Mf). \tag{4.25}$$

§ 5. Counterexample

Proposition 5.1. *Let $1 < r \le r_0 < \infty$ and $r < 1/\alpha$. Then, for any $c > 0$ we can find positive measurable functions f and g such that*

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r,r_0}} > c\|g\|_{\mathcal{M}^{r,1/\alpha}}\|f\|_{\mathcal{M}^{r,r_0}}.$$

Proof. The proof of this proposition is kind of lengthy and we use the predual of Morrey spaces investigated originally by Zorko [19]. Here we content ourselves with remarking that we chose $g = f = \chi_{E_j}$ with $j \in \mathbb{N}$, where E_j denotes the fractal set of this note (See Example 2.1) and $R > 1$ is appropriately chosen. □

§ 6. Applications

§ 6.1. Application to the variation problem

Here we apply Theorem 1.1 to a variation problem.

Denote by $X_q^{q_0}$ the closure of C_c^∞ with respect to $\mathcal{M}_q^{q_0}$.

Theorem 6.1. *Let $n \ge 3$ and $V \in \mathcal{M}_p^{n/2}$ with $1 < p \le n/2$. Let $W \in L^\infty$. Define a functional $\mathcal{E}_V$ and $\mathcal{E}_{V+W}$ by*

$$\mathcal{E}_V(\varphi) := \int_{\mathbb{R}^n} |\nabla\varphi(x)|^2 + V(x)|\varphi(x)|^2\,dx \quad (\varphi \in H^1)$$
$$\mathcal{E}_{V+W}(\varphi) := \int_{\mathbb{R}^n} |\nabla\varphi(x)|^2 + (V(x)+W(x))|\varphi(x)|^2\,dx \quad (\varphi \in H^1).$$

(a) *There exists a constant $\alpha > 0$ such that*

$$\|f \cdot \sqrt{|V|}\|_2 \le \alpha\sqrt{\|V\|_{\mathcal{M}_p^{n/2}}}\|\nabla f\|_2 \tag{6.1}$$

for all $f \in C_c^\infty$.

(b) *If* $\|\min(0,V)\|_{\mathcal{M}^{n/2}_p} < \alpha^{-2}$ *and* $2 < p \le 2n$, *then* $H = -\Delta + V$ *is a positive operator on* $L^2(\mathbb{R}^n)$.

(c) *Suppose that* $\|\min(0,V)\|_{\mathcal{M}^{n/2}_p} < \dfrac{1}{4\alpha^2}$. *If* $\{\varphi_j\}_{j\in\mathbb{N}}$ *is an* $L^2(\mathbb{R}^n)$*-sequence such that* $\|\varphi_j\|_2 \le 1$ *for each* $j \in \mathbb{N}$ *and that* $\sup_{j\in\mathbb{N}} \mathcal{E}_{V+W}(\varphi_j) < \infty$, *then* $\{\varphi_j\}_{j\in\mathbb{N}}$ *forms a bounded sequence in* $H^1(\mathbb{R}^n)$.

(d) *Suppose in addition that* $W \in L^\infty$. *Then we have*

$$\lim_{j\to\infty} \mathcal{E}_{V+W}(\varphi_j) \ge \mathcal{E}_{V+W}(\varphi) \tag{6.2}$$

if $\lim_{j\to\infty} \varphi_j = \varphi$ *in the weak topology of* H^1.

(e) *Let* $V \in X^{n/2}_p$ *and* $1 < p \le n/2$. *Suppose that* $W \in L^\infty$. *Assume in addition that* $\|\min(0,V)\|_{\mathcal{M}^{n/2}_p} < \dfrac{1}{4\alpha^2}$ *and that*

$$E_0 = \inf\{\mathcal{E}_{V+W}(\varphi) \,:\, \varphi \in H^1,\ \|\varphi\|_2 = 1\} < 0 \tag{6.3}$$

then there exists $\varphi_0 \in L^2$ *such that*

$$\mathcal{E}_{V+W}(\varphi_0) = \inf\{\mathcal{E}_{V+W}(\varphi) \,:\, \varphi \in H^1,\ \|\varphi\|_2 = 1\},\ \|\varphi_0\|_2 = 1. \tag{6.4}$$

Assertions (a) and (b) are easy consequences of Theorem 1.1. Here we prove (c), (d) and (e).

Proof of (c). By (a)

$$\|\,\varphi_j\sqrt{|\min(V,0)|}\,\|_2 \le \frac{1}{2}\|\nabla\varphi_j\|_2 \tag{6.5}$$

for each $j \in \mathbb{N}$. Thus, it follows that

$$T_{\varphi_j} = \int_{\mathbb{R}^n} |\nabla\varphi_j|^2 \ge -4\int_{\mathbb{R}^n} |\varphi_j(x)|^2 \min(V(x),0)\,dx. \tag{6.6}$$

This pointwise estimate yields

$$\frac{3}{4}T_{\varphi_j} = T_{\varphi_j} - \frac{1}{4}T_{\varphi_j} \le T_{\varphi_j} + \int_{\mathbb{R}^n} |\varphi_j(x)|^2 \min(V(x),0)\,dx \le \mathcal{E}_V(\varphi_j). \tag{6.7}$$

Since

$$\sup_j \mathcal{E}_V(\varphi_j) \le \|W\|_\infty + \sup_j \mathcal{E}_{V+W}(\varphi_j) < \infty, \tag{6.8}$$

it follows that $\{\varphi_j\}_{j\in\mathbb{N}}$ forms a bounded sequence in H^1. □

Proof of (d). Let us choose $\{V_k\}_{k\in\mathbb{N}} \subset C_c^\infty$ that approximates V in the $\mathcal{M}_p^{n/2}$ topology. Since by our theorem (see Theorem 1.1) we have

$$|\mathcal{E}_{V+W}(\varphi_j) - \mathcal{E}_{V_k+W}(\varphi_j)| \le C\,\|V - V_k\|_{\mathcal{M}_p^{n/2}}\|\varphi_j\|_{H^1} \tag{6.9}$$

and $\{\varphi_j\}_{j\in\mathbb{N}}$ forms a bounded family in H^1, we can assume that $V \in C_c^\infty$. Once we assume that $V \in C_c^\infty$, then we can use the compact embedding $H^1 \hookrightarrow L^{2+\varepsilon}$ to conclude the proof of (d). □

Proof of (e). Let us choose a sequence $\{\varphi_j\}_{j\in\mathbb{N}} \subset L^2$ so that

$$E_0 = \lim_{j\to\infty} \mathcal{E}_{V+W}(\varphi_j),\ \|\varphi_j\|_2 = 1. \tag{6.10}$$

By virtue of the weak compactness of H^1 and (c), we can assume that $\varphi_j \to \varphi$ in the weak topology of H^1.

Since $E_0 = \mathcal{E}_V(\varphi) < 0$ by virtue of (d), it follows that $\varphi \ne 0$. Now that φ attains E_0, its L^2-norm must be 1. Therefore, we obtain the desired result. □

As an example to which we can apply our Theorem 1.1, we have the following function V_N. Recall that we have defined E_N as a union of cubes of equal size in Proposition 2.1. Here the auxiliary parameter R is chosen for the function space $\mathcal{M}_p^{n/2}$, that is, we have chosen R so that

$$\left(\frac{2^n}{(1+R)^n}\right)^{1/p} = \left(\frac{1}{(1+R)^n}\right)^{2/n}. \tag{6.11}$$

Write $E_N = \bigcup_{j=1}^{2^{nN}} Q(z_{N,j}, r_{N,j})$, where

$$Q(z,r) := \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \,:\, |x_1 - z_1|, |x_2 - z_2|, \cdots, |x_n - z_n| \le r\}.$$

We define

$$V_N(x) := -\kappa \sum_{j=1}^{2^{nN}} \chi_{Q(z_{N,j},r_{N,j})}(x)|x - z_{N,j}|^{-2/n}. \tag{6.12}$$

Then there exists $\kappa_0 > 0$ with the following property; if $0 < \kappa < \kappa_0$ and $M \in \mathbb{R}$ satisfies

$$E_0 = \inf\{\mathcal{E}_{V_N+M}(\varphi) \,:\, \varphi \in H^1,\ \|\varphi\|_2 = 1\} < 0 \tag{6.13}$$

then for all $N \in \mathbb{N}$, there exists $\varphi \in H^1$ such that

$$\mathcal{E}_{V_N+W}(\varphi_0) = \inf\{\mathcal{E}_{V_N+W}(\varphi) \,:\, \varphi \in H^1,\ \|\varphi\|_2 = 1\},\ \|\varphi_0\|_2 = 1. \tag{6.14}$$

§ 6.2. Sobolev-Hardy inequality

Let $0 \le s \le 2$. Then we have

$$\|u\|_{L^{\frac{2n-2s}{n-2}}(|x|^{-s}\,dx)} \le C\,\|\nabla u\|_{L^2}. \tag{6.15}$$

Using Theorem 1.1, we can extend this theorem to some extent.

Here is our result.

Theorem 6.2. *Let $1 < p \le p_0 < n$, $1 < r \le r_0 < \infty$ and $0 < s < n$. Assume that*

$$\frac{1}{r_0} = \frac{s}{nr} + \frac{1}{p_0} - \frac{1}{n},\ \frac{r}{r_0} = \frac{p}{p_0},\ r \ge s. \tag{6.16}$$

Then we have

$$\left\|u|x|^{-\frac{s}{r}}\right\|_{\mathcal{M}^{r_0}_r} \le C\,\|\nabla u\|_{\mathcal{M}^{p_0}_p}.$$

Proof. This is just a rephrasement of Theorem 1.1 obtained by setting $q_0 = \dfrac{nr}{s}$ and $g(x) = |x|^{-s/r}$. Observe that $g \in \mathcal{M}^{q_0}_{q_0-\varepsilon}$ for all $0 < \varepsilon \ll 1$. □

§ 6.3. An extension of Olsen's result

Theorem 6.3. *Assume that*

$$0 < \alpha < n,\ 1 < \tilde{q} \le \frac{n}{2},\ 1 < q \le p < \infty,\ 1 < t \le s < \infty$$

and that

$$\tilde{q} > q,\ \frac{1}{s} = \frac{1}{p} - \frac{2}{n},\ \frac{t}{s} = \frac{q}{p}.$$

Denote by M_V the multiplication operator generated by the potential V. Then we have

$$\|VI_2W\|_{\mathcal{M}^p_q} \le C_0\,\|V\|_{\mathcal{M}^{n/2}_{\tilde{q}}}\|W\|_{\mathcal{M}^p_q} \tag{6.17}$$

and that

$$\|(I_2M_V)^nI_2W\|_{\mathcal{M}^s_t} \le C_1{C_0}^n\,{\|V\|_{\mathcal{M}^{n/2}_{\tilde{q}}}}^n\|W\|_{\mathcal{M}^p_q} \tag{6.18}$$

for some $C_0, C_1 > 1$.

In particular, if $V \in \mathcal{M}^{n/2}_{\tilde{q}} \cap \mathcal{M}^p_q$ and the norm is sufficiently small, then the formal solution (of $\Delta v + Vv = v$)

$$v = 1 + \sum_{j=1}^{\infty}(-1)^j(I_2M_V)^{j-1}(I_2V)$$

satisfies

$$\|v - 1\|_{\mathcal{M}^s_t} \le C. \tag{6.19}$$

Proof. Inequality (6.17) is just a repetition of our results. As for (6.18), we have

$$\|(I_2 M_V)^n I_2 W\|_{\mathcal{M}^s_t} \le C_1 \,\|(M_V I_2)^n W\|_{\mathcal{M}^p_q} \le C_1 {C_0}^n \|V\|_{\mathcal{M}^{n/2}_{\tilde{q}}}{}^n \|W\|_{\mathcal{M}^p_q}$$

using the first estimate n-times. Finally, the last assertion can be obtained directly from (6.18). □

Remark. Olsen postulated

$$0 < \alpha < n,\ 1 < \tilde{q} \le \frac{n}{2},\ 1 < t \le s < \infty,\ t < \frac{2}{n}\tilde{q}s \tag{6.20}$$

additionally on the parameters [10]. With this condition and the condition that all the functions are supported in a bounded set, he obtained $\|v - 1\|_{\mathcal{M}^s_t} \le C$. However, the last condition $t < \frac{2}{n}\tilde{q}s$ turned out to be superflous. We can replace this condition with $t < \frac{1}{p}\tilde{q}s$ with p given by $\frac{1}{s} = \frac{1}{p} - \frac{2}{n}$, keeping in mind that $p < \frac{n}{2}$.

§ 7. Full statement of our main results

Here we formulate our main theorem as the full statement.

§ 7.1. A passage from I_α to T_ρ and from $\mathcal{M}^p_q$ to generalized Morrey spaces

Here we describe what we actually obtained in our first paper [12]. Let $\rho :\ [0, \infty) \to [0, \infty]$ be a suitable function. We define the generalized fractional integral operator T_ρ and the generalized fractional maximal operator M_ρ by

$$T_\rho f(x) := \int_{\mathbb{R}^n} f(y) \frac{\rho(|x-y|)}{|x-y|^n}\, dy,$$
$$M_\rho f(x) := \sup_{x \in Q \in \mathcal{Q}} \rho(\ell(Q)) m_Q(|f|).$$

If $\rho(t) \equiv t^{n\alpha}$, $0 < \alpha < 1$, then $T_\rho = I_\alpha$ and $M_\rho = M_\alpha$. The Morrey norm $\|f\|_{p,\rho}$ is given by

$$\|f\|_{p,\rho} := \sup_{Q \in \mathcal{Q}} \rho(\ell(Q)) \left(\frac{1}{|Q|} \int_Q |f(x)|^p\, dx \right)^{1/p}. \tag{7.1}$$

In general, by the Dini condition we mean that

$$\int_0^1 \frac{\rho(s)}{s}\, ds < \infty, \tag{7.2}$$

while the doubling condition (with a doubling constant $C_0 > 0$) is that

$$\frac{1}{C_0} \le \frac{\rho(s)}{\rho(t)} \le C_0,\ \text{ if } \frac{1}{2} \le \frac{s}{t} \le 2. \tag{7.3}$$

A simple consequence that can be deduced from the doubling condition is

$$\frac{\log 2}{C_0}\rho(t) \le \int_{t/2}^{t} \frac{\rho(s)}{s}\,ds \le \log 2 \cdot C_0\rho(t) \text{ for all } t>0. \tag{7.4}$$

In the sequel, we always assume that ρ satisfies (7.2) and (7.3), and, then denote the set of all such functions by $\mathcal{G}_0$. We will write, when $\rho \in \mathcal{G}_0$,

$$\tilde\rho(t) := \int_0^t \frac{\rho(s)}{s}\,ds.$$

Let $\mathcal{G}_1$ be the set of all functions $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t)$ is nondecreasing but that $\phi(t)t^{-n}$ is nonincreasing. We notice that the condition $\phi \in \mathcal{G}_1$ is stronger than the doubling condition (7.3). More quantitatively, if we assume that $\phi \in \mathcal{G}_1$, then ϕ satisfies the doubling condition with the doubling constant 2^n.

Theorem 7.1. *Let $1<p<\infty$, $q>r$, $0 \le b \le 1$, $a>1$ and $(a+b-1)r = ap$. Suppose that ρ satisfies the Dini condition, (7.3) and that $\tilde\rho(t)^{\max(ap,bq)}t^{-n}$ is nonincreasing. The condition $\tilde\rho(t)^{\max(ap,bq)}t^{-n}$ is nonincreasing implies that $\tilde\rho(t)^{ap}t^{-n}$ and $\tilde\rho(t)^{bq}t^{-n}$ are nonincreasing, since*

$$\tilde\rho(t)^{\min(ap,bq)}t^{-n} = \tilde\rho(t)^{\min(ap,bq)-\max(ap,bq)} \cdot \tilde\rho(t)^{\max(ap,bq)}t^{-n}.$$

Then

$$\|g \cdot T_\rho f\|_{r,\tilde\rho^{a+b-1}} \le C\|g\|_{q,\tilde\rho^b}\|f\|_{p,\tilde\rho^a},$$

where the constant C is independent of f and g.

Theorem 7.1 generalizes of [10, Theorem 2] and [17, Theorem 1]. Theorem 7.1 is not longer true when $q=r$ (see Proposition 5.1).

Letting $b=0$ and $g \equiv 1$ in Theorem 7.1, we have the following:

Corollary 7.2. *Let $1<p<\infty$, $a>1$ and $(a-1)r = ap$. Then*

$$\|T_\rho f\|_{r,\tilde\rho^{a-1}} \le C\|f\|_{p,\tilde\rho^a}.$$

Corollary 7.2 generalizes [1, Theorem 1.3].

Theorem 7.3. *Let $1 \le p < \infty$, $\begin{cases} p \le q \text{ if } p=1, \\ p<q \text{ if } p>1, \end{cases}$ $0 \le b \le 1$ and $b<a$. Suppose that ρ satisfies the Dini condition, (7.3) and that $\tilde\rho(t)^{\max(ap,bq)}t^{-n}$ is nonincreasing. Then*

$$\|g \cdot T_\rho f\|_{p,\tilde\rho^a} \le C\|g\|_{q,\tilde\rho^b}\|M_{\tilde\rho^{1-b}}f\|_{p,\tilde\rho^a},$$

where the constant C is independent of f and g.

Corollary 7.4. *Let $1 \le p < \infty$ and $a > 0$. Then*

$$\|T_\rho f\|_{p,\tilde{\rho}^a} \le C\|M_{\tilde{\rho}} f\|_{p,\tilde{\rho}^a}.$$

Corollary 7.4 generalizes [2, Theorem 4.2].

Letting $b = 1$ in Theorem 7.3, we have the following too:

Corollary 7.5. *Let $1 \le p < \infty$, $\begin{cases} p \le q \text{ if } p = 1, \\ p < q \text{ if } p > 1, \end{cases}$ and $a > 1$. Then*

$$\|g \cdot T_\rho f\|_{p,\tilde{\rho}^a} \le C\|g\|_{q,\tilde{\rho}}\|Mf\|_{p,\tilde{\rho}^a}.$$

These results are somehow generalized. We generalized them in [13] as follows; we content ourselves with stating them.

Theorem 7.6. *Let $1 \le p < \infty$, $\begin{cases} p \le q \text{ if } p = 1, \\ p < q \text{ if } p > 1. \end{cases}$ Suppose that $\phi(t)$ and $\eta(t)$ are nondecreasing but that $\phi(t)^p t^{-n}$ and $\eta(t)^q t^{-n}$ are nonincreasing. Assume also that*

$$\int_t^\infty \frac{\rho(s)\eta(s)}{s\,\tilde{\rho}(s)\phi(s)}\,ds \le C\frac{\eta(t)}{\phi(t)} \text{ for all } t > 0. \tag{7.5}$$

Then

$$\|g \cdot T_\rho f\|_{p,\phi} \le C\|g\|_{q,\eta}\|M_{\tilde{\rho}/\eta} f\|_{p,\phi},$$

where the constant C is independent of f and g.

Theorem 7.7. *Let $1 < p \le r < q < \infty$. Suppose that $\phi(t)$ and $\eta(t)$ are nondecreasing but that $\phi(t)^p t^{-n}$ and $\eta(t)^q t^{-n}$ are nonincreasing. Suppose also that*

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\,\phi(s)}\,ds \le C\frac{\eta(t)}{\phi(t)^{p/r}} \text{ for all } t > 0. \tag{7.6}$$

Then

$$\|g \cdot T_\rho f\|_{r,\phi^{p/r}} \le C\|g\|_{q,\eta}\|f\|_{p,\phi},$$

where the constant C is independent of f and g.

Theorem 7.8. *Let $0 < p < \infty$. Suppose that ρ, η and ϕ are nondecreasing and that $\eta(t)^p t^{-n}$ and $\phi(t)^p t^{-n}$ are nonincreasing. Then*

$$\|g \cdot M_\rho f\|_{p,\phi} \le C\|g\|_{p,\eta}\|M_{\rho/\eta} f\|_{p,\phi},$$

where the constant C is independent of f and g.

§ 7.2. Extension of Theorem 1.1 to Orlicz-Morrey spaces

To describe Orlicz-Morrey spaces, we recall some definitions and notation. Here we follow [15].

A function $\Phi : [0, \infty) \to [0, \infty]$ is said to be a Young function if it is left-continuous, convex and increasing, and if $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. We say that Φ is a normalized Young function when Φ is a Young function and $\Phi(1) = 1$. It is easy to see that t^p, $1 \le p < \infty$, is a normalized Young function.

A Young function Φ is said to satisfy the Δ_2-condition, denoted $\Phi \in \Delta_2$, if for some $K > 1$

$$\Phi(2t) \le K\Phi(t) \text{ for all } t > 0.$$

Meanwhile, a Young function Φ is said to satisfy the ∇_2-condition, denoted $\Phi \in \nabla_2$, if for some $K > 1$

$$\Phi(t) \le \frac{1}{2K}\Phi(Kt) \text{ for all } t > 0.$$

The function $\Phi(t) \equiv t$ satisfies the Δ_2-condition but fails the ∇_2-condition. If $1 < p < \infty$, then $\Phi(t) \equiv t^p$ satisfies both conditions. The complementary function $\bar{\Phi}$ of a Young function Φ is defined by

$$\bar{\Phi}(t) := \sup\{ts - \Phi(s) : s \in [0, \infty)\}.$$

Then $\bar{\Phi}$ is also a Young function and $\bar{\bar{\Phi}} = \Phi$. Notice that $\Phi \in \nabla_2$ if and only if $\bar{\Phi} \in \Delta_2$. For the other properties of Young functions and the examples, see [8, p196].

Given a Young function Φ, define the Orlicz space $\mathcal{L}^\Phi(\mathbb{R}^n) = \mathcal{L}^\Phi$ by the Luxemberg norm

$$\|f\|_{\mathcal{L}^\Phi} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

When $\Phi(t) \equiv t^p$, $1 \le p < \infty$, $\|f\|_{\mathcal{L}^\Phi} = \|f\|_{L^p}$. We need the following basic two facts.

Generalized Hölder's inequality:

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \le C\|f\|_{\mathcal{L}^\Phi}\|g\|_{\mathcal{L}^{\bar{\Phi}}};$$

The dual equation:

$$\|f\|_{\mathcal{L}^\Phi} \approx \sup\left\{\|fg\|_{L^1} : \|g\|_{\mathcal{L}^{\bar{\Phi}}} \le 1\right\}.$$

Given a Young function Φ, define the mean Luxemburg norm of f on a cube $Q \in \mathcal{Q}$ by

$$\|f\|_{\Phi, Q} := \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

When $\Phi(t) \equiv t^p$, $1 \le p < \infty$,

$$\|f\|_{\Phi,\,Q} = \left(\frac{1}{|Q|}\int_Q |f(x)|^p\,dx\right)^{1/p},$$

that is, the mean Luxemburg norm coincides with the (normalized) L^p norm. It should be noticed that

$$\|f\|_{\Phi,\,Q} = \|\tau_{\ell(Q)}[f\chi_Q]\|_{\mathcal{L}^\Phi}, \tag{7.7}$$

where τ_δ, $\delta > 0$, is the dilation operator $\tau_\delta f(x) = f(\delta x)$. It follows from this relation and generalized Hölder's inequality that for any cube $Q \in \mathcal{Q}$

$$m_Q(|fg|) \le C\|f\|_{\Phi,\,Q}\|g\|_{\bar{\Phi},\,Q}. \tag{7.8}$$

The Orlicz maximal operator, for any Young function Ψ, is defined by

$$M^\Psi f(x) := \sup_{x\in Q\in\mathcal{Q}} \|f\|_{\Psi,\,Q}.$$

Now let us introduce Orlicz-Morrey spaces.

Definition 7.9. Let $\phi \in \mathcal{G}_1$ and let Φ be a Young function. The Orlicz-Morrey space $\mathcal{L}^{\Phi,\,\phi}(\mathbb{R}^n) = \mathcal{L}^{\Phi,\,\phi}$ consists of all locally integrable functions f on $\mathbb{R}^n$ for which the norm

$$\|f\|_{\mathcal{L}^{\Phi,\,\phi}} := \sup_{Q\in\mathcal{Q}} \phi(\ell(Q))\|f\|_{\Phi,\,Q}$$

is finite. In particular, in order that the characteristic function of the unit cubes belongs to $\mathcal{L}^{\Phi,\,\phi}$, it is always assumed that

$$\sup_{t>1} \frac{\phi(t)}{\Phi^{-1}(t^n)} < \infty.$$

Example We define

$$\|f\|_{\mathcal{M}^p_{\mathrm{L\log L}}} = \|f\|_{\Phi,\,\phi}, \tag{7.9}$$

when $\phi(t) = t^{n/p}$ and $\Phi(t) = t\log(3+t)$.

We have shown in [15] that

$$\|Mf\|_{\mathcal{M}^p_1} \sim \|f\|_{\mathcal{M}^p_{\mathrm{L\log L}}}. \tag{7.10}$$

If $\Phi(t) \equiv t^p$ and $\phi(t) \equiv t^{n/p_0}$, $1 \le p \le p_0 < \infty$, then $\mathcal{L}^{\Phi,\,\phi} = \mathcal{M}^{p,\,p_0}$. When $\Phi(t) \equiv t^p$, $1 \le p < \infty$, we will denote $\mathcal{L}^{\Phi,\,\phi}$ by $\mathcal{M}^{p,\,\phi}$. In this case we will call it the (generalized) Morrey space. We consider $\mathcal{M}^{p,\,\phi}$ even for $0 < p < 1$. We define an auxiliary space too.

Definition 7.10. Let $\phi \in \mathcal{G}_1$ and let Φ be a Young function. The space

$$\tilde{\mathcal{L}}^{\Phi,\,\phi}(\mathbb{R}^n) = \tilde{\mathcal{L}}^{\Phi,\,\phi}$$

consists of all locally integrable functions g on $\mathbb{R}^n$ for which the norm

$$\|g\|_{\tilde{\mathcal{L}}^{\Phi,\,\phi}} := \sup\left\{\|M_\phi[gw\chi_Q]\|_{\bar{\Phi},\,Q} \,:\, Q \in \mathcal{Q},\, \|w\|_{\bar{\Phi},\,Q} \le 1\right\}$$

is finite.

Related to the space $\tilde{\mathcal{L}}^{\Phi,\,\phi}$, we need the following notion too.

Definition 7.11. Let Φ and Ψ be Young functions. One says that "M^Ψ is locally bounded in the norm determined by Φ", when it satisfies

$$\|M^\Psi[g\chi_Q]\|_{\Phi,\,Q} \le C\|g\|_{\Phi,\,Q} \text{ for all cubes } Q \in \mathcal{Q}.$$

We now state our first results, which extend those in [12, 13] to Orlicz-Morrey spaces.

Theorem 7.12. *Let $\rho \in \mathcal{G}_0$, $\phi \in \mathcal{G}_1$ and $\Phi \in \nabla_2$. Suppose that the condition*

$$\int_t^\infty \frac{\rho(s)}{s\phi(s)}\,ds \le C\frac{\tilde{\rho}(t)}{\phi(t)} \text{ for all } t > 0. \tag{7.11}$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Phi,\,\phi}} \le C\|g\|_{\tilde{\mathcal{L}}^{\Phi,\,\tilde{\rho}}}\|f\|_{\mathcal{L}^{\Phi,\,\phi}}.$$

Theorem 7.13. *Let Ψ be a Young function. With the same condition posed in Theorem 7.12, if, in addition, $M^{\bar{\Psi}}$ is locally bounded in the norm determined by $\bar{\Phi}$, then we have*

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Phi,\,\phi}} \le C\|g\|_{\mathcal{L}^{\Psi,\,\tilde{\rho}}}\|f\|_{\mathcal{L}^{\Phi,\,\phi}}.$$

Theorems 7.12 and 7.13 are the trace inequalities of the generalized fractional integral operators for Orlicz-Morrey spaces.

Theorem 7.14. *Let $\rho \in \mathcal{G}_0$, $\phi, \psi \in \mathcal{G}_1$, $\Phi \in \nabla_2$ and $0 < a \le 1$. Set*

$$\eta(t) \equiv \phi(t)^a, \quad \Psi(t) \equiv \Phi(t^{1/a}).$$

Suppose that the condition

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)}\,ds \le C\frac{\psi(t)}{\eta(t)} \text{ for all } t > 0. \tag{7.12}$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Psi,\,\eta}} \le C\|g\|_{\tilde{\mathcal{L}}^{\Psi,\,\psi}}\|f\|_{\mathcal{L}^{\Phi,\,\phi}}.$$

Theorem 7.14 is a general form of Theorem 7.12 (letting $a \equiv 1$) and is the Olsen inequality of the generalized fractional integral operators for Orlicz-Morrey spaces.

Letting $g(x) \equiv 1$ and $\psi(t) \equiv 1$ in Theorem 7.14, we can recover the boundedness property of T_ρ.

Corollary 7.15. *Let $\rho \in \mathcal{G}_0$, $\phi \in \mathcal{G}_1$, $\Phi \in \nabla_2$ and $0 < a \le 1$. Set*

$$\eta(t) \equiv \phi(t)^a, \quad \Psi(t) \equiv \Phi(t^{1/a}).$$

Suppose that $\bar{\Psi} \in \nabla_2$ and that the condition

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)}\,ds \le \frac{C}{\eta(t)} \text{ for all } t > 0.$$

Then

$$\|T_\rho f\|_{\mathcal{L}^{\Psi,\,\eta}} \le C\|f\|_{\mathcal{L}^{\Phi,\,\phi}}.$$

However, in the next theorem we reproved this corollary directly without the assumption $\bar{\Psi} \in \nabla_2$.

Theorem 7.16. *Let $\rho \in \mathcal{G}_0$, $\phi \in \mathcal{G}_1$, $\Phi \in \nabla_2$ and $0 < a \le 1$. Set*

$$\eta(t) \equiv \phi(t)^a, \quad \Psi(t) \equiv \Phi(t^{1/a}).$$

Suppose that the condition

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)}\,ds \le \frac{C}{\eta(t)} \text{ for all } t > 0.$$

Then

$$\|T_\rho f\|_{\mathcal{L}^{\Psi,\,\eta}} \le C\|f\|_{\mathcal{L}^{\Phi,\,\phi}}.$$

Corollary 7.15 generalizes [13, Corollary 1.7]. In [7] Nakai studied the boundness of the generalized fractional integral operator T_ρ on Orlicz spaces. Since, we cannot recover Orlicz spaces as a special case of our Orlicz-Morrey spaces, we dare not compare Corollary 7.15 with [7, Theorem 3.1].

§ 8. Appendix-Boundedness of the fractional maximal operator

Lemma 8.1. *Let $p > 1$. Suppose that $\phi(t)$ is nondecreasing and $\phi(t)^p t^{-n}$ is nonincreasing. Then*

$$\|Mf\|_{p,\phi} \le C\|f\|_{p,\phi}.$$

Proof. Fix a cube Q_0. Let $f_1 = \chi_{3Q_0} f$ and $f_2 = f - f_1$. Then the subadditivity of M yields

$$Mf(x) \le Mf_1(x) + Mf_2(x).$$

It follows from the definition of M that for all $x \in Q_0$

$$Mf_2(x) = \sup_{x \in Q \in \mathcal{Q} \colon \ell(Q) \ge \ell(Q_0)} \frac{1}{|Q|} \int_Q |f(y)|\,dy.$$

Suppose that $x \in Q_0$, $x \in Q \in \mathcal{Q}$ and $\ell(Q) \ge \ell(Q_0)$. Then

$$\phi(\ell(Q_0)) m_Q(|f|) \le \phi(\ell(Q)) m_Q^{(p)}(|f|) \le \|f\|_{p,\phi},$$

where we have used Hölder's inequality and the fact that ϕ is nondecreasing.

This gives us that

$$\phi(\ell(Q_0)) Mf_2(x) \le \|f\|_{p,\phi} \text{ for all } x \in Q_0,$$

and that

$$\begin{aligned}\phi(\ell(Q_0)) m_{Q_0}^{(p)}(Mf)^p &\le \phi(\ell(Q_0)) m_{Q_0}^{(p)}(Mf_1)^p + \phi(\ell(Q_0)) m_{Q_0}^{(p)}(Mf_2)^p \\ &\le C\phi(\ell(3Q_0)) m_{3Q_0}^{(p)}(f) + \|f\|_{p,\phi} \le C\|f\|_{p,\phi},\end{aligned}$$

where we have used L^p boundedness of maximal operator M. This implies our desired inequality. □

Lemma 8.2. *Let $1 < p \le q < \infty$. Suppose that $\phi(t)$ is nondecreasing and $\phi(t)^p t^{-n}$ is nonincreasing. Then*

$$\|M_{\phi^{1-p/q}} f\|_{q,\phi^{p/q}} \le C\|f\|_{p,\phi}.$$

It is worth noting that the surjectivity of ϕ was superflous.

Proof. Let $x \in \mathbb{R}^n$ be a fixed point. For every cube $Q \ni x$ we see that

$$\begin{aligned}\phi(\ell(Q))^{1-p/q} m_Q(|f|) &\le \min(\phi(\ell(Q))^{1-p/q} Mf(x)\,,\, \phi(\ell(Q))^{-p/q}\|f\|_{p,\phi}) \\ &\le \sup_{t \ge 0} \min(t^{1-p/q} Mf(x)\,,\, t^{-p/q}\|f\|_{p,\phi}) \\ &= \|f\|_{p,\phi}^{1-p/q} Mf(x)^{p/q}.\end{aligned}$$

This implies

$$M_{\phi^{1-p/q}} f(x)^q \le \|f\|_{p,\phi}^{q-p} Mf(x)^p.$$

It follows from Lemma 8.1 that for every cube Q_0

$$m_{Q_0}^{(q)}(M_{\phi^{1-p/q}} f) \le \|f\|_{p,\phi}^{1-p/q} m_{Q_0}^{(p)}(Mf)^{p/q} \le C\|f\|_{p,\phi} \phi(\ell(Q_0))^{-p/q}.$$

The desired inequality then follows. □

§ 9. Acknowledgement

The authors are thankful to the anonymous referee for his/her careful reading of this paper. With the comments by the referee, the paper improved a lot. In particular, the comment on $\tilde{I}_\alpha$ should have been added before the paper was submitted for the first time; We took the reason of using $\tilde{I}_\alpha$ for totally granted.

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