Olsen’s inequality and its applications to Schrödinger equations

By

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Abstract

Morrey spaces turned out to be useful in that it grasps the subtle property of the fractional integral operators.

In 1995 Olsen obtained a bilinear estimate. Olsen applied his estimate called the Olsen inequality to the Schrödiger equation. We improve this estimate.

This Olsen inequality is a part of trace inequality that has kin relation with potential theory. Here we present some applications of our results to PDEs and the potential theory.

§ 1. Introduction

The aim of this note is to establish the following inequality.

\[ \| g \cdot I_\alpha f \|_{\mathcal{M}_{p_0}^{p_0}} \leq C \| g \|_{\mathcal{M}_{p_0}^{p_0}} \| f \|_{\mathcal{M}_p^p}, \]

where \( \| \cdot \|_{\mathcal{M}_p^p} \) denotes the Morrey (quasi-)norm given by

\[ \| f \|_{\mathcal{M}_p^p} = \sup_{Q \in \mathcal{D}} |Q|^{1/p - 1/p_0} \left( \int_Q |f(y)|^p \, dy \right)^{1/p}, \quad 0 < p \leq p_0 < \infty \]

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and \( I_\alpha \) denotes the fractional integral operator defined by
\[
I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy, \quad 0 < \alpha < n.
\]
Here \( \mathcal{D} \) denotes the set of all dyadic cubes in \( \mathbb{R}^n \). It is well known that \( I_2 \) is the inverse of \(-\Delta\) modulo multiplicative constants. In this note we shall prove the following theorem on the fractional integral operator \( I_\alpha \).

Here is a series of equivalent definitions of Morrey norms.
\[
\|f\|^{(1)}_{\mathcal{M}_q^p} = \sup_{Q \in \mathcal{Q}} |Q|^{1/p_0 - 1/p} \left( \int_Q |f(y)|^p \, dy \right)^{1/p},
\]
\[
\|f\|^{(2)}_{\mathcal{M}_q^p} = \sup_{Q \in \mathcal{Q}^*} |Q|^{1/p_0 - 1/p} \left( \int_Q |f(y)|^p \, dy \right)^{1/p},
\]
where \( \mathcal{Q} \) denotes the set of all cubes whose edges are parallel to the coordinate axis and \( \mathcal{Q}^* \) denotes the set of all cubes whose edges are not always parallel to the coordinate axis. In the present paper we identify \( \|f\|^{(1)}_{\mathcal{M}_q^p} \) with \( \|f\|_{\mathcal{M}_q^p} \) but we do not use \( \|f\|^{(2)}_{\mathcal{M}_q^p} \).

If we formally let \( p_0 = \infty \), then we obtain the \( L^\infty \) space.

**Theorem 1.1.** Let \( 0 < \alpha < n \), \( 1 < p \leq p_0 < \infty \), \( 1 < q \leq q_0 < \infty \) and \( 1 < r \leq r_0 < \infty \). Suppose that
\[
q > r, \quad \frac{1}{p_0} > \frac{\alpha}{n}, \quad \frac{1}{q_0} \leq \frac{\alpha}{n},
\]
and that
\[
\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n}, \quad \frac{r}{r_0} = \frac{p}{p_0}.
\]
Then
\[
\|g \cdot I_\alpha f\|_{\mathcal{M}_r^{r_0}} \leq C \|g\|_{\mathcal{M}_q^{q_0}} \cdot \|f\|_{\mathcal{M}_p^{p_0}},
\]
where the constant \( C \) is independent of \( f \) and \( g \).

**Remark.** Hölder’s inequality yields
\[
L^{p_0} = \mathcal{M}_{p_0}^{p_0} \hookrightarrow \mathcal{M}_{p_1}^{p_0} \hookrightarrow \mathcal{M}_{p_2}^{p_0}
\]
for all \( p_0 \geq p_1 \geq p_2 \geq 1 \).

Here is a precise result by Olsen.

**Theorem 1.2 ([10, Theorem 2]).** Let \( 0 < \alpha < n \), \( 1 < p \leq p_0 < \infty \), \( 1 < q \leq q_0 < \infty \) and \( 1 < r \leq r_0 < \infty \). Suppose that
\[
q > r, \quad \frac{1}{p_0} > \frac{\alpha}{n}, \quad \frac{1}{q_0} \leq \frac{\alpha}{n},
\]
and that
\begin{equation}
\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n}, \quad \frac{1}{r} = \frac{1}{q_0} + \frac{1}{p} - \frac{\alpha}{n}.
\end{equation}

Then
\[ \|g \cdot I_\alpha f\|_{\mathcal{M}_{r_0}^r} \leq C\|g\|_{\mathcal{M}_{q_0}^q} \cdot \|f\|_{\mathcal{M}_{p_0}^p}, \]
where the constant $C$ is independent of $f$ and $g$.

However, this result is not sharp as the following calculation shows.

\textbf{Remark.} Using naively the Adams theorem [1] and Hölder’s inequality, one can prove a minor part of $q$ in Theorem 1.1. That is, the proof of Theorem 1.1 is fundamental provided $\frac{p}{p_0} q_0 \leq q \leq q_0$. Indeed, by virtue of the Adams theorem we have, for any cube $Q \in \mathcal{Q}$,
\begin{equation}
|Q|^{1/s_0} \left( \frac{1}{|Q|} \int_Q |I_\alpha f(x)|^s dx \right)^{1/s} \leq C\|f\|_{\mathcal{M}_{p_0}^p}, \quad \frac{1}{s} = \frac{p_0}{p} \frac{1}{s_0}, \quad \frac{1}{s_0} = \frac{1}{p_0} - \frac{\alpha}{n}.
\end{equation}

The condition \(\frac{r}{r_0} = \frac{p}{p_0} \), \(\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n}\) reads
\[ \frac{1}{r} = \frac{p_0}{p} \left( \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n} \right) = \frac{p_0}{p} \frac{1}{q_0} + \frac{1}{s}. \]

These yield
\[ |Q|^{1/q_0+1/s_0} \left( \frac{1}{|Q|} \int_Q |g(x)I_\alpha f(x)|^r dx \right)^{1/r} \leq C\|g\|_{\mathcal{M}_{q_0}^q} \|f\|_{\mathcal{M}_{p_0}^p} \]
if $\frac{r}{r_0} = \frac{p}{p_0} = \frac{q}{q_0}$. In view of the inclusion (1.6), the same can be said when
\[ \frac{p}{p_0} q_0 \leq q \leq q_0. \]

Also observe that \(\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n} > \frac{1}{q_0}\). Hence we have $q_0 > r_0$. Thus, since the condition $q > r$, Theorem 1.1 is significant only when $\frac{p}{p_0} r_0 < q < \frac{p}{p_0} q_0$.

We generalized Theorem 1.1 in our subsequent papers [12, 13]. The motivation stemmed from the earlier works due to Sugano and Tanaka [17, 18]. To prove them we used some auxiliary results of maximal operators, which strengthen those by Nakai [8, 9] (see Lemmas 8.1 and 8.2).
As we have established in [14], the fractional maximal operator $I_{\alpha}$ is not surjective from $\mathcal{M}^p_q$ to $\mathcal{M}^t_s$ if

\begin{equation}
1 < q \leq p < \infty, 1 < t \leq s < \infty, \quad \frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{q}{p} = \frac{t}{s}.
\end{equation}

Therefore, one can expect such a strange phenomenon like Theorem 1.1.

\section{Fundamentals on Morrey spaces}

Here we shall present several examples of the functions in Morrey spaces. Here and below given a cube $Q$ and $\kappa > 0$, we denote by $\kappa Q$ the cube concentric to $Q$ and having volume $\kappa^n$.

Example Let $1 < q < q_0 < \infty$. Then $|x|^{-n/q_0} \in \mathcal{M}^{q_0}_q$.

The following proposition is due to Pérez [11]. It seems that we can construct counterexamples by using the following proposition.

\textbf{Proposition 2.1.} Let $p_0 > p > 0$ and $R > 1$ be fixed so that

\begin{equation}
(R + 1)^{-1/p_0} = 2^{1/p}(1 + R)^{-1/p}.
\end{equation}

For $\epsilon = \{\epsilon_a\}_{a=1}^n \in \{0, 1\}^n$, we define an Affine transformation $T_\epsilon$ by

\begin{equation}
T_\epsilon(x) = \frac{1}{R+1}x + \frac{R}{R+1}\epsilon \quad (x \in \mathbb{R}^n).
\end{equation}

Let $E_0 = [0, 1]^n$. Suppose that we have defined $E_0, E_1, E_2, \cdots, E_j$. Define

\begin{equation}
E_{j+1} = \bigcup_{\epsilon \in \{0,1\}^n} T_\epsilon(E_j).
\end{equation}

Then we have

\begin{equation}
\|\chi_{E_j}\|_{\mathcal{M}^{p_0}_p} \sim (1 + R)^{-jn/p_0},
\end{equation}

where the implicit constants in $\sim$ does not depend on $j$ but can depend on $p, p_0$.

Before the proof we make a preparatory observation. Note that

\begin{equation}
\|E_{j,0}\|_{L^{p_0}} = \|E_j\|_{L^p}(1 + R)^{-jn/p_0},
\end{equation}

where $E_{j,0} = [0, (1 + R)^{-j}]^n$.

\textbf{Proof.} Let us calculate

\begin{equation}
\|\chi_{E_j}\|_{\mathcal{M}^{p_0}_p} \sim \sup_{S \in \mathcal{Q}} |S|^{1/p_0 - 1/p}|S \cap E_j|^{1/p},
\end{equation}

where $\mathcal{Q}$ is the family of cubes.
where \( \mathcal{Q} \) denotes the set of all cubes.

Let us temporally say that \( Q \in \mathcal{Q} \) is wasteful, if there exists a cube \( S \in \mathcal{Q} \) such that

\[
|Q^{1/p_0 - 1/p} Q \cap E_j|^{1/p} < |S^{1/p_0 - 1/p} S \cap E_j|^{1/p}.
\]

Thus, by definition, any cube is clearly wasteful unless it is contained in \([0, 1]^n\). Also, if the side-length of a cube \( Q \) is less than \( (R + 1)^{-j} \), then it is wasteful. Indeed, then the equality

\[
\sup_{Q: \text{cube} \in \mathcal{Q}} |Q|^{1/p_0 - 1/p} |Q \cap E_j|^{1/p} = \sup_{Q: \text{cube} \subset E_j} |Q|^{1/p_0 - 1/p} |Q \cap E_j|^{1/p} = \sup_{Q: \text{cube} \subset E_j} |Q|^{1/p_0} = |E_{j,0}|^{1/p_0}
\]

holds. Thus, if the cube \( Q \) is not wasteful and \( (R + 1)^{-kn} \leq |Q| \leq (R + 1)^{-(k-1)n} \), then \( \frac{R + 1}{R - 1} Q \) contains a connected component of \( E_k \). Hence it follows that

\[
\| \chi_{E_j} \|_{\mathcal{M}^{p_0}_p} \sim \sup \{ \frac{|S \cap E_j|^{1/p}}{|S|^{1/p - 1/p_0}} : S \text{ contains a connected component of } E_k \}.
\]

Let us write

\[
\alpha = \sup \left\{ \frac{|S \cap E_j|^{1/p}}{|S|^{1/p_0 - 1/p_0}} : S \text{ contains a connected component of } E_k \right\}.
\]

Let \( S \) be a cube which contains a connected component of \( E_j \). We define

\[
S^* = \text{co} \left( \bigcup \{ W : W \text{ is a connected component of } E_j \text{ intersecting } S \} \right),
\]

where \( \text{co}(A) \) denotes the smallest convex set containing a set \( A \). Then a geometric observation shows that \( S^* \) engulfs \( k^n \) connected component of \( E_j \) for some \( 1 \leq k \leq 2^j \).

Take an integer \( l \) so that \( 2^{l-1} \leq k \leq 2^l \). Then we have

\[
|S^* \cap E_j| = k^n (1 + R)^{-jn}, \ |S^*| = (1 + R)^{-jn + ln}.
\]

Consequently, we obtain

\[
|S^*|^{1/p_0 - 1/p} |S^* \cap E_j|^{1/p} \sim 2^{ln/p} (1 + R)^{-jn/p} (1 + R)^{(-j+l)(n/p_0 - n/p)} \sim 2^{ln/p} (1 + R)^{-jn/p_0 + l(n/p_0 - n/p)} \sim (1 + R)^{-jn/p_0}.
\]

Thus, we obtain (2.4). \( \square \)

We remark that this example is essentially employed in our paper [12] to prove the sharpness of our result.
Proposition 2.2. Let $0 < \alpha < n$, $1 < q \leq p < \infty$ and $1 < t \leq s < \infty$. If there exists a constant $C > 0$

\begin{equation}
\|I_\alpha f\|_{\mathcal{M}_t^s} \leq C\|f\|_{\mathcal{M}_q^p}
\end{equation}

for all positive measurable functions $f$, it is necessary and sufficient that

\begin{equation}
\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{t}{s} \leq \frac{q}{p}.
\end{equation}

Proof. The sufficiency is well known as Adam’s theorem and we will use in this lecture. For the necessity, we first obtain

\begin{equation}
\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}
\end{equation}

by the dilation

\begin{equation}
\|f(\lambda \cdot)\|_{\mathcal{M}_q^p} = \lambda^{-\frac{\alpha}{n}}\|f\|_{\mathcal{M}_q^p}, \quad \|f(\lambda \cdot)\|_{\mathcal{M}_t^s} = \lambda^{-\frac{\alpha}{n}}\|f\|_{\mathcal{M}_t^s}, \quad I_\alpha[f(\lambda \cdot)] = \lambda^{-\alpha}I_\alpha f(\lambda \cdot)
\end{equation}

for $\lambda > 0$. Consequently, we obtain

\begin{equation}
\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}.
\end{equation}

We may assume that $q < p$ for the purpose of establishing $\frac{t}{s} = \frac{q}{p}$. Let $R > 1$ be the solution of $(1 + R)^{-\frac{1}{p}} = 2^{\frac{1}{q}}(1 + R)^{-\frac{1}{q}}$. Then we have

\begin{equation}
\|\chi_{E_j}\|_{\mathcal{M}_q^p} \sim (1 + R)^{-jn/p}.
\end{equation}

If we use $E_j$ above, then we have

\begin{equation}
I_\alpha \chi_{E_j}(x) \geq c(R + 1)^{-j\alpha} \chi_{E_j}(x).
\end{equation}

Indeed, as the equality

\begin{equation}
I_\alpha \chi_{E_j}(x) = \int_{x - E_j} \frac{dy}{|y|^{n-\alpha}}, \quad x - E_j := \{x - y : y \in E_j\}
\end{equation}

shows, $I_\alpha \chi_{E_j}$ is continuous. So we can take $c = \min_{x \in [0, 1]^n} I_\alpha \chi_{E_j}(x)$.

Also, from the definition of $\mathcal{M}_q^p$, we have

\begin{equation}
(R + 1)^{-\frac{jmq}{pt}} = \|\chi_{E_j}\|_{L^t} \leq \|\chi_{E_j}\|_{\mathcal{M}_q^p}
\end{equation}

from the definition of the Morrey norm. Consequently, it follows that

\begin{equation}
(R + 1)^{-\frac{jmq}{pt}} \leq C(R + 1)^{j\alpha}\|\chi_{E_j}\|_{\mathcal{M}_q^p} = C(R + 1)^{j(\alpha - \frac{n}{p})} = C(R + 1)^{-\frac{jm}{p}}.
\end{equation}
Since \( j \in \mathbb{N} \) is arbitrary, it follows that \( t \leq \frac{qs}{p} \) \( \square \).

### § 3. Boundedness of the operators on Morrey spaces

Now we present a typical argument proving the Morrey-boundedness. We accept

\[
\|Mf\|_q \leq C \|f\|_q, \quad 1 < q \leq \infty,
\]

where \( M \) denotes the Hardy-Littlewood maximal operator given by

\[
Mf(x) = \sup_{x \in Q, cube} \frac{1}{|Q|} \int_Q |f(y)| \, dy.
\]

Here and below given a cube \( Q \), we define

\[
\ell(Q) = |Q|^{1/n}.
\]

**Theorem 3.1.** Let \( 1 < q \leq p < \infty \). Then there exists \( C_{p,q} \) such that

\[
\|Mf\|_{\mathcal{M}^p_q} \leq C_{p,q} \|f\|_{\mathcal{M}^p_q}
\]

for all \( f \in \mathcal{M}^p_q \).

**Proof.** For the proof we shall show, from the definition, that

\[
|Q|^{\frac{1}{p} - \frac{1}{q}} \left( \int_Q Mf(y)^q \, dy \right)^{\frac{1}{q}} \leq C_{p,q} \|f\|_{\mathcal{M}^p_q}, Q \in \mathcal{Q}.
\]

Once this is achieve, we have only to consider the supremum over all \( Q \).

Write \( f = f_1 + f_2 \), where \( f_1 = f \) on \( 5Q \) and \( f_2 = f \) outside \( 5Q \). The estimate of (3.5) can be decomposed into

\[
|Q|^{\frac{1}{p} - \frac{1}{q}} \left( \int_Q Mf_1(y)^q \, dy \right)^{\frac{1}{q}} \leq C_{p,q} \|f\|_{\mathcal{M}^p_q},
\]

\[
|Q|^{\frac{1}{p} - \frac{1}{q}} \left( \int_Q Mf_2(y)^q \, dy \right)^{\frac{1}{q}} \leq C_{p,q} \|f\|_{\mathcal{M}^p_q}
\]

by virtue of the triangle inequality.

As we have seen in (3.1), \( M \) is \( L^q \)-bounded. Thus (3.6) can be shown easily;

\[
|Q|^{\frac{1}{p} - \frac{1}{q}} \left( \int_Q Mf_1(y)^q \, dy \right)^{\frac{1}{q}} \leq C_{p,q} \|f\|_{\mathcal{M}^p_q},
\]

\[
\leq C_{p,q} |Q|^{\frac{1}{p} - \frac{1}{q}} \left( \int_{\mathbb{R}^n} Mf_1(y)^q \, dy \right)^{\frac{1}{q}}
\]

\[
\leq C_{p,q} |5Q|^{\frac{1}{p} - \frac{1}{q}} \left( \int_{5Q} |f(y)|^q \, dy \right)^{\frac{1}{q}}
\]

\[
\leq C_{p,q} \|f\|_{\mathcal{M}^p_q}.
\]
Thus, we obtain (3.6).

To prove (3.7) we have to keep in mind the following fundamental geometric observation.

If $R \in Q$ is a cube that meets both $Q$ and $\mathbb{R}^n \setminus 5Q$, then $\ell(R) \geq 2\ell(Q)$ and $2R \supset Q$.

This geometric observation yields

$$
M f_2(y) \leq 2^n \sup_{Q \subset R \in Q} \frac{1}{|R|} \int_R |f(y)| \, dy. \tag{3.8}
$$

If we insert this inequality to (3.7), then we obtain

$$
|Q|^\frac{1}{p} \left( \int_Q M f_2(y)^q \, dy \right)^{\frac{1}{q}} \leq 2^n |Q|^\frac{1}{p} \sup_{Q \subset R \in \mathcal{Q}} \frac{1}{|R|} \int_R |f(y)| \, dy.
$$

Taking into account $\mathcal{M}_q^p \subset \mathcal{M}_1^p$, we see that

$$
|Q|^\frac{1}{p} \left( \int_Q M f_2(y)^q \, dy \right)^{\frac{1}{q}} \leq 2^n \sup_{R \in \mathcal{Q}} |R|^\frac{1}{p-1} \int_R |f(y)| \, dy = 2^n \|f\|_{\mathcal{M}_1^p} \leq 2^n \|f\|_{\mathcal{M}_q^p}. \tag{3.9}
$$

Estimate (3.7) is therefore proved. With (3.6) and (3.7) proved, we obtain (3.4). \hfill \square

We now turn to the boundedness of the fractional integral operators. The following lemma is named the comparison lemma by Volberg.

**Lemma 3.2** (Comparison lemma). Let $0 < \alpha < n$.

$$
\int_0^\infty \chi_{B(x,l)}(y) \frac{dl}{l^{n+1-\alpha}} = \frac{|x-y|^{-n+\alpha}}{n-\alpha}. \tag{3.10}
$$

**Proof.** The proof is simple. Indeed, we have

$$
\int_0^\infty \chi_{B(x,l)}(y) \frac{dl}{l^{n+1-\alpha}} = \int_0^\infty \chi_{(|x-y|,\infty)}(l) \frac{dl}{l^{n+1-\alpha}} = \int_{|x-y|}^\infty \frac{dl}{l^{n+1-\alpha}} = \frac{|x-y|^{-n+\alpha}}{n-\alpha}, \tag{3.13}
$$

which is the desired result. \hfill \square

We use a notation; if we are given a cube $Q$ and a locally integrable function $f$, then we write

$$
m_Q(f) = \frac{1}{|Q|} \int_Q f(x) \, dx,
$$
the average of \( f \) over a cube \( Q \).

Morrey spaces, the BMO space and Hölder spaces lie in a line. More precisely we have the following.

**Theorem 3.3** (\( I_\alpha \) and the modified fractional maximal operators \( \tilde{I}_\alpha \)). *Suppose that* \( 0 < \alpha < n \). *We define*

\[
I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy
\]

\[
\tilde{I}_\alpha f(x) := \int_{\mathbb{R}^n} \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{\chi_{Q_0}}{|x_0-y|^{n-\alpha}} \right) f(y) dy,
\]

*where* \( Q_0 \) *is a fixed cube centered at* \( x_0 \).

1. (Subcritical case) *Let* \( p < \frac{n}{\alpha} \). *Assume that the parameters* \( s, t \) *satisfy*

\[
1 < q \leq p < \infty, 1 < t \leq s < \infty, \frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \frac{t}{s} = \frac{q}{p}.
\]

*Then*

\[
\|I_\alpha f\|_{\mathcal{M}_t^s} \leq C_{p,q,\alpha}\|f\|_{\mathcal{M}_q^p}
\]

*for every* (positive) \( f \in \mathcal{M}_q^p \).

2. (Critical case) *Assume that* \( 1 \leq q \leq p = \frac{n}{\alpha} \). *Then*

\[
\|\tilde{I}_\alpha f\|_* \leq C_{p,q,\alpha}\|f\|_{\mathcal{M}_q^p}
\]

*for every* \( f \in \mathcal{M}_q^p \), *where* \( \| \cdot \|_* \) *denotes the BMO norm given by*

\[
\|g\|_* = \sup_{Q \in \mathcal{Q}} m_Q(|g - m_Q(g)|).
\]

3. (Supercritical case) *Assume that* \( 1 \leq q \leq p < \infty \) *and that* \( 0 < \alpha - \frac{n}{p} < 1 \). *Then*

\[
\|\tilde{I}_\alpha f\|_{\text{Lip}(\alpha - \frac{n}{p})} \leq C_{p,q,\alpha}\|f\|_{\mathcal{M}_q^p}
\]

*for every* \( f \in \mathcal{M}_q^p \), *where we denoted*

\[
\|g\|_{\text{Lip}(\theta)} = \sup \left\{ \frac{|g(x) - g(y)|}{|x-y|^\theta} : x, y \in \mathbb{R}^n, x \neq y \right\}, \quad 0 < \theta < 1
\]

*for a function* \( g \).
Before we come to the proof of this theorem, let us remark that the integral kernel of $\tilde{I}_\alpha$ is better than that of $I_\alpha$. With the better kernel, we can apply $\tilde{I}_\alpha$ for $\mathcal{M}_q^{n/\alpha}$ functions.

Here we content ourselves with proving (2), other results being proved similarly.

**Proof.** We have to prove

\begin{equation}
(3.22) \quad m_Q(|\tilde{I}_\alpha f - m_Q(\tilde{I}_\alpha f)|) \leq c \|f\|_{\mathcal{M}_q^{n/\alpha}}
\end{equation}

for all cubes $Q$. For the proof we may assume $q < \frac{n}{\alpha}$ because we always have

\begin{equation}
(3.23) \quad \|f\|_{\mathcal{M}_q^{n/\alpha}} \leq \|f\|_{\mathcal{M}_n^{n/\alpha}} = \|f\|_{L_n^{n/\alpha}}.
\end{equation}

We decompose $f$ according to $2Q$ as usual. That is, we split $f = f_1 + f_2$ with $f_1 = \chi_{2Q} \cdot f$ and $f_2 = f - f_1$. By virtue of the triangle inequality our present task is partitioned into proving

\begin{equation}
(3.24) \quad m_Q(|\tilde{I}_\alpha f_1 - m_Q(\tilde{I}_\alpha f_1)|) + m_Q(|\tilde{I}_\alpha f_2 - m_Q(\tilde{I}_\alpha f_2)|) \leq c \|f\|_{\mathcal{M}_q^{n/\alpha}}.
\end{equation}

Then the estimate for $f_1$ is simple. Indeed, to estimate $f_1$, we define an auxiliary index $w \in (q, \infty)$ by $\frac{1}{w} = \frac{1}{q} - \frac{\alpha}{n}$. By the triangle inequality and the Hölder inequality, we have

\begin{equation}
(3.25) \quad m_Q(|\tilde{I}_\alpha f_1 - m_Q(\tilde{I}_\alpha f_1)|) \leq 2m_Q(|\tilde{I}_\alpha f_1|) \leq 2m_Q^{(w)}(|\tilde{I}_\alpha f_1|),
\end{equation}

where we wrote

\begin{equation}
(3.26) \quad m_Q^{(w)}(F) := \left( \frac{1}{|Q|} \int_Q F(x)^w \, dx \right)^{1/w}
\end{equation}

for positive measurable functions $F$. By using the $L^q$-$L^w$ boundedness of the fractional integral operator we obtain

\begin{align*}
m_Q(|\tilde{I}_\alpha f_1 - m_Q(\tilde{I}_\alpha f_1)|) &\leq c \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} |\tilde{I}_\alpha f_1(x)|^w \, dx \right)^{1/w} \\
&\leq c m_{2Q}(|f|^q)^{1/w} \\
&\leq c \|f\|_{\mathcal{M}_q^{n/\alpha}}.
\end{align*}

For the proof of the second inequality, we write the left-side out in full.

\begin{align*}
m_Q(|\tilde{I}_\alpha f_2 - m_Q(\tilde{I}_\alpha f_2)|) \\
= \frac{1}{|Q|^2} \int_Q \left| \int_{\mathbb{R}^n \setminus 2Q} \left( \frac{f(z)}{|x-z|^{n-\alpha}} - \frac{f(z)}{|y-z|^{n-\alpha}} \right) dy \, dz \right| \, dx.
\end{align*}
First, we bound the right-hand side with the triangle inequality and arrange it. Then
the right-hand side is majorized by

\[
\frac{1}{|Q|^2} \iiint_{Q \times Q \times (\mathbb{R}^n \setminus 2Q)} |f(z)| \cdot \left| \frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|y-z|^{n-\alpha}} \right| \, dx \, dy \, dz.
\]

Denote by \( c(Q) \) the center of the cube \( Q \). By virtue of the mean value theorem, we have

\[
\left| \frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|y-z|^{n-\alpha}} \right| \leq c \frac{|x-y|}{|z-c(Q)|^{n-\alpha+1}} \leq c \frac{\ell(Q)}{|z-c(Q)|^{n-\alpha+1}}.
\]

Thus, inserting this inequality gives us

\[
m_Q(\|I_\alpha f_2 - m_Q(I_\alpha f_2)\|) \leq c \ell(Q) \int_{\mathbb{R}^n \setminus 2Q} \frac{|f(z)|}{|z-c(Q)|^{n-\alpha+1}} \, dz.
\]

By the comparison lemma, the integral of the right-hand side is bounded by

\[
\int_{2\ell(Q)}^\infty \left( \int_{B(c(Q), \ell)} |f(z)| \, dz \right) \frac{d\ell}{\ell^{n-\alpha+2}} \leq c \|f\|_{\mathcal{M}^{n/\alpha}_1} \cdot \int_{\ell(Q)}^\infty \frac{d\ell}{\ell^2} = c \frac{\|f\|_{\mathcal{M}^{n/\alpha}_q}}{\ell(Q)}.
\]

Thus, the estimate of the second inequality is complete and the proof is concluded. \( \square \)

§ 4. Proof of Theorem 1.1

In [10] the proof was rather lengthy. But our proof is a little shorter than that by Olsen.

We need the following observation below. This lemma dates back to [18] for example.

**Lemma 4.1.** For a nonnegative function \( h \) in \( L^\infty(Q_0) \) we let \( \gamma_0 = m_{Q_0}(h) \) and \( c = 2^{n+1} \). For \( k = 1, 2, \ldots \) let

\[
D_k := \bigcup \left\{ Q : Q \in \mathcal{D}_1(Q_0) : m_Q(h) > \gamma_0 c^k \right\} \subset \mathbb{R}^n.
\]

(0) Define

\[
\mathcal{Z}_k = \left\{ Q : Q \in \mathcal{D}_1(Q_0) : m_Q(h) > \gamma_0 c^k \right\}.
\]

Considering the maximal cubes in \( \mathcal{Z}_k \) with respect to inclusion, we can write

\[
D_k = \bigcup_j Q_{k,j},
\]

where the cubes \( \{Q_{k,j}\} \subset \mathcal{D}_1(Q_0) \) are nonoverlapping.
(1) By virtue of the maximality of $Q_{k,j}$ one has that
\[ \gamma_0 c^k < m_{Q_{k,j}}(h) \leq 2^n \gamma_0 c^k. \]

(2) Let
\[ E_0 = Q_0 \setminus D_1 \quad \text{and} \quad E_{k,j} = Q_{k,j} \setminus D_{k+1}. \]

Then \( E_0 \cup \{ E_{k,j} \} \) is a disjoint family of sets which decomposes $Q_0$ and satisfies
\[
|Q_0| \leq 2|E_0| \quad \text{and} \quad |Q_{k,j}| \leq 2|E_{k,j}|.
\]

(3) Also, we set
\[
\mathcal{D}_0 := \{ Q \in \mathcal{D}_1(Q_0) : m_{Q}(h) \leq \gamma_0 c \} \subset \mathcal{D} \\
\mathcal{D}_{k,j} := \{ Q \in \mathcal{D}_1(Q_0) : Q \subset Q_{k,j}, \gamma_0 c^k < m_{Q}(h) \leq \gamma_0 c^{k+1} \} \subset \mathcal{D}.
\]

Then $\mathcal{D}_1(Q_0)$ is partitioned as follows:
\[
\mathcal{D}_1(Q_0) = \mathcal{D}_0 \cup \bigcup_{k,j} \mathcal{D}_{k,j}.
\]

**Proof.** We choose $Q_{k,j}$ as is indicated in (0).

(1) The left inequality is a consequence of the fact that we chose $Q_{k,j}$ from $\mathcal{Z}_k$. If we consider the dyadic parent $R$ of $Q_{k,j}$, then we have
\[
m_R(h) \leq \gamma_0 c^k.
\]

Otherwise, instead of $Q_{k,j}$ $R$ would have been chosen as an element of $\mathcal{Z}_k$. Since
\[
m_{Q_{j,k}}(h) \leq 2^n m_R(h),
\]
we obtain the right inequality.

(2) A geometric observation shows that two dyadic cube never intersect unless one is not included in the other. Let $j$ and $k$ be freezed. We write
\[
D_{k+1} = \bigcup_{j^*} Q_{k+1,j^*}, \quad Q_{k+1,j^*} \in \mathcal{Z}_{k+1}.
\]

From the observation above, we have
\[
D_{k+1} \cap Q_{k,j} = \bigcup \{ Q_{k+1,j^*} : Q_{k+1,j^*} \subset Q_{k,j} \}
\]
because $Q_{k+1,j^*} \supseteq Q_{k,j}$ never happens thanks to the maximality of $Q_{k,j}$. Note that

\[ |Q_{k+1,j^*}| \gamma_0 c^{k+1} \leq \int_{Q_{k+1,j^*}} h(x) \, dx \]

since $Q_{k+1,j^*} \in \mathcal{Z}_{k+1}$. If we add this estimate for all $j^*$ such that $Q_{k+1,j^*} \subset Q_{k,j}$, we obtain

\[ |D_{k+1} \cap Q_{k,j}| \gamma_0 c^{k+1} \leq \int_{Q_{k,j}} h(x) \, dx \leq |Q_{k,j}| 2^n \gamma_0. \]

So, if we consider the complement set of $D_{k+1}$, then we obtain the desired result.

(3) By maximality if $R \subset Q_0$ is a dyadic cube such that $m_R(h) > \gamma_0$, then $R$ is contained in some $Q_{k,j}$. So this assertion follows.

\[ \square \]

We begin by discretizing the operator $I_\alpha f$ following the idea of C. Pérez (see [11]):

\[ I_\alpha f(x) = \sum_{\nu \in \mathbb{Z}} \int_{2^{-\nu-1} < |x-y| \leq 2^{-\nu}} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy \leq C \sum_{\nu \in \mathbb{Z}} 2^{\nu(n-\alpha)} \int_{B(x,2^{-\nu})} f(y) \, dy. \]

Denote by $\mathcal{D}_\nu$ the set of all dyadic cubes of volume $2^{-\nu n}$:

\[ \mathcal{D}_\nu = \{2^{-\nu}m + 2^{-\nu}[0, 1)^n : m \in \mathbb{Z}^n \}. \]

Then we have

\[ I_\alpha f(x) \leq C \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu} \frac{\ell(Q)^\alpha}{|Q|} \int_{3Q} f(y) \, dy \]

\[ = C \sum_{Q \in \mathcal{D}} \frac{\ell(Q)^\alpha}{|Q|} \int_{3Q} f(y) \, dy \cdot \chi_Q(x) \]

\[ = C \sum_{Q \in \mathcal{D}} \ell(Q)^\alpha m_{3Q}(f) \cdot \chi_Q(x). \]

It suffices, from the definition of the Morrey norm, to show that

\[ \left( \int_{Q_0} (g(x)I_\alpha f(x))^r \, dx \right)^{1/r} \leq C \|g\|_{\mathcal{M}_{r_0}} \cdot \|M_\alpha f\|_{\mathcal{M}_{r_0}} \cdot |Q_0|^{1/r - 1/r_0}, \]

for a given dyadic cube $Q_0$, where $M_\alpha$ denotes the fractional maximal operator given by

\[ M_\alpha f(x) = \sup_{x \in Q \in \mathcal{Q}} |Q|^{\alpha/n} \left( \frac{1}{|Q|} \int_Q |f(y)| \, dy \right). \]
Indeed, a geometric observation shows

\[(4.12) \quad |Q|^{\alpha/n-1}\chi_Q(y) \leq c|x-y|^{-n+\alpha}, \quad x \in Q.\]

So the pointwise estimate $M_{\alpha}f(x) \leq cI_{\alpha}f(x)$ follows. If we invoke the Adams theorem, then we obtain the desired result. Furthermore, by a simple limiting argument, we can assume that $g \in L^\infty$. Hereafter, we write

\[
\begin{cases}
\mathcal{D}_1(Q_0) = \{ Q \in \mathcal{D} : Q \subset Q_0 \}, \\
\mathcal{D}_2(Q_0) = \{ Q \in \mathcal{D} : Q \supset Q_0 \}.
\end{cases}
\]

Let us define for $i = 1, 2$

\[F_i(x) = \sum_{Q \in \mathcal{D}_i(Q_0)} \ell(Q)^{\alpha}m_{3Q}(f)\chi_Q(x)\]

and we shall estimate

\[
\left( \int_{Q_0} (g(x)F_i(x))^r \right)^{1/r}.
\]

First, we establish

\[(4.13) \quad \left( \int_{Q_0} (g(x)F_1(x))^r \right)^{1/r} \leq C\|g\|_{\mathcal{M}_{\alpha-n/q_0}} \left( \int_{Q_0} M_{\alpha-n/q_0}f(x)^r dx \right)^{1/r},\]

by the duality argument. Take a nonnegative function $w \in L^{r'}$, $1/r + 1/r' = 1$, satisfying that $w$ is supported on $\overline{Q_0}$, that $\|w\|_{L^{r'}(Q_0)} = 1$ and that

\[
\left( \int_{Q_0} (g(x)F_1(x))^r \right)^{1/r} = \int_{Q_0} g(x)F_1(x)w(x) dx.
\]

Letting $h = g \cdot w$, we shall apply Lemma 4.1 to estimate this quantity. It follows that

\[
\int_{Q_0} g(x)F_1(x)w(x) dx = \sum_{Q \in \mathcal{D}_1(Q_0)} \ell(Q)^{\alpha}m_{3Q}(f) \int_{Q} g(x)w(x) dx
\]

\[
= \sum_{Q \in \mathcal{D}_0} \ell(Q)^{\alpha}m_{3Q}(f) \int_{Q} g(x)w(x) dx
\]

\[
+ \sum_{j=1}^{\infty} \sum_{k} \sum_{Q \in \mathcal{D}_{j,k}} \ell(Q)^{\alpha}m_{3Q}(f) \int_{Q} g(x)w(x) dx.
\]

First, we evaluate

\[(4.14) \quad \sum_{Q \in \mathcal{D}_{k,j}} \ell(Q)^{\alpha}m_{3Q}(f) \int_{Q} g(x)w(x) dx.
\]
In view of the definition of $D_{k,j}$, we have
\[
\text{R.H.S. of (4.14)} = \sum_{l=0}^{\infty} \sum_{Q \in D_{k,j}} \ell(Q) \alpha m_{3Q}(f) \int_Q g(x)w(x) \, dx
\]
\[
= 3^n \sum_{l=0}^{\infty} \sum_{Q \in D_{k,j}} \ell(Q) \alpha m_Q(f) \int_Q f(x) \, dx
\]

Let $Q \in D_{k,j}$. Now that we have
\[
\gamma_0 c^k \leq m_Q(gw) \leq \gamma_0 c^{k+1}, \quad \gamma_0 c^k \leq m_{Q_{k,j}}(gw) \leq 2^n \gamma_0 c^k, \quad Q \subset Q_{k,j},
\]
we obtain $m_Q(gw) \sim m_{Q_{k,j}}(gw)$. Hence it follows that
\[
\text{R.H.S. of (4.14)} \leq C_{p,q,\alpha} m_{Q_{k,j}}(gw) \sum_{l=0}^{\infty} \sum_{Q \in D_{k,j}} \ell(Q) \alpha \int_{3Q} f(x) \, dx
\]
\[
\leq C_{p,q,\alpha} \sum_{l=0}^{\infty} 2^{-l} \ell(Q_{k,j}) \alpha \int_{3Q_{k,j}} f(x) \, dx
\]
\[
\leq C_{p,q,\alpha} \ell(Q_{k,j}) \alpha m_{Q_{k,j}}(gw) \int_{3Q_{k,j}} f(x) \, dx.
\]

In summary, we have obtained
\[
\sum_{Q \in D_{k,j}} \ell(Q) \alpha m_{3Q}(f) \int_Q g(x)w(x) \, dx \leq C_{p,q,\alpha} \ell(Q_{k,j}) \alpha m_{3Q_{k,j}}(f) m_{Q_{k,j}}(gw) |E_{k,j}|.
\]

Using Hölder’s inequality, we have
\[
m_{Q_{k,j}}(gw) \leq m_{Q_{k,j}}^{(q)}(g)m_{Q_{k,j}}^{(q')}(w),
\]
and
\[
m_{Q_{k,j}}^{(q)}(g) \leq \|g\|_{\mathcal{M}_q^{q_0}} \ell(Q_{k,j})^{-\frac{n}{q_0}}
\]

We denote
\[
M^{(q')}w(x) = \sup_{x \in Q \in \mathcal{Q}} \left( \frac{1}{|Q|} \int_Q w(x)^{q'} \, dx \right)^{1/q'}.
\]

Estimates (4.17) and (4.18) yield
\[
(4.16) \leq C \|g\|_{\mathcal{M}_q^{q_0}} \ell(Q_{k,j})^{-\frac{n}{q_0}} m_{3Q_{k,j}}(f) m_{Q_{k,j}}^{(q')}(w) |E_{k,j}|
\]
\[
\leq C \|g\|_{\mathcal{M}_q^{q_0}} \ell(Q_{k,j})^{-\frac{n}{q_0}} m_{3Q_{k,j}}(f) \inf_{x \in E_{k,j}} M^{(q')}w(x)
\]
\[
\leq C \|g\|_{\mathcal{M}_q^{q_0}} \int_{E_{k,j}} M_{\alpha-n/q_0} f(x) M^{(q')}w(x) \, dx.
\]
Similarly, we have
\[
\sum_{Q \in D_0} \ell(Q)^{\alpha} m_{3Q}(f) \int_{Q} g(x) w(x) \, dx \leq C \|g\|_{\mathcal{M}_{q_{0}}^{q_{0}}} \int_{E_0} M_{\alpha-n/q_{0}} f(x) M^{(q')} w(x) \, dx.
\]
Summing up all factors we obtain
\[
(4.14) \leq C \|g\|_{\mathcal{M}_{q_{0}}^{q_{0}}} \int_{Q_0} M_{\alpha-n/q_{0}} f(x) M^{(q')} w(x) \, dx.
\]
Another application of Hölder’s inequality gives us that
\[
(4.14) \leq C \|g\|_{\mathcal{M}_{q_{0}}^{q_{0}}} \left( \int_{Q_0} M_{\alpha-n/q_{0}} f(x)^{r} \, dx \right)^{\frac{1}{r}} \left( \int_{Q_0} M^{(q')} w(x)^{r'} \, dx \right)^{\frac{1}{r'}}.
\]
The fact \( r' > q' \) and the \( L^{r'/q'} \)-boundedness of maximal operator \( M \) yield
\[
(4.14) \leq C \|g\|_{\mathcal{M}_{q_{0}}^{q_{0}}} \left( \int_{Q_0} M_{\alpha-n/q_{0}} f(x)^{r} \, dx \right)^{\frac{1}{r}} \left( \int_{Q_0} M^{(q')} w(x)^{r'} \, dx \right)^{\frac{1}{r'}}
= C \|g\|_{\mathcal{M}_{q_{0}}^{q_{0}}} \left( \int_{Q_0} M_{\alpha-n/q_{0}} f(x)^{r} \, dx \right)^{\frac{1}{r}}.
\]
This is our desired inequality.

**The case** \( i = 2 \) A cruder estimate suffices in this case. By a property of the dyadic cubes, for all \( x \in Q_0 \) we have
\[
F_2(x) = \sum_{Q \in \mathcal{D}_2(Q_0)} \ell(Q)^{\alpha} m_{3Q}(f) \leq \sum_{Q \in \mathcal{D}_2(Q_0)} \ell(Q)^{\alpha-rac{n}{p_0}} f \|f\|_{\mathcal{M}_{p_0}^{p}}.
\]
In view of the definition of \( \mathcal{D}_2(Q_0) \), we have
\[
F_2(x) \leq \mu \ell(Q_0)^{\alpha-rac{n}{p_0}} f \|f\|_{\mathcal{M}_{p_0}^{p}} \left( \mu = \sum_{j=0}^{\infty} 2^{j(\alpha-n/p_0)} \right)
\]
Thus, for all \( x \in Q_0 \) we obtain
\[
F_2(x) \leq C \|f\|_{\mathcal{M}_{p_0}^{p}} \ell(Q_0)^{\alpha-rac{n}{p_0}}
\]
and
\[
\left( \int_{Q_0} (g(x) F_2(x))^{r} \, dx \right)^{\frac{1}{r}} \leq C m_{Q_0}^{(q)}(g) \|f\|_{\mathcal{M}_{p_0}^{p}} \ell(Q_0)^{\alpha-rac{n}{p_0} + \frac{n}{r}}.
\]
This is our desired inequality.
Remark. In the course of the proof we have proved

\begin{equation}
I_\alpha f(x) \leq C \sum_{Q \in \mathcal{D}} \ell(Q)^{\alpha} m_{3Q}(f) \chi_Q(x).
\end{equation}

Here is a converse inequality:

\begin{equation}
C^{-1} \sum_{Q \in \mathcal{D}} \ell(Q)^{\alpha} m_{3Q}(f) \chi_Q \leq \sum_{Q \in \mathcal{D}} \ell(Q)^{\alpha} \left( \inf_Q Mf \right) \chi_Q \leq CI_\alpha(Mf).
\end{equation}

§ 5. Counterexample

**Proposition 5.1.** Let $1 < r \leq r_0 < \infty$ and $r < 1/\alpha$. Then, for any $c > 0$ we can find positive measurable functions $f$ and $g$ such that

$$
\|g \cdot I_\alpha f\|_{\mathcal{M}^{r,r_0}} > c \|g\|_{\mathcal{M}^{r,1/\alpha}} \|f\|_{\mathcal{M}^{r,r_0}}.
$$

**Proof.** The proof of this proposition is kind of lengthy and we use the predual of Morrey spaces investigated originally by Zorko [19]. Here we content ourselves with remarking that we chose $g = f = \chi_{E_j}$ with $j \in \mathbb{N}$, where $E_j$ denotes the fractal set of this note (See Example 2.1) and $R > 1$ is appropriately chosen. \(\square\)

§ 6. Applications

§ 6.1. Application to the variation problem

Here we apply Theorem 1.1 to a variation problem.

Denote by $X_{q^{0}}^{q}$ the closure of $C_{c}^{\infty}$ with respect to $\mathcal{M}_{q}^{q}$.

**Theorem 6.1.** Let $n \geq 3$ and $V \in \mathcal{M}_{p}^{n/2}$ with $1 < p \leq n/2$. Let $W \in L^\infty$.

Define a functional $\mathcal{E}_V$ and $\mathcal{E}_{V+W}$ by

\begin{align*}
\mathcal{E}_V(\varphi) &:= \int_{\mathbb{R}^n} |\nabla \varphi(x)|^2 + V(x)|\varphi(x)|^2 \, dx \quad (\varphi \in H^1) \\
\mathcal{E}_{V+W}(\varphi) &:= \int_{\mathbb{R}^n} |\nabla \varphi(x)|^2 + (V(x) + W(x))|\varphi(x)|^2 \, dx \quad (\varphi \in H^1).
\end{align*}

(a) There exists a constant $\alpha > 0$ such that

\begin{equation}
\|f \cdot \sqrt{|V|}\|_2 \leq \alpha \sqrt{\|V\|_{\mathcal{M}_{p}^{n/2}}^2 \|\nabla f\|_2}
\end{equation}

for all $f \in C_c^\infty$. 

(b) If \( \| \min(0, V) \|_{M_p^{n/2}} < \alpha^{-2} \) and \( 2 < p \leq 2n \), then \( H = -\Delta + V \) is a positive operator on \( L^2(\mathbb{R}^n) \).

(c) Suppose that \( \| \min(0, V) \|_{M_p^{n/2}} < \frac{1}{4\alpha^2} \). If \( \{ \varphi_j \}_{j \in \mathbb{N}} \) is an \( L^2(\mathbb{R}^n) \)-sequence such that \( \| \varphi_j \|_2 \leq 1 \) for each \( j \in \mathbb{N} \) and that \( \sup_{j \in \mathbb{N}} \mathcal{E}_{V+W}(\varphi_j) < \infty \), then \( \{ \varphi_j \}_{j \in \mathbb{N}} \) forms a bounded sequence in \( H^1(\mathbb{R}^n) \).

(d) Suppose in addition that \( W \in L^\infty \). Then we have

\[
\lim_{j \to \infty} \mathcal{E}_{V+W}(\varphi_j) \geq \mathcal{E}_{V+W}(\varphi)
\]

if \( \lim_{j \to \infty} \varphi_j = \varphi \) in the weak topology of \( H^1 \).

(e) Let \( V \in X_p^{n/2} \) and \( 1 < p \leq n/2 \). Suppose that \( W \in L^\infty \). Assume in addition that

\[
\| \min(0, V) \|_{M_p^{n/2}} < \overline{4\alpha^2}
\]

and that

\[
E_0 = \inf\{\mathcal{E}_{V+W}(\varphi) : \varphi \in H^1, \| \varphi \|_2 = 1 \} < 0
\]

then there exists \( \varphi_0 \in L^2 \) such that

\[
\mathcal{E}_{V+W}(\varphi_0) = \inf\{\mathcal{E}_{V+W}(\varphi) : \varphi \in H^1, \| \varphi \|_2 = 1 \}, \| \varphi_0 \|_2 = 1.
\]

Assertions (a) and (b) are easy consequences of Theorem 1.1. Here we prove (c), (d) and (e).

Proof of (c). By (a)

\[
\| \varphi_j \sqrt{\min(V, 0)} \|_2 \leq \frac{1}{2} \| \nabla \varphi_j \|_2
\]

for each \( j \in \mathbb{N} \). Thus, it follows that

\[
T_{\varphi_j} = \int_{\mathbb{R}^n} |\nabla \varphi_j|^2 \geq -4 \int_{\mathbb{R}^n} |\varphi_j(x)|^2 \min(V(x), 0) \, dx.
\]

This pointwise estimate yields

\[
\frac{3}{4} T_{\varphi_j} = T_{\varphi_j} - \frac{1}{4} T_{\varphi_j} \leq T_{\varphi_j} + \int_{\mathbb{R}^n} |\varphi_j(x)|^2 \min(V(x), 0) \, dx \leq \mathcal{E}_V(\varphi_j).
\]

Since

\[
\sup_{j} \mathcal{E}_V(\varphi_j) \leq \| W \|_\infty + \sup_{j} \mathcal{E}_{V+W}(\varphi_j) < \infty,
\]

it follows that \( \{ \varphi_j \}_{j \in \mathbb{N}} \) forms a bounded sequence in \( H^1 \). \( \square \)
Proof of (d). Let us choose \( \{V_k\}_{k \in \mathbb{N}} \subset C_c^\infty \) that approximates \( V \) in the \( M_p^{n/2} \) topology. Since by our theorem (see Theorem 1.1) we have

\[
|\mathcal{E}_{V+W}(\varphi_j) - \mathcal{E}_{V_k+W}(\varphi_j)| \leq C \|V - V_k\|_{M_p^{n/2}} \|\varphi_j\|_{H^1}
\]

and \( \{\varphi_j\}_{j \in \mathbb{N}} \) forms a bounded family in \( H^1 \), we can assume that \( V \in C_c^\infty \). Once we assume that \( V \in C_c^\infty \), then we can use the compact embedding \( H^1 \to L^{2+\varepsilon} \) to conclude the proof of (d).

Proof of (e). Let us choose a sequence \( \{\varphi_j\}_{j \in \mathbb{N}} \subset L^2 \) so that

\[
E_0 = \lim_{j \to \infty} \mathcal{E}_{V+W}(\varphi_j), \|\varphi_j\|_2 = 1.
\]

By virtue of the weak compactness of \( H^1 \) and (c), we can assume that \( \varphi_j \to \varphi \) in the weak topology of \( H^1 \).

Since \( E_0 = \mathcal{E}_V(\varphi) < 0 \) by virtue of (d), it follows that \( \varphi \neq 0 \). Now that \( \varphi \) attains \( E_0 \), its \( L^2 \)-norm must be 1. Therefore, we obtain the desired result.

As an example to which we can apply our Theorem 1.1, we have the following function \( V_N \). Recall that we have defined \( E_N \) as a union of cubes of equal size in Proposition 2.1. Here the auxiliary parameter \( R \) is chosen for the function space \( M_p^{n/2} \), that is, we have chosen \( R \) so that

\[
\left( \frac{2^n}{(1+R)^n} \right)^{1/p} = \left( \frac{1}{(1+R)^n} \right)^{2/n}
\]

Write \( E_N = \bigcup_{j=1}^{2^{nN}} Q(z_{N,j}, r_{N,j}) \), where

\[
Q(z, r) := \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : |x_1 - z_1|, |x_2 - z_2|, \cdots, |x_n - z_n| \leq r\}.
\]

We define

\[
V_N(x) := -\kappa \sum_{j=1}^{2^{nN}} \chi_{Q(z_{N,j}, r_{N,j})}(x)|x - z_{N,j}|^{-2/n}.
\]

Then there exists \( \kappa_0 > 0 \) with the following property; if \( 0 < \kappa < \kappa_0 \) and \( M \in \mathbb{R} \) satisfies

\[
E_0 = \inf \{\mathcal{E}_{V_N+M}(\varphi) : \varphi \in H^1, \|\varphi\|_2 = 1\} < 0
\]

then for all \( N \in \mathbb{N} \), there exists \( \varphi \in H^1 \) such that

\[
\mathcal{E}_{V_N+W}(\varphi_0) = \inf \{\mathcal{E}_{V_N+W}(\varphi) : \varphi \in H^1, \|\varphi\|_2 = 1\}, \|\varphi_0\|_2 = 1.
\]
§ 6.2. Sobolev-Hardy inequality

Let \(0 \leq s \leq 2\). Then we have

\[
\|u\|_{L_{n-2}^{\frac{2n-2s}{n-2}}(|x|^{-s} \, dx)} \leq C \|\nabla u\|_{L^2}.
\]

Using Theorem 1.1, we can extend this theorem to some extent.

Here is our result.

**Theorem 6.2.** Let \(1 < p \leq p_0 < n\), \(1 < r \leq r_0 < \infty\) and \(0 < s < n\). Assume that

\[
\frac{1}{r_0} = \frac{s}{nr} + \frac{1}{p_0} - \frac{1}{n}, \quad \frac{r}{r_0} = \frac{p}{p_0}, \quad r \geq s.
\]

Then we have

\[
\|u|x|^{-\frac{s}{r}}\|_{\mathcal{M}_{r}^{r_0}} \leq C \|\nabla u\|_{\mathcal{M}_{p}^{p_0}}.
\]

**Proof.** This is just a rephrasement of Theorem 1.1 obtained by setting \(q_0 = \frac{nr}{s}\) and \(g(x) = |x|^{-s/r}\). Observe that \(g \in \mathcal{M}_{q_0}^{q_0}\) for all \(0 < \varepsilon \ll 1\). \(\square\)

§ 6.3. An extension of Olsen’s result

**Theorem 6.3.** Assume that

\(0 < \alpha < n\), \(1 < \bar{q} \leq \frac{n}{2}\), \(1 < q \leq p \leq \infty\), \(1 < t \leq s < \infty\)

and that

\[
\frac{1}{s} = \frac{1}{p} - \frac{2}{n}, \quad t = \frac{q}{p}.
\]

Denote by \(M_V\) the multiplication operator generated by the potential \(V\). Then we have

\[
\|VI_2W\|_{\mathcal{M}_{\bar{q}}^{p}} \leq C_0 \|V\|_{\mathcal{M}_{\frac{n}{2}}^{n/2}}\|W\|_{\mathcal{M}_{\bar{q}}^{p}}
\]

and that

\[
\|(I_2M_V)^nI_2W\|_{\mathcal{M}_{t}^{s}} \leq C_1 C_0^n \|V\|_{\mathcal{M}_{\frac{n}{2}}^{n/2}}\|W\|_{\mathcal{M}_{\bar{q}}^{p}}
\]

for some \(C_0, C_1 > 1\).

In particular, if \(V \in \mathcal{M}_{\frac{n}{2}}^{n/2} \cap \mathcal{M}_{\bar{q}}^{p}\) and the norm is sufficiently small, then the formal solution (of \(\Delta v + Vv = v\))

\[
v = 1 + \sum_{j=1}^{\infty} (-1)^j (I_2M_V)^{j-1}(I_2V)
\]

satisfies

\[
\|v - 1\|_{\mathcal{M}_{t}^{s}} \leq C.
\]
Olsen’s inequality and its applications to Schrödinger equations

Proof. Inequality (6.17) is just a repetition of our results. As for (6.18), we have
\[ \|(I_2 M_V)^n I_2 W\|_{\mathcal{M}_t^s} \leq C_1 \|(M_V I_2)^n W\|_{\mathcal{M}_q^p} \leq C_1 C_0^n \|V\|_{\mathcal{M}_{q/2}^{n/2}} \|W\|_{\mathcal{M}_q^p} \]
using the first estimate \(n\)-times. Finally, the last assertion can be obtained directly from (6.18).

\[ \square \]

Remark. Olsen postulated
\[ 0 < \alpha < n, \ 1 < \tilde{q} \leq \frac{n}{2}, \ 1 < t \leq s < \infty, \ t < \frac{2}{n} \tilde{q}s \]
additionally on the parameters [10]. With this condition and the condition that all the functions are supported in a bounded set, he obtained \(\|v - 1\|_{\mathcal{M}_t^s} \leq C\). However, the last condition \(t < \frac{2}{n} \tilde{q}s\) turned out to be superfluous. We can replace this condition with \(t < \frac{1}{p} \tilde{q}s\) with \(p\) given by \(\frac{1}{s} = \frac{1}{p} - \frac{2}{n}\), keeping in mind that \(p < \frac{n}{2}\).

\[ \square \]

§ 7. Full statement of our main results

Here we formulate our main theorem as the full statement.

§ 7.1. A passage from \(I_\alpha\) to \(T_\rho\) and from \(\mathcal{M}_q^p\) to generalized Morrey spaces

Here we describe what we actually obtained in our first paper [12]. Let \(\rho : [0, \infty) \to [0, \infty]\) be a suitable function. We define the generalized fractional integral operator \(T_\rho\) and the generalized fractional maximal operator \(M_\rho\) by
\[ T_\rho f(x) := \int_{\mathbb{R}^n} f(y) \frac{\rho(|x - y|)}{|x - y|^n} \, dy, \]
\[ M_\rho f(x) := \sup_{x \in Q \in \mathcal{Q}} \rho(\ell(Q)) m_Q(|f|). \]
If \(\rho(t) \equiv t^{n\alpha}, \ 0 < \alpha < 1\), then \(T_\rho = I_\alpha\) and \(M_\rho = M_\alpha\). The Morrey norm \(\|f\|_{p, \rho}\) is given by
\[ \|f\|_{p, \rho} := \sup_{Q \in \mathcal{Q}} \rho(\ell(Q)) \left( \frac{1}{|Q|} \int_Q |f(x)|^p \, dx \right)^{1/p}. \]
In general, by the Dini condition we mean that
\[ \int_0^1 \frac{\rho(s)}{s} \, ds < \infty, \]
while the doubling condition (with a doubling constant \(C_0 > 0\)) is that
\[ \frac{1}{C_0} \leq \frac{\rho(s)}{\rho(t)} \leq C_0, \text{ if } \frac{1}{2} \leq \frac{s}{t} \leq 2. \]
A simple consequence that can be deduced from the doubling condition is

\[ \frac{\log 2}{C_0} \rho(t) \leq \int_{t/2}^{t} \frac{\rho(s)}{s} ds \leq \log 2 \cdot C_0 \rho(t) \quad \text{for all } t > 0. \]

In the sequel, we always assume that \( \rho \) satisfies (7.2) and (7.3), and, then denote the set of all such functions by \( \mathcal{G}_0 \). We will write, when \( \rho \in \mathcal{G}_0 \),

\[ \tilde{\rho}(t) := \int_{0}^{t} \frac{\rho(s)}{s} ds. \]

Let \( \mathcal{G}_1 \) be the set of all functions \( \phi : [0, \infty) \to [0, \infty) \) such that \( \phi(t) \) is nondecreasing but that \( \phi(t)t^{-n} \) is nonincreasing. We notice that the condition \( \phi \in \mathcal{G}_1 \) is stronger than the doubling condition (7.3). More quantitatively, if we assume that \( \phi \in \mathcal{G}_1 \), then \( \phi \) satisfies the doubling condition with the doubling constant \( 2^n \).

**Theorem 7.1.** Let \( 1 < p < \infty, q > r, 0 \leq b \leq 1, a > 1 \) and \( (a + b - 1)r = ap \). Suppose that \( \rho \) satisfies the Dini condition, (7.3) and that \( \tilde{\rho}(t)^{\max(ap,bq)}t^{-n} \) is nonincreasing. The condition \( \tilde{\rho}(t)^{\max(ap,bq)}t^{-n} \) is nonincreasing implies that \( \tilde{\rho}(t)^{ap}t^{-n} \) and \( \tilde{\rho}(t)^{bq}t^{-n} \) are nonincreasing, since

\[ \tilde{\rho}(t)^{\min(ap,bq)}t^{-n} = \tilde{\rho}(t)^{\min(ap,bq) - \max(ap,bq)} \cdot \tilde{\rho}(t)^{\max(ap,bq)}t^{-n}. \]

Then

\[ \|g \cdot T_{\rho}f\|_{r, \tilde{\rho}^{a+b-1}} \leq C \|g\|_{q, \tilde{\rho}^{b}} \|f\|_{p, \tilde{\rho}^{a}}, \]

where the constant \( C \) is independent of \( f \) and \( g \).

Theorem 7.1 generalizes of [10, Theorem 2] and [17, Theorem 1]. Theorem 7.1 is not longer true when \( q = r \) (see Proposition 5.1).

Letting \( b = 0 \) and \( g \equiv 1 \) in Theorem 7.1, we have the following:

**Corollary 7.2.** Let \( 1 < p < \infty, a > 1 \) and \( (a-1)r = ap \). Then

\[ \|T_{\rho}f\|_{r, \tilde{\rho}^{a-1}} \leq C \|f\|_{p, \tilde{\rho}^{a}}. \]

Corollary 7.2 generalizes [1, Theorem 1.3].

**Theorem 7.3.** Let \( 1 \leq p < \infty \), \[ \begin{cases} p \leq q \text{ if } p = 1, & 0 \leq b \leq 1 \text{ and } b < a, \\ p < q \text{ if } p > 1, & \end{cases} \]

Suppose that \( \rho \) satisfies the Dini condition, (7.3) and that \( \tilde{\rho}(t)^{\max(ap,bq)}t^{-n} \) is nonincreasing. Then

\[ \|g \cdot T_{\rho}f\|_{p, \tilde{\rho}^{a}} \leq C \|g\|_{q, \tilde{\rho}^{b}} \|M_{\tilde{\rho}^{1-b}}f\|_{p, \tilde{\rho}^{a}}, \]

where the constant \( C \) is independent of \( f \) and \( g \).
Corollary 7.4. Let $1 \leq p < \infty$ and $\alpha > 0$. Then
\[ \|T_\rho f\|_{p,\hat{\rho}^\alpha} \leq C\|M_{\hat{\rho}} f\|_{p,\hat{\rho}^\alpha}. \]

Corollary 7.4 generalizes [2, Theorem 4.2].
Letting $b = 1$ in Theorem 7.3, we have the following too:

Corollary 7.5. Let $1 \leq p < \infty$.
\[ \|g \cdot T_\rho f\|_{p,\hat{\rho}^\alpha} \leq C\|g\|_{q,\hat{\rho}}\|M_{\hat{\rho}} f\|_{p,\hat{\rho}^\alpha}. \]

These results are somehow generalized. We generalized them in [13] as follows; we content ourselves with stating them.

Theorem 7.6. Let $1 \leq p < \infty$.
\[ \|g \cdot T_\rho f\|_{p,\phi} \leq C\|g\|_{q,\eta}\|M_{\rho/\eta} f\|_{p,\phi}, \]
where the constant C is independent of $f$ and $g$.

Theorem 7.7. Let $1 < p \leq r < q < \infty$. Suppose that $\phi(t)$ and $\eta(t)$ are nondecreasing but that $\phi(t)^p t^{-n}$ and $\eta(t)^q t^{-n}$ are nonincreasing. Suppose also that
\[ \int_t^\infty \frac{\rho(s)\eta(s)}{s \hat{\rho}(s)\phi(s)} \ ds \leq C\frac{\eta(t)}{\phi(t)} \text{ for all } t > 0. \]
Then
\[ \|g \cdot T_\rho f\|_{r,\phi^{p/r}} \leq C\|g\|_{q,\eta}\|f\|_{p,\phi}, \]
where the constant C is independent of $f$ and $g$.

Theorem 7.8. Let $0 < p < \infty$. Suppose that $\rho$, $\eta$ and $\phi$ are nondecreasing and that $\eta(t)^p t^{-n}$ and $\phi(t)^p t^{-n}$ are nonincreasing. Then
\[ \|g \cdot M_\rho f\|_{p,\phi} \leq C\|g\|_{p,\eta}\|M_{\rho/\eta} f\|_{p,\phi}, \]
where the constant C is independent of $f$ and $g.$
§7.2. Extension of Theorem 1.1 to Orlicz-Morrey spaces

To describe Orlicz-Morrey spaces, we recall some definitions and notation. Here we follow [15].

A function $\Phi : [0, \infty) \to [0, \infty]$ is said to be a Young function if it is left-continuous, convex and increasing, and if $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. We say that $\Phi$ is a normalized Young function when $\Phi$ is a Young function and $\Phi(1) = 1$. It is easy to see that $t^p$, $1 \leq p < \infty$, is a normalized Young function.

A Young function $\Phi$ is said to satisfy the $\Delta_2$-condition, denoted $\Phi \in \Delta_2$, if for some $K > 1$

$$\Phi(2t) \leq K\Phi(t) \text{ for all } t > 0.$$ 

Meanwhile, a Young function $\Phi$ is said to satisfy the $\nabla_2$-condition, denoted $\Phi \in \nabla_2$, if for some $K > 1$

$$\Phi(t) \leq \frac{1}{2K}\Phi(2Kt) \text{ for all } t > 0.$$ 

The function $\Phi(t) \equiv t$ satisfies the $\Delta_2$-condition but fails the $\nabla_2$-condition. If $1 < p < \infty$, then $\Phi(t) \equiv t^p$ satisfies both conditions. The complementary function $\overline{\Phi}$ of a Young function $\Phi$ is defined by

$$\overline{\Phi}(t) := \sup\{ts - \Phi(s) : s \in [0, \infty)\}.$$ 

Then $\overline{\Phi}$ is also a Young function and $\overline{\overline{\Phi}} = \Phi$. Notice that $\Phi \in \nabla_2$ if and only if $\overline{\Phi} \in \Delta_2$. For the other properties of Young functions and the examples, see [8, p196].

Given a Young function $\Phi$, define the Orlicz space $\mathcal{L}^\Phi(\mathbb{R}^n) = \mathcal{L}^\Phi$ by the Luxemburg norm

$$\|f\|_{\mathcal{L}^\Phi} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) \, dx \leq 1 \right\}.$$ 

When $\Phi(t) \equiv t^p$, $1 \leq p < \infty$, $\|f\|_{\mathcal{L}^\Phi} = \|f\|_{L^p}$. We need the following basic two facts.

Generalized Hölder’s inequality:

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq C\|f\|_{\mathcal{L}^\Phi}\|g\|_{\mathcal{L}^\Phi};$$

The dual equation:

$$\|f\|_{\mathcal{L}^\Phi} \approx \sup \{ \|fg\|_{L^1} : \|g\|_{\mathcal{L}^\Phi} \leq 1 \}.$$ 

Given a Young function $\Phi$, define the mean Luxemburg norm of $f$ on a cube $Q \in \mathcal{Q}$ by

$$\|f\|_{\Phi, Q} := \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) \, dx \leq 1 \right\}.$$
When $\Phi(t) \equiv t^p$, $1 \leq p < \infty$,
\[
\|f\|_{\Phi, Q} = \left( \frac{1}{|Q|} \int_Q |f(x)|^p \, dx \right)^{1/p},
\]
that is, the mean Luxemburg norm coincides with the (normalized) $L^p$ norm. It should be noticed that
\[
(7.7) \quad \|f\|_{\Phi, Q} = \|\tau_\delta(Q)[f\chi_Q]\|_{L^\Phi},
\]
where $\tau_\delta$, $\delta > 0$, is the dilation operator $\tau_\delta f(x) = f(\delta x)$. It follows from this relation and generalized Hölder’s inequality that for any cube $Q \in \mathcal{Q}$
\[
(7.8) \quad m_Q(|fg|) \leq C\|f\|_{\Phi, Q}\|g\|_{\overline{\Phi}, Q}.
\]

The Orlicz maximal operator, for any Young function $\Psi$, is defined by
\[
M^\Psi f(x) := \sup_{x \in Q \in \mathcal{Q}} \|f\|_{\Psi, Q}.
\]
Now let us introduce Orlicz-Morrey spaces.

**Definition 7.9.** Let $\phi \in G_1$ and let $\Phi$ be a Young function. The Orlicz-Morrey space $\mathcal{L}^{\Phi, \phi}(\mathbb{R}^n) = \mathcal{L}^{\Phi, \phi}$ consists of all locally integrable functions $f$ on $\mathbb{R}^n$ for which the norm
\[
\|f\|_{\mathcal{L}^{\Phi, \phi}} := \sup_{Q \in \mathcal{Q}} \phi(\ell(Q))\|f\|_{\Phi, Q}
\]
is finite. In particular, in order that the characteristic function of the unit cubes belongs to $\mathcal{L}^{\Phi, \phi}$, it is always assumed that
\[
\sup_{t>1} \frac{\phi(t)}{\Phi^{-1}(t^n)} < \infty.
\]

**Example** We define
\[
(7.9) \quad \|f\|_{\mathcal{M}_{L \log L}^p} = \|f\|_{\Phi, \phi},
\]
when $\phi(t) = t^{n/p}$ and $\Phi(t) = t \log(3 + t)$.

We have shown in [15] that
\[
(7.10) \quad \|Mf\|_{\mathcal{M}_p} \sim \|f\|_{\mathcal{M}_{L \log L}^p}.
\]

If $\Phi(t) \equiv t^p$ and $\phi(t) \equiv t^{n/p}$, $1 \leq p \leq p_0 < \infty$, then $\mathcal{L}^{\Phi, \phi} = \mathcal{M}^{p, p_0}$. When $\Phi(t) \equiv t^p$, $1 \leq p < \infty$, we will denote $\mathcal{L}^{\Phi, \phi}$ by $\mathcal{M}_p$. In this case we will call it the (generalized) Morrey space. We consider $\mathcal{M}_p$ even for $0 < p < 1$. We define an auxiliary space too.
**Definition 7.10.** Let $\phi \in \mathcal{G}_1$ and let $\Phi$ be a Young function. The space

$$\tilde{\mathcal{L}}^{\Phi, \phi}(\mathbb{R}^n) = \tilde{\mathcal{L}}^{\Phi, \phi}$$

consists of all locally integrable functions $g$ on $\mathbb{R}^n$ for which the norm

$$\|g\|_{\tilde{\mathcal{L}}^{\Phi, \phi}} := \sup \left\{ \|M_{\phi}[gw\chi_Q]\|_{\tilde{\Phi}, \Phi} : Q \in \mathcal{Q}, \|w\|_{\tilde{\Phi}, \Phi} \leq 1 \right\}$$

is finite.

Related to the space $\tilde{\mathcal{L}}^{\Phi, \phi}$, we need the following notion too.

**Definition 7.11.** Let $\Phi$ and $\Psi$ be Young functions. One says that “$M^\Psi$ is locally bounded in the norm determined by $\Phi$”, when it satisfies

$$\|M^\Psi[g\chi_Q]\|_{\Phi, \Phi} \leq C\|g\|_{\Phi, \Phi}$$

for all cubes $Q \in \mathcal{Q}$.

We now state our first results, which extend those in [12, 13] to Orlicz-Morrey spaces.

**Theorem 7.12.** Let $\rho \in \mathcal{G}_0$, $\phi \in \mathcal{G}_1$ and $\Phi \in \nabla_2$. Suppose that the condition

$$\int_t^\infty \frac{\rho(s)}{s\phi(s)} \, ds \leq C \frac{\tilde{\rho}(t)}{\phi(t)} \text{ for all } t > 0. \tag{7.11}$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Phi, \phi}} \leq C\|g\|_{\mathcal{L}^{\Phi, \phi}} \|f\|_{\mathcal{L}^{\Phi, \phi}}.$$

**Theorem 7.13.** Let $\Psi$ be a Young function. With the same condition posed in Theorem 7.12, if, in addition, $M^\Psi$ is locally bounded in the norm determined by $\Phi$, then we have

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Phi, \phi}} \leq C\|g\|_{\mathcal{L}^{\Phi, \phi}} \|f\|_{\mathcal{L}^{\Phi, \phi}}.$$

Theorems 7.12 and 7.13 are the trace inequalities of the generalized fractional integral operators for Orlicz-Morrey spaces.

**Theorem 7.14.** Let $\rho \in \mathcal{G}_0$, $\phi, \psi \in \mathcal{G}_1$, $\Phi \in \nabla_2$ and $0 < a \leq 1$. Set

$$\eta(t) \equiv \phi(t)^{a}, \quad \Psi(t) \equiv \Phi(t^{1/a}).$$

Suppose that the condition

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)} \, ds \leq C \frac{\psi(t)}{\eta(t)} \text{ for all } t > 0. \tag{7.12}$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Phi, \phi}} \leq C\|g\|_{\mathcal{L}^{\Phi, \phi}} \|f\|_{\mathcal{L}^{\Phi, \phi}}.$$
Theorem 7.14 is a general form of Theorem 7.12 (letting $a \equiv 1$) and is the Olsen inequality of the generalized fractional integral operators for Orlicz-Morrey spaces.

Letting $g(x) \equiv 1$ and $\psi(t) \equiv 1$ in Theorem 7.14, we can recover the boundedness property of $T_\rho$.

**Corollary 7.15.** Let $\rho \in \mathcal{G}_0$, $\phi \in \mathcal{G}_1$, $\Phi \in \nabla_2$ and $0 < a \leq 1$. Set

$$
\eta(t) \equiv \phi(t)^a, \quad \Psi(t) \equiv \Phi(t^{1/a}).
$$

Suppose that $\Psi \in \nabla_2$ and that the condition

$$
\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)} \, ds \leq \frac{C}{\eta(t)} \quad \text{for all } t > 0.
$$

Then

$$
\|T_\rho f\|_{\mathcal{L}^\Psi, \eta} \leq C\|f\|_{\mathcal{L}^\Phi, \phi}.
$$

However, in the next theorem we reproved this corollary directly without the assumption $\Psi \in \nabla_2$.

**Theorem 7.16.** Let $\rho \in \mathcal{G}_0$, $\phi \in \mathcal{G}_1$, $\Phi \in \nabla_2$ and $0 < a \leq 1$. Set

$$
\eta(t) \equiv \phi(t)^a, \quad \Psi(t) \equiv \Phi(t^{1/a}).
$$

Suppose that the condition

$$
\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)} \, ds \leq \frac{C}{\eta(t)} \quad \text{for all } t > 0.
$$

Then

$$
\|T_\rho f\|_{\mathcal{L}^\Psi, \eta} \leq C\|f\|_{\mathcal{L}^\Phi, \phi}.
$$

Corollary 7.15 generalizes [13, Corollary 1.7]. In [7] Nakai studied the boundness of the generalized fractional integral operator $T_\rho$ on Orlicz spaces. Since, we cannot recover Orlicz spaces as a special case of our Orlicz-Morrey spaces, we dare not compare Corollary 7.15 with [7, Theorem 3.1].

§ 8. Appendix—Boundedness of the fractional maximal operator

**Lemma 8.1.** Let $p > 1$. Suppose that $\phi(t)$ is nondecreasing and $\phi(t)^pt^{-n}$ is nonincreasing. Then

$$
\|Mf\|_{p, \phi} \leq C\|f\|_{p, \phi}.
$$
Proof. Fix a cube $Q_0$. Let $f_1 = \chi_{3Q_0}f$ and $f_2 = f - f_1$. Then the subadditivity of $M$ yields
\[ Mf(x) \leq Mf_1(x) + Mf_2(x). \]
It follows from the definition of $M$ that for all $x \in Q_0$
\[ Mf_2(x) = \sup_{x \in Q : \ell(Q) \geq \ell(Q_0)} \frac{1}{|Q|} \int_Q |f(y)| \, dy. \]
Suppose that $x \in Q_0$, $x \in Q \in \mathcal{Q}$ and $\ell(Q) \geq \ell(Q_0)$. Then
\[ \phi(\ell(Q_0)) m_Q(|f|) \leq \phi(\ell(Q)) m_Q^{(p)}(|f|) \leq \|f\|_{p, \phi}, \]
where we have used Hölder’s inequality and the fact that $\phi$ is nondecreasing.

This gives us that
\[ \phi(\ell(Q_0)) Mf_2(x) \leq \|f\|_{p, \phi} \text{ for all } x \in Q_0, \]
and that
\[ \phi(\ell(Q_0)) m_Q^{(p)}(Mf)^p \leq \phi(\ell(Q_0)) m_Q^{(p)}(Mf_1)^p + \phi(\ell(Q_0)) m_Q^{(p)}(Mf_2)^p \]
\[ \leq C \phi(\ell(3Q_0)) m_{3Q_0}^{(p)}(f) + \|f\|_{p, \phi} \leq C \|f\|_{p, \phi}, \]
where we have used $L^p$ boundedness of maximal operator $M$. This implies our desired inequality. \hfill \square

Lemma 8.2. Let $1 < p \leq q < \infty$. Suppose that $\phi(t)$ is nondecreasing and $\phi(t)^p t^{-n}$ is nonincreasing. Then
\[ \|M_{\phi^{1-p/q}}f\|_{q, \phi^{p/q}} \leq C \|f\|_{p, \phi}. \]
It is worth noting that the surjectivity of $\phi$ was superfluous.

Proof. Let $x \in \mathbb{R}^n$ be a fixed point. For every cube $Q \ni x$ we see that
\[ \phi(\ell(Q))^{1-p/q} m_Q(|f|) \leq \min(\phi(\ell(Q))^{1-p/q} Mf(x), \phi(\ell(Q))^{-p/q} \|f\|_{p, \phi}) \]
\[ \leq \sup_{t \geq 0} \min(t^{1-p/q} Mf(x), t^{-p/q} \|f\|_{p, \phi}) \]
\[ = \|f\|_{p, \phi}^{1-p/q} Mf(x)^{p/q}. \]
This implies
\[ M_{\phi^{1-p/q}}f(x)^q \leq \|f\|_{p, \phi}^{q-p} Mf(x)^p. \]
It follows from Lemma 8.1 that for every cube $Q_0$
\[ m_{Q_0}^{(q)}(M_{\phi^{1-p/q}}f) \leq \|f\|_{p, \phi}^{1-p/q} m_{Q_0}^{(p)}(Mf)^{p/q} \leq C \|f\|_{p, \phi} \phi(\ell(Q_0))^{-p/q}. \]
The desired inequality then follows.

\[\square\]

§ 9. Acknowledgement

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