# Necessary and Sufficient Conditions for the Fractional Gagliardo-Nirenberg Inequalities and Applications to Navier-Stokes and Generalized Boson Equations

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**Abstract**

Necessary and sufficient conditions for the generalized Gagliardo-Nirenberg inequalities are obtained. For $0 < q < \infty$, $0 < p, p_0, p_1 \leqslant \infty$, $s, s_0, s_1 \in \mathbb{R}$, $\theta \in (0, 1)$,

$$\|u\|_{\dot{B}^s_{p,q}} \lesssim \|u\|^{1-\theta}_{\dot{B}^{s_0}_{p_0,\infty}} \|u\|^{\theta}_{\dot{B}^{s_1}_{p_1,\infty}} \tag{0.1}$$

holds if and only if $n/p-s = (1-\theta)(n/p_0-s_0)+\theta(n/p_1-s_1)$, $s_0-n/p_0 \neq s_1-n/p_1$, $s \leqslant (1-\theta)s_0 + \theta s_1$, and $p_0 = p_1$ if $s = (1-\theta)s_0 + \theta s_1$. Applying this inequality, we show that the solution of the Navier-Stokes equation at finite blowup time $T_m$ has a concentration phenomena in the critical space $L^3(\mathbb{R}^3)$. Moreover, we consider the minimization problem for the variational problem

$$M_c = \inf \left\{ E(u) : \ \|u_i\|_2^2 = c_i > 0, \ i = 1, ..., L \right\},$$

where

$$E(u) = \frac{1}{2}\|u\|^2_{\dot{H}^s} - \int_{\mathbb{R}^{2n}} G(u(x))V(x-y)G(u(y))dxdy$$

for $u = (u_1, ..., u_L) \in (H^s)^L$ and show that $M_c$ admits a radial and radially decreasing minimizer under suitable assumptions on $s$, $G$ and $V$.

*Keywords.* Fractional Gagaliardo-Nirenberg inequality, Besov spaces, Triebel-Lizorkin spaces, boson equation, minimizer.

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# 1 Introduction

The Gagliardo-Nirenberg (GN) inequality is a fundamental tool in the study of nonlinear partial differential equations, which was discovered by Gagliardo [27], Nirenberg [51] (see also [36]) in some special cases. Throughout this paper, we denote by $L^p := L^p(\mathbb{R}^n)$ the Lebesgue space, $\|\cdot\|_p := \|\cdot\|_{L^p}$. $C > 1$ will denote positive universal constants, which can be different at different places. $a \lesssim b$ stands for $a \leqslant Cb$ for some constant $C > 1$, $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. We write $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$. The classical integer version of the GN inequality can be stated as follows (see [25] for instance):

**Theorem 1.1** *Let $1 \leqslant p, p_0, p_1 \leqslant \infty$, $\ell, m \in \mathbb{N} \cup \{0\}$, $\ell < m$, $\ell/m \leqslant \theta \leqslant 1$, and*

$$\frac{n}{p} - \ell = (1-\theta)\frac{n}{p_0} + \theta\left(\frac{n}{p_1} - m\right). \tag{1.1}$$

*Then we have for all $u \in C_0^\infty(\mathbb{R}^n)$,*

$$\sum_{|\alpha|=\ell} \|\partial^\alpha u\|_p \lesssim \|u\|_{p_0}^{1-\theta} \sum_{|\alpha|=m} \|\partial^\alpha u\|_{p_1}^{\theta}, \tag{1.2}$$

*where we further assume $\ell/m \leqslant \theta < 1$ if $m - \ell - n/p_1$ is an integer.*

The classical proof of the GN inequality is based on the global derivative analysis in $L^p$ spaces, whose proof is rather complicated, cf. [25, 29]. On the basis of the harmonic analysis techniques, there are some recent works devoted to generalizations of the GN inequality, cf. [5, 9, 10, 15, 16, 17, 22, 25, 29, 30, 38, 42, 50, 52, 55].

Now we introduce some function spaces which will be frequently used, cf. [57]. We denote by $\dot H^s_p := (-\triangle)^{s/2}L^p$ the Riesz potential space, $\dot H^s = \dot H^s_2$, $H^s = L^2 \cap \dot H^s$ for any $s \geqslant 0$. Let $\psi$ be a smooth cut-off function supported in the ball $\{\xi : |\xi| \leqslant 2\}$, $\varphi = \psi(\cdot) - \psi(2\,\cdot)$. We write $\varphi_k(\xi) = \varphi(2^{-k}\xi)$, $k \in \mathbb{Z}$. We see that

$$\sum_{k\in\mathbb{Z}} \varphi_k(\xi) = 1, \quad \xi \in \mathbb{R}^n \setminus \{0\}. \tag{1.3}$$

We introduce the homogeneous dyadic decomposition operators $\triangle_k = \mathscr{F}^{-1}\varphi_k\mathscr{F}$, $\quad k \in \mathbb{Z}$. Let $-\infty < s < \infty$, $1 \leqslant p, q \leqslant \infty$. The space $\dot B^s_{p,q}$ equipped with norm

$$\|f\|_{\dot B^s_{p,q}} := \left(\sum_{k=-\infty}^{\infty} 2^{ksq}\|\triangle_k f\|_p^q\right)^{1/q} \tag{1.4}$$

is said to be a homogeneous Besov space (a tempered distribution $f \in \dot B^s_{p,q}$ modulo polynomials). Let

$$-\infty < s < \infty, \quad 1 \leqslant p < \infty, \quad 1 \leqslant q \leqslant \infty. \tag{1.5}$$

The space $\dot{F}^s_{p,q}$ equipped with norm

$$\|f\|_{\dot{F}^s_{p,q}} := \left\| \Big( \sum_{k=-\infty}^{\infty} 2^{ksq} |\triangle_k f|^q \Big)^{1/q} \right\|_p \tag{1.6}$$

is said to be a homogeneous Triebel-Lizorkin space (a tempered distribution $f \in \dot{F}^s_{p,q}$ modulo polynomials).

## 2 Fractional GN inequalities

In this paper we will obtain necessary and sufficient conditions for the GN inequality in homogeneous Besov spaces $\dot{B}^s_{p,q}$ and Triebel-Lizorkin spaces $\dot{F}^s_{p,q}$. As a corollary, we obtain that the GN inequality also holds in fractional Sobolev spaces $\dot{H}^s_p$. The fractional GN inequalities in Theorems 2.1, 2.2 and 2.3 below cover all of the available GN inequalities in [5, 9, 10, 15, 16, 17, 22, 25, 29, 30, 38, 42, 50, 52, 55] for both integer and fractional versions. Moreover, our results below clarify how the third indices $q$ in $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ contribute the validity of the GN inequalities. We have

**Theorem 2.1** *Let $0 < p, p_0, p_1, q, q_0, q_1 \leqslant \infty$, $s, s_0, s_1 \in \mathbb{R}$, $0 \leqslant \theta \leqslant 1$. Then the fractional GN inequality of the following type*

$$\|u\|_{\dot{B}^s_{p,q}} \lesssim \|u\|^{1-\theta}_{\dot{B}^{s_0}_{p_0,q_0}} \|u\|^{\theta}_{\dot{B}^{s_1}_{p_1,q_1}} \tag{2.1}$$

*holds for all $u \in \dot{B}^{s_0}_{p_0,q_0} \cap \dot{B}^{s_1}_{p_1,q_1}$ if and only if*

$$\frac{n}{p} - s = (1-\theta)\left(\frac{n}{p_0} - s_0\right) + \theta\left(\frac{n}{p_1} - s_1\right), \tag{2.2}$$

$$s \leqslant (1-\theta)s_0 + \theta s_1, \tag{2.3}$$

$$\frac{1}{q} \leqslant \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \textit{if } \ p_0 \neq p_1 \ \textit{and } \ s = (1-\theta)s_0 + \theta s_1, \tag{2.4}$$

$$s_0 \neq s_1 \quad \textit{or} \quad \frac{1}{q} \leqslant \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \textit{if } \ p_0 = p_1 \ \textit{and } \ s = (1-\theta)s_0 + \theta s_1, \tag{2.5}$$

$$s_0 - \frac{n}{p_0} \neq s - \frac{n}{p} \quad \textit{or} \quad \frac{1}{q} \leqslant \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \textit{if } \ s < (1-\theta)s_0 + \theta s_1. \tag{2.6}$$

**Theorem 2.2** *Let $0 < q < \infty$, $0 < p$, $p_0$, $p_1 \leqslant \infty$, $0 < \theta < 1$, $s, s_0, s_1 \in \mathbb{R}$. Then the fractional GN inequality of the following type*

$$\|u\|_{\dot{B}^s_{p,q}} \lesssim \|u\|^{1-\theta}_{\dot{B}^{s_0}_{p_0,\infty}} \|u\|^{\theta}_{\dot{B}^{s_1}_{p_1,\infty}} \tag{2.7}$$

*holds if and only if*

$$\frac{n}{p} - s = (1-\theta)\left(\frac{n}{p_0} - s_0\right) + \theta\left(\frac{n}{p_1} - s_1\right), \tag{2.8}$$

$$s_0 - \frac{n}{p_0} \neq s_1 - \frac{n}{p_1}, \tag{2.9}$$

$$s \leqslant (1-\theta)s_0 + \theta s_1, \tag{2.10}$$

$$p_0 = p_1 \quad if \quad s = (1-\theta)s_0 + \theta s_1. \tag{2.11}$$

In homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,q}$, we have the following

**Theorem 2.3** *Let $0 < p, p_i, q < \infty$, $s, s_0, s_1 \in \mathbb{R}$, $0 < \theta < 1$. Then the fractional GN inequality of the following type*

$$\|u\|_{\dot{F}^s_{p,q}} \lesssim \|u\|^{1-\theta}_{\dot{F}^{s_0}_{p_0,\infty}} \|u\|^{\theta}_{\dot{F}^{s_1}_{p_1,\infty}} \tag{2.12}$$

*holds if and only if*

$$\frac{n}{p} - s = (1-\theta)\left(\frac{n}{p_0} - s_0\right) + \theta\left(\frac{n}{p_1} - s_1\right), \tag{2.13}$$

$$s \leqslant (1-\theta)s_0 + \theta s_1, \tag{2.14}$$

$$s_0 \neq s_1 \quad if \quad s = (1-\theta)s_0 + \theta s_1. \tag{2.15}$$

The following is the GN inequality with fractional derivatives.

**Corollary 2.4** *Let $1 < p, p_0, p_1 < \infty$, $s$, $s_1 \in \mathbb{R}$, $0 \leqslant \theta \leqslant 1$. Then the fractional GN inequality of the following type*

$$\|u\|_{\dot{H}^s_p} \lesssim \|u\|^{1-\theta}_{L^{p_0}} \|u\|^{\theta}_{\dot{H}^{s_1}_{p_1}} \tag{2.16}$$

*holds if and only if*

$$\frac{n}{p} - s = (1-\theta)\frac{n}{p_0} + \theta\left(\frac{n}{p_1} - s_1\right), \quad s \leqslant \theta s_1. \tag{2.17}$$

# 3 Corollaries of the GN inequalities

In this section we give some corollaries of our main results. Noticing that $BMO = \dot{F}^0_{\infty,2} \subset \dot{B}^0_{\infty,\infty}$ and $\|\nabla^s u\|_{\dot{B}^0_{p,\infty}} \lesssim \|\nabla^s u\|_p$, we can deduce the following useful interpolation inequalities:

$$\|u\|_{L^{10}(\mathbb{R}^3)} \leqslant C\|u\|^{2/3}_{\dot{B}^{-1/2}_{\infty,\infty}(\mathbb{R}^3)} \|u\|^{1/3}_{\dot{B}^1_{10/3,10/3}(\mathbb{R}^3)}, \tag{3.1}$$

$$\|u\|_{L^4} \lesssim \|\nabla u\|^{1/2}_{L^2} \|u\|^{1/2}_{\dot{B}^{-1}_{\infty,\infty}}, \tag{3.2}$$

$$\|\nabla u\|_{L^4} \lesssim \|\nabla^2 u\|^{1/2}_{L^2} \|u\|^{1/2}_{BMO}, \tag{3.3}$$

$$\|u\|_{L^q} \lesssim \|\nabla u\|^{\theta}_{L^p} \|u\|^{1-\theta}_{\dot{B}^{-\theta/(1-\theta)}_{\infty,\infty}}, \quad 1 \le p < q < \infty, \theta = p/q. \tag{3.4}$$

$$\|\nabla^m u\|_{L^q} \lesssim \|\nabla^k u\|_{L^p}^{\theta} \|u\|_{BMO}^{1-\theta}, \quad 1 \leqslant m < k,\ q = kp/m,\ \theta = m/k. \tag{3.5}$$

Following Bourgain [8], we can show (3.1), which is useful to obtain the concentration phenomena of the solutions of the nonlinear Schrödinger equation. Meyer and Rivière [50] studied the partial regularity of solutions for the stationary Yang-Mills fields by using (3.2) and (3.3). (3.4) and (3.5) are generalized versions of (3.2) and (3.3), respectively (see Ledoux [38], Strzelecki [55]). Machihara and Ozawa [42] showed that

**Proposition 3.1** *Let* $1 \leq p_0 \vee p_1 \leqslant p \leqslant \infty$, $0 < \theta < 1$, $s_0, s_1 \in \mathbb{R}$. *Assume that*

$$\begin{gathered} \frac{n}{p} - s = (1-\theta)\left(\frac{n}{p_0} - s_0\right) + \theta\left(\frac{n}{p_1} - s_1\right), \\ s_0 < \frac{n}{p_0} - \frac{n}{p}, \quad s_1 > \frac{n}{p_1} - \frac{n}{p}. \end{gathered} \tag{3.6}$$

*Then*

$$\|u\|_{\dot{B}^0_{p,1}} \lesssim \|u\|^{1-\theta}_{\dot{B}^{s_0}_{p_0,\infty}} \|u\|^{\theta}_{\dot{B}^{s_1}_{p_1,\infty}} \tag{3.7}$$

Oru [52] obtained that (see also [11])

**Proposition 3.2** *Let* $0 < p_0, p_1, p < \infty$, $0 < r < \infty$, $-\infty < s_0, s_1, s < \infty$, $0 < \theta < 1$ *and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad s = (1-\theta)s_0 + \theta s_1, \quad s_0 \neq s_1. \tag{3.8}$$

*Then*

$$\|u\|_{\dot{F}^s_{p,r}(\mathbb{R}^n)} \leqslant C\|u\|^{1-\theta}_{\dot{F}^{s_0}_{p_0,\infty}(\mathbb{R}^n)} \|u\|^{\theta}_{\dot{F}^{s_1}_{p_1,\infty}(\mathbb{R}^n)}. \tag{3.9}$$

The following interpolation inequality was shown in [58].

**Proposition 3.3** *Let* $0 < p_0 < p < \infty$, $0 < r \leqslant \infty$, $-\infty < s_1 < s < s_0 < \infty$, $0 < \theta < 1$ *and*

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{\infty}, \qquad s = \theta s_0 + (1-\theta)s_1. \tag{3.10}$$

*Then*

$$\|u\|_{\dot{F}^s_{p,r}(\mathbb{R}^n)} \leqslant C\|u\|^{1-\theta}_{\dot{B}^{s_1}_{\infty,\infty}(\mathbb{R}^n)} \|u\|^{\theta}_{\dot{B}^{s_0}_{p_0,p_0}(\mathbb{R}^n)}. \tag{3.11}$$

# 4 Concentration of solutions of NS equation

In the second part of this paper we consider some applications of the fractional GN inequality. First, We study the Cauchy problem for the Navier-Stokes (NS) equation

$$u_t - \Delta u + (u \cdot \nabla) u + \nabla p = 0, \quad \operatorname{div} u = 0, \quad u(0,x) = u_0(x), \tag{4.1}$$

where $\Delta = \sum_{i=1}^n \partial_{x_i}^2$, $\nabla = (\partial_{x_1}, ..., \partial_{x_n})$, $\operatorname{div} u = \partial_{x_1} u_1 + ... + \partial_{x_n} u_n$, $u = (u_1, ..., u_n)$ and $p$ are real-valued unknown functions of $(t,x) \in [0, T_m) \times \mathbb{R}^n$ for some $T_m > 0$, $u_0 = (u_0^1, ..., u_0^n)$ denotes the initial value of $u$ at $t = 0$. It is known that NS equation is local well posed in $L^n$, namely, for initial data $u_0 \in L^n(\mathbb{R}^n)$, there exists a unique local solution $u \in C([0,T_m); L^n) \cap L^{2+n}_{\mathrm{loc}}(0, T_m; L^{2+n})$ (cf. [33, 34]). Whether the local solution can be extended to a global one is still open. Recently, Escauriaza, Seregin and Šverák [21] showed that any "Leray-Hopf" weak solution in 3D which remains bounded in $L^3(\mathbb{R}^3)$ cannot develop a singularity in finite time. Kenig and Koch [34] gave an alternative approach to this problem by substituting $L^3$ with $\dot{H}^{1/2}$. Dong and Du [19] generalized their results in higher spatial dimensions $n \geqslant 3$. Noticing that $L^3 \subset B^{-1}_{\infty,\infty}$ in 3D is a sharp embedding, for any solution $u$ of the NS equation in $C([0,T^*); L^3)$, we see that $u \in C([0,T^*); B^{-1}_{\infty,\infty})$. May [49] prove that if $T^* < \infty$, then there exists a constant $c > 0$ independent of the solution of NS equation such that $\limsup_{t \to T^*} \|u(t) - \omega\|_{B^{-1}_{\infty,\infty}} \geqslant c$ for all $\omega \in \mathscr{S}$. In this paper we will use the fractional GN inequality to study the finite time blowup solution and we have the following concentration result:

**Theorem 4.1** *Let $n = 3$ and $u \in C([0,T_m); L^n \cap L^2) \cap L^{2+n}_{\mathrm{loc}}(0, T_m; L^{2+n})$ be the solution of NS equation with maximal existing time $T_m < \infty$. Then there exist $c_0 > 0$ and $\delta > 0$ such that*

$$\overline{\lim_{t \nearrow T_m}} \sup_{x_0 \in \mathbb{R}^n} \int_{|x - x_0| \leq (T_m - t)^\delta} |u(t,\, x - x_0)|^n dx \geqslant c_0, \tag{4.2}$$

*where the constant $c_0 > 0$ only depends on $\|u_0\|_n$, $\delta$ can be chosen as any positive constant less than $2/n^2$.*

# 5 Minimizing problem

As the second application to the fractional GN inequality, we consider the existence of the radial and radially decreasing non-negative solutions for the following system:

$$(m^2 - \triangle)^s u_i - [G(u) * V]\, \partial_i G(u) + r_i u_i = 0, \quad i = 1, ..., L, \tag{5.1}$$

where $m^2 \geqslant 0$, $u = (u_1, ..., u_L)$, $u_i \geqslant 0$ and $u \neq 0$, $G: \mathbb{R}^L_+ \to \mathbb{R}_+ = [0, \infty)$ is a differentiable function, $\partial_i G(v_1, ..., v_L) := \partial G(v_1, ..., v_L)/\partial v_i$. $V(x) = |x|^{-(n-\beta)}$, $*$ denotes the convolution in $\mathbb{R}^n$, $r_i > 0$. In order to work out a desired solution of (5.1), it suffices to consider the existence of the radial and radially decreasing non-negative and non-zero minimizers of the following variational problem. We write for $c_1, ..., c_L > 0$,

$$S_c = \left\{u = (u_1, ..., u_L) \in (H^s)^L : \ \|u_i\|_2^2 = c_i, \ i = 1, ..., L\right\}. \tag{5.2}$$

We will consider the variation problem

$$M_c = \inf\{E(u) : \ u \in S_c, \ c_1, ..., c_L > 0\}, \tag{5.3}$$

where

$$E(u) = \frac{1}{2}\sum_{i=1}^{L} \|(m^2 + |\xi|^2)^{s/2}\widehat{u}_i\|_2^2 - \int\int G(u(x))V(|x-y|)G(u(y))dxdy. \tag{5.4}$$

Fractional calculus has gained tremendous popularity during the last two decades thanks to its applications in widespread domains of sciences, economics and engineering, see [1, 6, 35, 37]. Fractional powers of the Laplacian arise in many areas. Some of the fields of applications of fractional Laplacian models include medicine where the equation of motion of semilunar heart value vibrations and stimuli of neural systems are modeled by a Capulo fractional Laplacian; cf. [20, 41]. It also appears in modeling populations [53], flood flow, material viscoelastic theory, biology dynamics, earthquakes, chemical physics, electromagnetic theory, optic, signal processing, astrophysics, water wave, bio-sciences dynamical process and turbulence; cf. [1, 2, 6, 7, 13, 14, 18, 24, 23, 35, 37, 39, 43, 44, 56].

In [39], Lieb and Yau studied the existence and symmetry of ground state solutions for the boson equation in three dimensions:

$$(m^2 - \triangle)^{1/2}u - \left(|x|^{-1} * u^2\right)u + ru = 0, \tag{5.5}$$

Taking $G(u) = u^2$ and $V(x) = |x|^{-1}$ in three dimensions, (5.1) is reduced to (5.5). The variational problem associated with (5.5) is

$$M_c^{(3)} = \inf\left\{\frac{1}{2}\|(m^2 + |\xi|^2)^{1/4}\widehat{u}\|_2^2 - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}dxdy : \ u \in H^{1/2}, \ \|u\|_2^2 = c\right\}. \tag{5.6}$$

As indicated in [39], (5.5) and (5.6) play a fundamental role in the mathematical theory of gravitational collapse of boson stars. Indeed, Lieb and Yau essentially showed that

- For $m^2 = 0$, $s = 1/2$, there exists $c_* > 0$, such that (5.5) has a non-negative radial solution if and only if $c = c_*$.

- For $m^2 > 0$, $s = 1/2$, (5.5) has a non-negative radial solution if and only if $0 < c < c_*$, where $c^*$ is the same as in the case $m^2 = 0$.

It was proven in [39] that boson stars with total mass strictly less than $c^*$ are gravitationally stable, whereas boson stars whose total mass exceed $c^*$ may undergo a "gravitational collapse" based on variational arguments and many-body quantum theory. The main tools used by Lieb and Yau are the Hardy-Littlewood-Sobolev inequality together with some rearrangement inequalities. Inspired and motivated by Lieb and Yau's work, Frank and Lenzemann [26] recently showed the uniqueness of ground states to (5.5).

Taking $G(u) = u^2$ and $V(x) = |x|^{-(n-2)}$ in $n$-dimensions with $n \geqslant 3$, (5.1) is reduced to the general Choquard-Peckard equation

$$(m^2 - \triangle)^s u - \left(|x|^{-(n-2)} * u^2\right) u + ru = 0. \tag{5.7}$$

The variational problem associated with (5.7) is

$$M_c^{(n)} = \inf \left\{ \frac{1}{2} \|(m^2 + |\xi|^2)^{s/2} \widehat{u}\|_2^2 - \Upsilon_2(u) : \, u \in H^s, \; \|u\|_2^2 = c \right\}, \tag{5.8}$$

where

$$\Upsilon_\beta(u) = \int \frac{|u(x)|^2 |u(y)|^2}{|x-y|^{n-\beta}} dxdy. \tag{5.9}$$

Taking $G(u) = u_1^2 + u_2^2$ and $V(x) = |x|^{-1}$ in 3-dimensions, (5.1) is reduced to the following system

$$(m^2 - \triangle)^s u_i - \left(|x|^{-1} * (u_1^2 + u_2^2)\right) u_i + r_i u_i = 0, \;\; i = 1, 2, \tag{5.10}$$

which was studied in [4] and [26] in the cases $s = 1$ and $s = 1/2$, respectively. If we treat $u = (u_1, u_2)$ and $\|u\|_X^2 = \|u_1\|_X^2 + \|u_2\|_X^2$, we see that the variational problem associated with (5.10) is the same as in (5.8) if one constraint $\|u_1\|_2^2 + \|u_2\|_2^2 = c$ is considered.

Now we state our main result on the existence of the minimizer of (5.3). There are two kinds of basic nonlinearities, one is $G(u) = u_1^{\mu_1} ... u_L^{\mu_L}$ and another is $G(u) = u_1^\mu + ... + u_L^\mu$. For the former case, we need to use $m$-constraints $\|u_i\|_2^2 = c_i > 0$ to prevent the situation that the second term of $E(u)$ in (5.4) vanishes. For the later case, one can use one constraint $\|u_1\|_2^2 + ... + \|u_L\|_2^2 = c$. Let $s \geqslant (n-\beta)/2$. We first consider the former case and our main assumptions on $G$ are the following:

(G1) $G : \mathbb{R}_+^L \ni (v_1, ..., v_L) \to G(v_1, ..., v_L) \in \mathbb{R}_+$ is a continuous function and there exists $\mu \in [2, 1 + (2s+\beta)/n)$ such that

$$G(v) \leqslant C(|v|^2 + |v|^\mu), \;\; v = (v_1, ..., v_L). \tag{5.11}$$

Moreover, there exist $\alpha_i > 0$ such that for all $0 < v_1, ..., v_L \ll 1$,

$$G(v) \geqslant c v_1^{\alpha_1} v_2^{\alpha_2} ... v_L^{\alpha_L}. \tag{5.12}$$

where $0 < n + \beta - n(\alpha_1 + ... + \alpha_L) + 2s$.

(G2) If $v$ has a zero component, then $G(v) = 0$. The function $G \otimes G : \mathbb{R}_+^L \times \mathbb{R}_+^L \ni (u, v) \to G(u)G(v) \in \mathbb{R}_+$ is a super-modular[1].

(G3) $G(t_1 v_1, ..., t_L v_L) \geqslant t_{\max} G(v_1, ..., v_L)$ for any $t_i \geqslant 1$, where $t_{\max} = \max(t_1, ..., t_L)$.

Noticing that $v_1^{\alpha_1} v_2^{\alpha_2} ... v_L^{\alpha_L} \leq |v|^{\alpha_1 + ... + \alpha_L}$, we see that condition (5.11) covers the nonlinearity $G(v) = v_1^{\alpha_1} v_2^{\alpha_2} ... v_L^{\alpha_L}$ if $\alpha_1 + ... + \alpha_L \in [2, \mu]$. Our main result on the existence of the minimizer of (5.3) is the following:

**Theorem 5.1** *Let $m^2 \geqslant 0$, $0 < \beta < n$, $s > (n - \beta)/2$. Assume that conditions* (G1)–(G3) *are satisfied. Then* (5.3) *admits a radial and radially decreasing minimizer in $(H^s)^L$.*

We point out that both conditions $s \geqslant (n-\beta)/2$ and $0 < n + \beta - n(\alpha_1 + ... + \alpha_L) + 2s$ are necessary for Theorem 5.1. Indeed, we can give a counterexample to show that $M_c = -\infty$ if $s < (n - \beta)/2$ or $0 > n + \beta - n(\alpha_1 + ... + \alpha_L) + 2s$ for a class of nonlinearities $G(u)$.

The endpoint case $s = (n - \beta)/2$ can not be handled in Theorem 5.1. Note that for $s = (n - \beta)/2$, we have $\mu = 2$ in (5.11), a basic example is $G(u) = u_1^2 + ... + u_L^2$. Now we consider the variational problem

$$M_{c,\beta}^{(n)} = \inf \left\{ \frac{1}{2} \|(m^2 + |\xi|^2)^{s/2} \widehat{u}\|_2^2 - \Upsilon_\beta(u) : \, u \in (H^s)^L, \; \|u\|_2^2 = c > 0 \right\}. \tag{5.13}$$

where $u = (u_1, ..., u_L)$, $|u|^2 = u_1^2 + ... + u_L^2$ and $\|u\|_X^2 = \|u_1\|_X^2 + ... + \|u_L\|_X^2$. Using the definition of the Riesz potential, the Plancherel identity, the Hardy-Littlewood-Sobolev, and fractional GN inequalities, we have

$$\begin{aligned} \Upsilon_\beta(u) &= C(n, \beta) \int |u(x)|^2 [(-\Delta)^{-\beta/2} |u|^2](x) dx = \|(-\Delta)^{-\beta/4} |u|^2\|_2^2 \\ &\leqslant C \left( \|u_1\|_{4n/(n+\beta)}^2 + ... + \|u_L\|_{4n/(n+\beta)}^2 \right)^2 \\ &\leqslant C \left( \|u_1\|_2 \|u_1\|_{\dot{H}^{(n-\beta)/2}} + ... + \|u_L\|_2 \|u_L\|_{\dot{H}^{(n-\beta)/2}} \right)^2 \\ &\leqslant C \|u\|_2^2 \|u\|_{\dot{H}^{(n-\beta)/2}}^2. \end{aligned} \tag{5.14}$$

Define

$$C^* = \sup_{u \in H^{(n-\beta)/2} \setminus \{0\}} \frac{\Upsilon_\beta(u)}{\|u\|_2^2 \|u\|_{\dot{H}^{(n-\beta)/2}}^2}. \tag{5.15}$$

**Theorem 5.2** *Let $m^2 = 0$, $0 < \beta < n$, $s = (n - \beta)/2$, $G(u) = u_1^2 + ... + u_L^2$. Then* (5.13) *admits a radial and radially decreasing minimizer in $(H^s)^L$ if and only if $c = 1/2C^*$.*

[1] $F$ is said to be a supermodular if ([40])

$$F(y + he_i + ke_j) + F(y) \geqslant F(y + he_i) + F(y + ke_j) \; (i \neq j, \; h, k > 0),$$

where $y = (y_1, ..., y_L)$, and $e_i$ denotes the $i$-th standard basis vector in $\mathbb{R}^L$. It is known that a smooth function is a supermodular if all its mixed second partial derivatives are nonnegative.

As a straightforward consequence of Theorem 5.1, we see that (5.8) admits a radial and radially decreasing minimizer in $H^{(n-2)/2}$ if and only if $c = 1/2C^*$, where $\beta = 2$ in the definition of $C^*$.

In the case $m^2 > 0$ we have the following

**Theorem 5.3** *Let $m^2 > 0$, $0 < \beta < n$, $s = (n-\beta)/2$, $c > 0$. Then we have*

(1) *If $n > 2+\beta$, then (5.13) has no minimizer in in $(H^s)^L$.*

(2) *If $n < 2+\beta$, then (5.13) admits a radial and radially decreasing minimizer in $(H^s)^L$ if and only if $0 < c < 1/2C^*$.*

(3) *If $n = 2+\beta$, then (5.13) admits a radial and radially decreasing minimizer in $(H^s)^L$ if and only if $c = 1/2C^*$.*

## 6 Sketch Proofs of the GN inequalities

Let us start with an interpolation inequality in Besov spaces, see [28, 30].

**Proposition 6.1 (Convexity Hölder's inequality)** *Let $0 < p_i, q_i \leqslant \infty$, $0 \leqslant \theta_i \leqslant 1$, $\sigma_i, \sigma \in \mathbb{R}$ $(i = 1, \dots, N)$, $\sum_{i=1}^N \theta_i = 1$, $\sigma = \sum_{i=1}^N \theta_i\sigma_i$, $1/p = \sum_{i=1}^N \theta_i/p_i$, $1/q = \sum_{i=1}^N \theta_i/q_i$. Then $\cap_{i=1}^N \dot B^{\sigma_i}_{p_i,q_i} \subset \dot B^{\sigma}_{p,q}$ and for any $v \in \cap_{i=1}^N \dot B^{\sigma_i}_{p_i,q_i}$,*

$$\|v\|_{\dot B^{\sigma}_{p,q}} \leqslant \prod_{i=1}^N \|v\|^{\theta_i}_{\dot B^{\sigma_i}_{p_i,q_i}}.$$

*This estimate also holds if one substitutes $\dot B^{\sigma}_{p,q}$ by $\dot F^{\sigma}_{p,q}$ $(p, p_i \neq \infty)$.*

In the convexity Hölder inequality, condition $1/q = \sum_{i=1}^N \theta_i/q_i$ can be replaced by $1/q \leqslant \sum_{i=1}^N \theta_i/q_i$. Indeed, noticing that $\ell^q \subset \ell^p$ for all $q \leqslant p$, we see that Proposition 6.1 still holds if $1/q < \sum_{i=1}^N \theta_i/q_i$. In [28, 30], Proposition 6.1 was stated as the case $1 \leq p_i, q_i \leq \infty$, however, the proof in [30] is also adapted to the case $0 < p_i, q_i \leq \infty$.

**Sketch Proof of Theorem 2.1** (Sufficiency) First, we consider the case $1/q \leqslant (1-\theta)/q_0 + \theta/q_1$. By (2.3), we have

$$\frac{1}{p} - \frac{1-\theta}{p_0} - \frac{\theta}{p_1} = \frac{s}{n} - (1-\theta)\frac{s_0}{n} - \theta\frac{s_1}{n} := -\eta \leqslant 0. \tag{6.1}$$

Take $p^*$ and $s^*$ satisfying

$$\frac{1}{p^*} = \frac{1}{p} + \eta, \quad s^* = s + n\eta.$$

Applying the convexity Hölder inequality, we have

$$\|f\|_{\dot B^{s^*}_{p^*,q}} \leqslant \|f\|^{1-\theta}_{\dot B^{s_0}_{p_0,q_0}} \|f\|^{\theta}_{\dot B^{s_1}_{p_1,q_1}}. \tag{6.2}$$

Using the inclusion $\dot{B}^{s^*}_{p^*,q} \subset \dot{B}^s_{p,q}$, we get the conclusion.

Next, we need to consider the following two cases: (i) $s = (1-\theta)s_0 + \theta s_1$, $p_0 = p_1$ and $s_0 \neq s_1$; (ii) $s < (1-\theta)s_0 + \theta s_1$ and $s - n/p \neq s_0 - n/p_0$. We can show that

$$\|f\|_{\dot{B}^s_{p,q}} \leqslant \|f\|^{1-\theta}_{\dot{B}^{s_0}_{p_0,\infty}} \|f\|^{\theta}_{\dot{B}^{s_1}_{p_1,\infty}}, \tag{6.3}$$

(6.3) implies the result, as desired.

(Necessity) By scaling

$$\|f(\lambda\cdot)\|_{\dot{B}^s_{p,q}} \sim \lambda^{s-n/p}\|f\|_{\dot{B}^s_{p,q}}, \quad \lambda \in 2^{\mathbb{Z}},$$

We immediately obtain that $s - n/p - [(1-\theta)(s_0 - n/p_0) + \theta(s_1 - n/p_1) = 0$.

Next, we show that $s - s_0 \leqslant \theta(s_1 - s_0)$. Assume on the contrary that $s - s_0 > \theta(s_1 - s_0)$. Assume that $s_0 = 0$. Let $\varphi$ satisfy supp $\varphi \subset \{\xi : 1/2 \leqslant |\xi| \leqslant 3/2\}$ and $\varphi(\xi) = 1$ for $3/4 \leqslant |\xi| \leqslant 1$. So, $\varphi(2^{-j}\xi) = 1$ if $3 \cdot 2^{j-2} \leqslant |\xi| \leqslant 2^j$. Denoting

$$\rho_j(\xi) = \varphi(2(\xi - \xi^{(j)})), \quad \xi^{(j)} = (7 \cdot 2^{j-3}, 0, ..., 0). \tag{6.4}$$

and for sufficiently small $\varepsilon > 0$, we write

$$\hat{f}(\xi) = \sum_{j=100}^{N} 2^{\varepsilon j} \rho_j(\xi). \tag{6.5}$$

We see that

$$\|f\|_{\dot{B}^s_{p,q}} \sim 2^{(s+\varepsilon)N}, \quad \|f\|_{\dot{B}^0_{p_0,q_0}} \sim 2^{\varepsilon N}, \quad \|f\|_{\dot{B}^{s_1}_{p_1,q_1}} \sim 2^{(s_1+\varepsilon)N}.$$

By (2.1), we obtain that $2^{(s+\varepsilon)N} < 2^{\varepsilon N} 2^{s_1 \theta N}$. However, for sufficiently large $N$, it contradicts the fact $s > \theta s_1$. Substituting $s$ by $s - s_0$, we get the proof in the case $s_0 \neq 0$.

Thirdly, we consider the case $p_0 \neq p_1$ and $s = (1-\theta)s_0 + \theta s_1$ and show that $1/q \leqslant (1-\theta)/q_0 + \theta/q_1$. Put

$$\lambda = \frac{s_1 - s_0}{n(1/p_0 - 1/p_1)}. \tag{6.6}$$

We see that

$$s + n\lambda\left(\frac{1}{p} - 1\right) = s_0 + n\lambda\left(\frac{1}{p_0} - 1\right) = s_1 + n\lambda\left(\frac{1}{p_1} - 1\right). \tag{6.7}$$

*Case 1.* We consider the case $\lambda \geqslant 0$. Let $\varphi$ and $\xi^{(j)}$ be as in (6.4). Put

$$\varrho^\lambda_j := \varphi(2^{\lambda j}(\xi - \xi^{(j)}))$$

and

$$\widehat{F} = \sum_{j=100}^{J} 2^{-sj - n\lambda(1/p-1)j} \varrho^\lambda_j. \tag{6.8}$$

We have

$$\|F\|_{\dot{B}^s_{p,q}} \sim J^{1/q}, \quad \|F\|_{\dot{B}^{s_0}_{p_0,q_0}} \sim J^{1/q_0}, \quad \|F\|_{\dot{B}^{s_1}_{p_1,q_1}} \sim J^{1/q_1}, \tag{6.9}$$

By (2.1), we have $J^{1/q} \lesssim J^{(1-\theta)/q_0} J^{\theta/q_1}$ for any $J \gg 1$. It follows that $1/q \leqslant (1-\theta)/q_0 + \theta/q_1$.

*Case 2.* We consider the case $\lambda < 0$. Denote

$$\varphi^{(N)} = \varphi(2^{-N}\,\cdot), \quad \varphi^{(N)}_j = \varphi(2^{-j-N}\,\cdot), \quad \triangle_{j,N} = \mathscr{F}^{-1}\varphi^{(N)}_j \mathscr{F}.$$

It is easy to see that

$$\|f\|^{(N)}_{\dot{B}^s_{p,q}} = \left(\sum_j (2^{sj}\|\triangle_{j,N}\|_p)^q\right)^{1/q}$$

is an equivalent norm on $\dot{B}^s_{p,q}$ (see also [57]). Let

$$\widehat{F} = \sum_{j=100}^{J} 2^{-sj-n\lambda(1/p-1)j}\varphi(2^{\lambda j}\,\cdot). \tag{6.10}$$

Assuming that $N \geqslant 100(|\lambda|+1)$, analogously to the above, we have from the definition of $\|\cdot\|^{(N)}_{\dot{B}^s_{p,q}}$ that

$$\|F\|^{(N)}_{\dot{B}^s_{p,q}} \sim J^{1/q}, \quad \|F\|^{(N)}_{\dot{B}^{s_0}_{p_0,q_0}} \sim J^{1/q_0}, \quad \|F\|^{(N)}_{\dot{B}^{s_1}_{p_1,q_1}} \sim J^{1/q_1}. \tag{6.11}$$

By (2.1) we have $1/q \leqslant (1-\theta)/q_0 + \theta/q_1$.

Fourthly, we show the necessity of (2.5). If not, then we have $p_0 = p_1 = p$, $s_0 = s_1 = s$ and $1/q > (1-\theta)/q_0 + \theta/q_1$. Let

$$\widehat{F} = \sum_{j=100}^{J} 2^{-sj+n(1/p-1)j}\varphi(2^{-j}\,\cdot). \tag{6.12}$$

We easily see that for $N \gg 1$,

$$\|F\|^{(N)}_{\dot{B}^s_{p,q}} \sim J^{1/q}, \quad \|F\|^{(N)}_{\dot{B}^s_{p,q_0}} \sim J^{1/q_0}, \quad \|F\|^{(N)}_{\dot{B}^s_{p,q_1}} \sim J^{1/q_1}. \tag{6.13}$$

We have $1/q \leqslant (1-\theta)/q_0 + \theta/q_1$, which is a contradiction.

Finally, we show the necessity of (2.6). Assume for a contrary that $s - n/p = s_0 - n/p_0$ and $1/q > (1-\theta)/q_0 + \theta/q_1$. Using the same way as in (6.12) and (6.13), we have a contraction. $\square$

## 7 Sketch Proof of Theorem 5.2

(Necessity) Put $u_\lambda = \lambda^{n/2}u(\lambda\,\cdot)$, $s = (n-\beta)/2$. For any $\phi \in (H^s)^L$, we write

$$I_{c,\beta}^{(n)}(\phi) = \frac{1}{2}\|\phi\|_{\dot{H}^s}^2 - \Upsilon_\beta(\phi). \tag{7.1}$$

we have

$$I_{c,\beta}^{(n)}(\phi_\lambda) = \lambda^{n-\beta}\left(\frac{1}{2}\|\phi\|_{\dot{H}^s}^2 - \Upsilon_\beta(\phi)\right). \tag{7.2}$$

By (5.15),

$$\Upsilon_\beta(u) = \int_{\mathbb{R}^{2n}} \frac{|u(x)|^2|u(y)|^2}{|x-y|^{n-\beta}}dxdy \leq C^*c\|u\|_{\dot{H}^s}^2.$$

Using the scaling argument, we can show that $C^*c = 1/2$.

(Sufficiency) First, we show that $M_{c,\beta}^{(n)} = 0$. Since $C^*c = 1/2$, we have

$$\Upsilon_\beta(u) \leqslant \frac{1}{2}\|u\|_{\dot{H}^s}^2.$$

It follows that $M_{c,\beta}^{(n)} \geqslant 0$. On the other hand, for any $\varepsilon > 0$, we find some $\phi \in (\dot{H}^s)^L$ satisfying

$$\Upsilon_\beta(\phi) \geqslant \frac{1-\varepsilon}{2}\|\phi\|_{\dot{H}^s}^2.$$

For $s = (n-2)/2$, the above inequality is invariant under the scaling $\phi \mapsto \lambda^{n/2}\phi(\lambda\,\cdot)$, which implies that we can assume that $\|\phi\|_{\dot{H}^s} = 1$. It follows that $I_{c,\beta}^{(n)}(\phi) \leqslant \varepsilon$. Hence $M_c = 0$.

Now, let $u_k$ be a sequence verifying

$$\frac{\Upsilon_\beta(u_k)}{\|u_k\|_2^2\|u_k\|_{\dot{H}^s}^2} \geqslant C^*\left(1 - \frac{1}{k}\right). \tag{7.3}$$

Let $u_k^*$ be the rearrangement of $u_k$. Using the fact that

$$\Upsilon_\beta(u_k) \leq \Upsilon_\beta(u_k^*),\ \|u^*\|_{\dot{H}^s} \leqslant \|u\|_{\dot{H}^s},\ \|u^*\|_2 = \|u\|_2,$$

we see that (7.3) also holds if $u_k$ is replaced by $u_k^*$. One can find $\lambda_k > 0$ such that $\|\lambda_k^{n/2}u_k^*(\lambda_k\,\cdot)\|_{\dot{H}^s} = 1$. Since (7.3) is invariant under the scaling $u_k^* \mapsto \lambda^{n/2}u_k^*(\lambda\,\cdot)$, we see that for $v_k = \lambda_k^{n/2}u_k^*(\lambda_k\,\cdot)$,

$$\frac{\Upsilon_\beta(v_k)}{\|v_k\|_2^2\|v_k\|_{\dot{H}^s}^2} \geqslant C^*\left(1 - \frac{1}{k}\right) \tag{7.4}$$

and $\|v_k\|_2^2 = c$, $\|v_k\|_{\dot{H}^s} = 1$. The inequality (7.4) also implies that $I_{c,\beta}^{(n)}(v_k) \leqslant 1/2k \to 0$. It follows that $v_k$ is a radial and radially decreasing minimizing sequence. In view of

$\|v_k\|_{H^s}^2 \leqslant 1+c$ we see that $v_k$ has a subsequence which is still written by $v_k$ such that $v_k$ converges to $v$ with respect to the weak topology in $(H^s)^L$. On the other hand, the embedding $H^s \subset L^q$ with $s=(n-\beta)/2$, $2<q<2n/\beta$ is compact for the class of radial functions, we see that $v_k$ strongly converges to $v$ (up to a subsequence) in $(L^q)^m$ for all $2<q<2n/\beta$. By (7.4) and Theorem 2.2, we have for $k \geqslant 2$,

$$1/4 \leqslant \Upsilon_\beta(v_k) \leqslant C\||v_k|^2\|_{2n/(n+\beta)}^2 \leq C\|v_k\|_{B^s_{2,\infty}}^2 \|v_k\|_{B^{-n/2}_{\infty,\infty}}^2. \tag{7.5}$$

It follows that $\|v_k\|_{B^{-n/2}_{\infty,\infty}} \geqslant c_0$, where $c_0 := 1/2\sqrt{C}$ is independent of $k$. Let $v_k = (v_k^1, ..., v_k^L)$. It is easy to see that there exist $i \in \{1,2,...,L\}$ and a subsequence of $v_k^i$ which is still written by $v_k^i$ verifying $\|v_k^i\|_{B^{-n/2}_{\infty,\infty}} \geqslant c_0/L$. From the definition of $B^a_{\infty,\infty}$ we can choose $j_k \in \mathbb{Z}_+$ and $x_k \in \mathbb{R}^n$,

$$c_0/2L \leqslant 2^{-nj_k/2}|(\triangle_{j_k} v_k^i)(x_k)|. \tag{7.6}$$

By (7.6), we find some $A \gg 1$ such that We have

$$\|v_k^i\|_{L^2(|\cdot - x_k| \leq A)} \geqslant c_0/4C.$$

Since $v_k^i$ is radial, we have $|x_k| \leq X_0 := X_0(c_0, C, A)$. Indeed, in the opposite case we will have $\|v_k^i\|_2^2 > c$ if $|x_k| \gg 1$. So, we further have

$$\|v_k^i\|_{L^2(|\cdot| \leq X_0 + A)} \geqslant c_0/4C.$$

By Hölder's inequality,

$$\|v_k^i\|_{L^q(|\cdot| \leq X_0 + A)} \geqslant \tilde{c}_0, \quad \tilde{c}_0 := \tilde{c}_0(A, X_0, c_0).$$

Since $v_k \to v$ in $(L^q)^L$, $2<q<2n/\beta$, we immediately have $v \neq 0$. Using the same way as in the proof of Theorem 5.1, we can get that

$$0 \leq I^{(n)}_{c,\beta}(v) \leq I^{(n)}_{c,\beta}(v_k) \to 0.$$

It follows that $I^{(n)}_{c,\beta}(v) = 0$. To finish the proof, it suffices to show that $\|v\|_2^2 = c$. If not, then we have $\|v\|_2^2 < c$. Putting $\tilde{v} = \sqrt{c}v/\|v\|_2$, we have

$$I^{(n)}_{c,\beta}(\tilde{v}) < 0, \tag{7.7}$$

which contradicts the fact that $I^{(n)}_{c,\beta}(u) \geqslant 0$ for all $u \in (H^s)^L$.

# References

[1] Ravi. P. Agarwal, M. Ben Chohra, J. J. Nieto, A. Ouhab, *Fractional Differential Equations and Inclusions*, In Press.

[2] L. Abdelouaheb, J. L. Bona, M. Felland, J. C. Saut, Nonolocal models for nonlinear dispersive waves, Phys D **40** (1989), 360-393.

[3] F. J. Almgren, E. H. Lieb : Symmetric decreasing rearragement is sometimes continous, J. Amer. Math. Soc., **2** ( 1989), 683-773.

[4] W. H. Aschbacher, M. Squassina, On phase segregation in nonlocal two-particle Hartree systems, Cent. Eur. J. Math., **7** (2009)2, 30-248

[5] H. Bahouri, P. Gérard and C. J. Xu, Espace de Besov et estimations de Strichartz généralisées sur le group de Heisenberg, J. d'Anal. Math., **82** (2000), 93–118.

[6] D. Balenu, K. Diethelm, E. Scalas, J. J. Trujillo, Fractional calculus models and numerical methods. Amsterdam, Word Science Publish, 450 pp. Nonlinear Science and complexity.

[7] J. P. Bouchard, A. Georges, Anomalous diffusion in disordered media, Statistical Mechanics, models and physical applications, Phy. Reports **195**, 1990.

[8] J. Bourgain, Global well posedness of defocusing critical nonlinear Schrödinger equation in the radial case, J. Amer. Math. Soc., **12** (1999), 145–171.

[9] J. Bourgain, H. Brézis, P. Mironescu, Another look at Sobolev spaces. Optimal control and Partial Differential Equations. A volume dedicated in the honour of A . Bensoussan's 60th birthday. Ios Press, Amsterdam 2001, 349-355.

[10] J. Bourgain, H. Brézis, P. Mironescu, Limiting embedding theorems for $W_p^s$ when $s \to 1$, J Anal. Math., **87** (2002), 439- 455.

[11] H. Brezis and P. Mironescu, Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces, J. Evol. Equ. **1** (2001) 387-404.

[12] A . Burchard, H. Hajaiej, Rearrangement inequalities for functional with monotone integrands. J. of Funct. Anal., **233** (2006), 561-582.

[13] L. Caffarelli, L. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. To appear.

[14] A. Castro, D. Cordoba, F. Ganceo, R. Orive, Incompressible flow in porous media with fractional diffusion. Preprint.

[15] A. Cohen, R. DeVore, P. Petrushev and H. Xu, Nonlinear approximation and the space BV($\mathbb{R}^2$), Amer. J. Math. **121** (1999) 587–628.

[16] A. Cohen, W. Dahmen, I. Daubechies and R. De Vore, Harmonic analysis of the space BV, Rev. Mat. Iberoamericana **19** (2003), 235–263.

[17] A. Cohen, Y. Meyer and F. Oru, Improved Sobolev embedding theorem, Séminaire sur les Équations aux Dérivées Partielles, 1997–1998 (École Polytech., Palaiseau, 1998) Exp. No. XVI, 16 pp.

[18] P. Constantin, *Euler Equations, Navier Stokes Equations and Turbulence*, Mathematical foundation of turbulence viscous flows, Vol 1871, Lectures Notes in Math, 1-43, 2006.

[19] H. Dong, D. Du, The Navier-Stokes equations in the critical Lebesgue space, Comm. Math. Phys. **292** (2009), 811–827.

[20] E. Elshahed, A fractional calculus model in semilunar heart valvevibrations, International Mathematica symposium 2003.

[21] L. Escauriaza, G. A. Seregin, and V. $\check{S}$verák, $L^{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness, Uspekhi Mat. Nauk, **58** (2003), 3–44.

[22] M. Escobedo and L. Vega, A semilinear Dirac equation in $H^s(\mathbb{R}^3)$ for $s > 1$, SIAM J. Math. Anal., **28** (1997), 338–362.

[23] R. Frank, E. Lenzemann, Uniqueness of ground states of the $L^2$ critical Boson star equations, preprint.

[24] R. Frank, R. Seringer, Nonlinear ground state representation and sharp Hardy inequalities, J. Funct. Ana **255** (2008), 3407-3430.

[25] A. Friedmann, *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1969.

[26] J. Fröhich, E. Lenzmann, Dynamical collapse of white dwarfs in Hartree and Hartree-Fock theory, Commun. Math. Phys., **274** (2007), 737–750.

[27] E. Gagliardo, Proprieta di alcune classi di funzioni in pia variabili, Richerche Mat., **7** (1958), 102–137; **9** (1959), 24–51.

[28] J. Ginibre and G. Velo, Time decay of finite energy solutions of the nonlinear Klein-Gordon and Schrödinger equations, Ann. Inst. H. Poincare. Phys. Theor., **43** (1985), 399-442.

[29] B. L. Guo, *Viscosity Elimination Method and the Viscosity of Difference Scheme*, Chinese Sci. Publ., 2004.

[30] B. L. Guo and B. X. Wang, The Cauchy problem for the Davey-Stewartson systems, Comm.on Pure Appl.Math., **52** (1999), 1477–1490.

[31] H. Hajaiej and S.Kromer: A weak-strong convergence property and symmetry of minimizers of constrainted variational problems in $\mathbb{R}^N$, arXiv:1008.1939v1.

[32] I.W. Herbst : Spectral Theory of the operator $(p^2 + m^2)^1/2 - Ze^2/r$. Commun. Math. Phys. **53** (1977), 285-294.

[33] T. Kato, Strong $L^p$ solutions of the Navier-Stokes equations in $\mathbb{R}^m$ with applications to weak solutions, Math. Z. **187** (1984), 471–480.

[34] C. E. Kenig. G. S. Koch, An alternative approach to regularity for the Navier–Stokes equations in critical spaces, Preprint.

[35] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North Holland Mathematical Studies, Vol **204**, 540 pages.

[36] O. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow,* Second English edition, Mathematics and its Applications, Vol. 2, Gordon and Breach, Science Publishers, New York-London-Paris, 1969.

[37] Lakshmikantham, *Theory of Fractional Dynamic Systems*, Cambridge Sc Publ, 2009.

[38] M. Ledoux, On Improved Sobolev embedding theorems, Math. Res. Lett., **10** (2003), 659–669.

[39] E.H .Lieb, H. T. Yau, The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics, Commun. Math. Phys., **112** (1987), 147-174.

[40] G.G. Lorentz, An inequality for rearrangements, Amer. Math. Monthly, **60** (1953), 176C179.

[41] B. Lundstrom, M.Higgs, W. Spain, A. Fairhall, Fractional differentiation by neocortical pyramidal neurons. Nature Neuroscience **11** (2008), 1335- 1342.

[42] S. Machihara and T. Ozawa, Interpolation inequalities in Besov spaces, Proc. Amer. Math. Soc., **131** (2002), 1553–1556.

[43] A. J. Majda, D.W. Mc Laughin, E. G. Tabak, A one dimensional model for dispersive wave turbulence. J. Nonlinear Sci., **7** (1997), 9-44.

[44] R. L. Magin, Fractional calculus in bioenginering 1, 2, 3. Critical Reviews in Biomedical Engineering, **32** (2004), 1-104, **32** (2004), 105-193, **32** (2004), 194-377.

[45] R.L. Magin, S. Boregowda, and C. Deodhar, Modeling of pulsating peripheral bioheat transf using fractional calculus and constructal theory, Journal of Design & Nature, **1** (2007), 18-33.

[46] R.L. Magin, X. Feng, and D. Baleanu, Fractional calculus in NMR, Magnetic Resonance Engineering, **34** (2009), 16-23.

[47] R.L. Magin and M. Ovadia, Modeling the cardiac tissue electrode interface using fractional calculus, Journal of Vibration and Control, **19** (2009), 1431-1442.

[48] J. Martin and M. Milman, Sharp Gagliardo–Nirenberg inequalities via symmetrization, Math. Res. Lett. **14** (2006), 49–62.

[49] R. May, Rôle de l'space de Besov $B^{-1}_{\infty,\infty}$ dans le contrôle de l'explosion évetuelle en temps fini des solutions réguli'eres des équations de Navier-Stokes, C. R. Acad. Sci. Paris. **323** (2003), 731–734.

[50] Y. Meyer and T. Rivière, A partial regularity result for a class of stationary Yang–Mills fields, Rev. Mat. Iberoamericana **19** (2003), 195–219.

[51] L. Nirenberg, On elliptic partial differential equations, Ann. Sc. Norm. Sup. Pisa, Ser. III, **13**, (1959), 115–162.

[52] F. Oru, Rôle des oscillations dans quelques problémes d'nalyse non-linéaire, Doctorat de Ecole Normale Supérieure de Cachan, 1998.

[53] M. Rivero, J. J. Trujillo, L. Vasquez, M. P. Valesco, Fractional dynamic systems and anomalous growing of populations. Elec J. Diff. Equa. In press.

[54] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ 1970.

[55] P. Strzelecki, Gagliardo–Nirenberg inequalities with a BMO term, Bull. London Math. Soc. **38** (2006), 294–300.

[56] V. E. Tarasov, G. M. Zaslavsky, Fractional dynamics of systems with long-range interaction. Commun. Nonlinear Sci. Numer. Sim. **11** (2006), 885-889.

[57] H. Triebel, *Theory of Function Spaces,* Birkhäuser-Verlag, 1983.

[58] B. X. Wang, Concentration phenomenon for the $L^2$ critical and cuper critical nonlinear Schrödinger equation in energy spaces, Commun. Contemp. Math., **8** (2006), 309-330.