Uniform convergence of spectral shift functions

By

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Abstract

The spectral shift function $\xi_L(E)$ for a Schrödinger operator restricted to a finite cube of length L in multi-dimensional Euclidean space, with Dirichlet boundary conditions, counts the number of eigenvalues less than or equal to $E \in \mathbb{R}$ created by a perturbation potential V. We study the behavior of this function $\xi_L(E)$ as $L \to \infty$ for the case of a compactly-supported and bounded potential V. After reviewing results of Kirsch [Proc. Amer. Math. Soc. **101**, 509–512 (1987)], and our recent pointwise convergence result for the Cesàro mean [Proc. Amer. Math. Soc. **138**, 2141–2150 (2010)], we present a new result on the convergence of the energy-averaged spectral shift function that is uniform with respect to the location of the potential V within the finite box.

§ 1.   Statement of the Problem and Result

In the late eighties, W. Kirsch [11, 12] investigated the relative eigenvalue counting function $\xi_L(E)$ for compactly-supported nonnegative perturbations V of the nonnegative Laplacian $-\Delta_L \geqslant 0$ on cubes $\Lambda_L :=]-L/2, L/2[^d \subset \mathbb{R}^d$ with Dirichlet boundary conditions. The cubes Λ_L of edge length $L > 0$ are centered about the origin in d-dimensional Euclidean space $\mathbb{R}^d$. The real-valued local perturbation V is a compactly-supported, nonnegative potential $0 \leqslant V \in \mathrm{L}^\infty(\mathbb{R}^d)$. We emphasize that V does not depend on L and, of course, L is large enough so that $\mathrm{supp}(V) \subset \Lambda_L$.

The finite-volume spectral shift function (SSF) or relative eigenvalue counting function $\xi_L(E)$ for the pair of self-adjoint operators $(-\Delta_L/2 + V, -\Delta_L/2)$ on the Hilbert

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space $L^2(\Lambda_L)$, with Dirichlet boundary conditions on $\partial\Lambda_L$, is defined as the real-valued function of $E \in \mathbb{R}$ given by

$$(1.1) \quad \xi_L(E) := \#\left\{\begin{array}{c}\text{eigenvalues of } -\Delta_L/2 \\ \text{(counting multipl.)} \leqslant E\end{array}\right\} - \#\left\{\begin{array}{c}\text{eigenvalues of } -\Delta_L/2 + V \\ \text{(counting multipl.)} \leqslant E\end{array}\right\}.$$

Kirsch was interested in the limit of $\xi_L(E)$ as $L \to \infty$. Perhaps surprisingly, Kirsch proved [11] that if $d \in \mathbb{N} \setminus \{1\}$ and if $\int_{\mathbb{R}^d} \mathrm{d}x\, V(x) > 0$, then

$$(1.2) \qquad\qquad \sup_{L>0} \xi_L(E) = \infty \qquad \text{for every } E > 0.$$

Furthermore, he proved the existence of a countable dense set of energies $\mathcal{E} \subset [0, \infty[$ so that

$$(1.3) \qquad\qquad \sup_{L\in\mathbb{N}} \xi_L(E) = \infty \qquad \text{for every } E \in \mathcal{E}.$$

Clearly, the reason for this divergence is the growing degeneracy of eigenvalues of the Laplacian in $d \geqslant 2$ dimensions as $L \to \infty$, which is lifted by the perturbation V. In contrast, in $d = 1$ space dimension, and also for corresponding lattice systems in arbitrary dimension, the spectral shift function remains bounded in this limit, as follows from a finite-rank-perturbation argument.

   More generally, if one replaces the sequence $L_n = n \in \mathbb{N}$ in (1.3) by another diverging sequence of lengths, one would expect the set of "bad" energies $\mathcal{E}$ to change. One might conjecture, however, that the largest set of energies $\mathcal{E}$, on which ξ_L explodes, still has zero Lebesgue measure. Although still unproven in this generality, this conjecture is strongly supported by Theorems 1.1 and 1.2 below.

   In order to state these theorems, we need to list the hypotheses. We write $\mathcal{K}(\mathbb{R}^d)$ and $\mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$ to denote the Kato class and the local Kato class, respectively [1, 15]. We say U is Kato decomposable, $U \in \mathcal{K}_{\pm}(\mathbb{R}^d)$, if $\max\{0, U\} \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$ and $\max\{0, -U\} \in \mathcal{K}(\mathbb{R}^d)$. In the following we consider two real-valued potential functions U and V on $\mathbb{R}^d$ such that

$$(\star) \qquad U \in \mathcal{K}_{\pm}(\mathbb{R}^d), \qquad 0 \leqslant V \in \mathcal{K}(\mathbb{R}^d), \quad \mathrm{supp}(V) \subseteq \Lambda_\ell \text{ for some } \ell > 0.$$

We also introduce the corresponding infinite-volume self-adjoint Schrödinger operators $H_0 := -(\Delta/2) + U$ and $H_1 := H_0 + V$ on $L^2(\mathbb{R}^d)$. Their self-adjoint finite-volume Dirichlet restrictions $H_0^{(L)}$ and $H_1^{(L)}$ to $L^2(\Lambda_L)$ have compact resolvents and, therefore, discrete spectrum. For a given energy $E \in \mathbb{R}$, let $N_0^{(L)}(E)$, resp. $N_1^{(L)}(E)$, denote the number of eigenvalues, including multiplicity, for $H_0^{(L)}$, resp. $H_1^{(L)}$, less than or equal to E. These are both monotone increasing functions of the energy E. We define the relative eigenvalue counting function by

$$(1.4) \qquad E \mapsto \xi_L(E) \equiv \xi(E; H_1^{(L)}, H_0^{(L)}) := N_0^{(L)}(E) - N_1^{(L)}(E) \geqslant 0$$

for all $E \in \mathbb{R}$. It is known that this function is equal to the (more generally defined) *spectral shift function* for the pair $(H_1^{(L)}, H_0^{(L)})$, see e.g. [16, 2] or Eq. (5.1) in the Appendix of [10].

Theorem 1.1. *Let $d \in \mathbb{N}$ and assume $(\star)$. Then, we have*

$$(1.5) \qquad \lim_{L \to \infty} \int_{\mathbb{R}} \mathrm{d}E \, \xi_L(E) \, f(E) = \int_{\mathbb{R}} \mathrm{d}E \, \xi(E) \, f(E)$$

for every function f of the form $f = \chi_I g$, where $g \in C(\mathbb{R})$ is continuous and χ_I is the indicator function of a (finite) interval $I \subset \mathbb{R}$. In particular, for Lebesgue-almost all $E \in \mathbb{R}$ we have

$$(1.6) \qquad \lim_{\delta \downarrow 0} \lim_{L \to \infty} \frac{1}{\delta} \int_E^{E+\delta} \mathrm{d}E' \, \xi_L(E') = \xi(E).$$

We refer to [10] for a proof of the theorem, see also [6]. Kirsch's result (1.2) shows that one cannot get rid of the energy smoothing in (1.6), that is, the limits $\delta \downarrow 0$ and $L \to \infty$ must not be interchanged. The best one could hope for is convergence Lebesgue-almost everywhere of $(\xi_{L_j})_{j \in \mathbb{N}}$ for sequences of diverging lengths. The next theorem is a partial result in this direction.

Theorem 1.2. *Let $d \in \mathbb{N}$ and assume $(\star)$. Then, for every sequence of lengths $(L_j)_{j \in \mathbb{N}} \subset {]0, \infty[}$ with $\lim_{j \to \infty} L_j = \infty$ there exists a subsequence $(j_i)_{i \in \mathbb{N}} \subset \mathbb{N}$ with $\lim_{i \to \infty} j_i = \infty$ such that for every subsequence $(i_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ with $\lim_{k \to \infty} i_k = \infty$ we have*

$$(1.7) \qquad \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} \xi_{\widetilde{L}_k}(E) \leqslant \xi(E)$$

for Lebesgue-almost all $E \in \mathbb{R}$. Here we have set $\widetilde{L}_k := L_{j_{i_k}}$ for all $k \in \mathbb{N}$.

The simple proof [10] of Theorem 1.2 relies on Theorem 1.1, the fact that V, and hence ξ_L, has a definite sign, and a deep result of Komlós. For one-dimensional systems, equality in (1.7) has recently been shown in [3]. We refer to the literature cited in [10] for further estimates of spectral shift functions.

In this note we are interested in refining the convergence in Theorem 1.1 for the case where the background potential U is periodic with period $p = (p_1, \ldots, p_d) \in \mathbb{R}^d$. In this situation, one can consider the potential V centered at any point $x_0 + x \in \Lambda_L$, with $x \in p\mathbb{Z}^d$ and $x_0 \in \Lambda_L$, so that $\Lambda_\ell(x_0) \subset \Lambda_L$. One expects that the limit in (1.5) is independent of the shift $x \in p\mathbb{Z}^d$. We prove that this is true and that the convergence in (1.5) is uniform in the shift vectors $x \in p\mathbb{Z}^d$.

To formulate this precisely, we introduce some more notation. Recall that $\ell > 0$ was defined by $\mathrm{supp}(V) \subseteq \Lambda_\ell$. We choose a security distance function D with the properties $D(L) \leqslant (L - \ell)/2$ for all $L \geqslant \ell$ and $\lim_{L \to \infty} D(L) = \infty$. For $x_0 \in \mathbb{R}^d$ we define the set of allowed shifts

$$(1.8) \qquad A_L(x_0) := \left\{ x_0 + x : x \in p\mathbb{Z}^d \text{ and } \mathrm{dist}\left(\Lambda_\ell(x_0 + x), \mathbb{R}^d \setminus \Lambda_L\right) > D(L) \right\}$$

in the box Λ_L. Here we used the notation $\Lambda_\ell(y)$ to indicate that the cube is centered about $y \in \mathbb{R}^d$ instead of the origin. Shifts in $A_L(x_0)$ differ only by a multiple of the period of U, and it is enough to consider $x_0 \in \times_{j=1}^d [0, p_j[$ in the periodicity cell of U. Given $y \in \mathbb{R}^d$ we write $V_y := V(\cdot - y)$ for the shifted perturbation potential and note that if $y \in A_L(x_0)$, then $\mathrm{supp}(V_y)$ respects the security distance $D(L)$ to the boundary of Λ_L. Finally we define the Schrödinger operator $H_{1,y} := H_0 + V_y$ with the shifted perturbation potential along with its finite-volume Dirichlet restrictions $H_{1,y}^{(L)}$. We do not require $V \geqslant 0$ any more.

Theorem 1.3. *Let $d \in \mathbb{N}$ and let $U, V \in \mathcal{K}_\pm(\mathbb{R}^d)$. Assume that U is periodic with period $p \in \mathbb{R}^d$ and $\mathrm{supp}(V) \subseteq \Lambda_\ell$ for some $\ell > 0$. Fix an arbitrary point $x_0 \in \times_{j=1}^d [0, p_j[$ in the periodicity cell of U and a sequence $(L_n)_{n \in \mathbb{N}} \subset [\ell, \infty[$ of diverging lengths, $\lim_{n \to \infty} L_n = \infty$. Then,*

$$(1.9) \qquad \lim_{n \to \infty} \int_\mathbb{R} dE\, \xi\left(E; H_{1,x_{L_n}}^{(L_n)}, H_0^{(L_n)}\right) f(E) = \int_\mathbb{R} dE\, \xi(E; H_{1,x_0}, H_0)\, f(E)$$

holds uniformly in $(x_{L_n})_{n \in \mathbb{N}} \in \times_{n \in \mathbb{N}} A_{L_n}(x_0)$ for every given function f of the form $f = \chi_I g$, where $g \in C(\mathbb{R})$ is continuous and χ_I is the indicator function of a (finite) interval $I \subset \mathbb{R}$.

The proof of Theorem 1.3 relies on a suitable continuity theorem for Laplace transforms, Theorem 2.3, proven in the next section. We prove Theorem 1.3 in Section 3 by verifying the assumptions of Theorem 2.3 using the Feynman-Kac representation for Schrödinger semigroups.

Remark 1.4.   Theorem 1.3 also works in the context of ergodic random Schrödinger operators, if we replace the periodic potential U by a random potential U_ω that is ergodic with respect to a subgroup G of the translation group $\mathbb{R}^d$. In this case the group G replaces $p\mathbb{Z}^d$ in the definition (1.8) of the sets $A_L(x_0)$ of allowed shift vectors. In addition, we have to replace the spectral shift functions in (1.9) by their expectations $\mathbb{E}$ over randomness. In fact, our original motivation for Theorem 1.3 stems from the construction of a strictly positive lower bound on the density of states for alloy-type random Schrödinger operators in the continuum [8], where it will be needed. As is

apparent already in the lattice case of this problem [9], one needs to control a Cesàro sum

$$(1.10) \qquad \frac{1}{(L/\ell)^d} \sum_{j \in \ell\mathbb{Z}^d \cap \Lambda_L} \int_{\mathbb{R}} \mathrm{d}E \, f(E) \, \mathbb{E}\left[\xi\big(E; H_{1,j}^{(L)}, H_0^{(L)}\big)\right]$$

in the limit $L \to \infty$ for some given function f as allowed by Theorem 1.3. We apply Theorem 1.3 with the distance function $D(L) = (1/2)\log(L - \ell + 1)$ to (1.10). Then the asserted uniformity of the convergence in j implies

(1.11)

$$\lim_{L \to \infty} \frac{\ell^d}{L^d} \sum_{j \in \ell\mathbb{Z}^d \cap \Lambda_L} \int_{\mathbb{R}} \mathrm{d}E \, f(E) \, \mathbb{E}\left[\xi\big(E; H_{1,j}^{(L)}, H_0^{(L)}\big)\right] = \int_{\mathbb{R}} \mathrm{d}E \, f(E) \, \mathbb{E}\left[\xi\big(E; H_{1,0}, H_0\big)\right].$$

§ 2.   An abstract uniform convergence result for measures

The uniform convergence of the finite-volume spectral shift functions will follow from a more general result, Theorem 2.3, on the continuity of the Laplace transform of Borel measures. We begin with a simple observation:

Lemma 2.1.   *For given sets A_n, $n \in \mathbb{N}$, define $\mathbb{A} := \times_{n \in \mathbb{N}} A_n$. Consider a family of sequences in $\mathbb{C}$ which is indexed by $a \equiv (a_n)_{n \in \mathbb{N}} \in \mathbb{A}$ and has the property that the n-th sequence element depends only on a_n, but not on a_m for $m \neq n$. We denote a sequence of this family by $(x_n^{a_n})_{n \in \mathbb{N}}$. Suppose that for every $a \in \mathbb{A}$ the limit*

$$(2.1) \qquad x := \lim_{n \to \infty} x_n^{a_n}$$

exists in $\mathbb{C}$ and is independent of $a \in \mathbb{A}$. Then the sequence $(x_n^{a_n})_{n \in \mathbb{N}}$ converges to x uniformly in $a \in \mathbb{A}$.

Proof.   We argue by contradiction. Assume that the limit x is not approached uniformly in a. Then there is $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there is $n \geqslant N$ and $\alpha_n \in A_n$ such that $|x_n^{\alpha_n} - x| > \varepsilon$. In other words, there exists $\varepsilon > 0$, a subsequence $(n_j)_{j \in \mathbb{N}}$ with $\lim_{j \to \infty} n_j = \infty$ and $\alpha_{n_j} \in A_{n_j}$ for all $j \in \mathbb{N}$ such that

$$(2.2) \qquad |x_{n_j}^{\alpha_{n_j}} - x| > \varepsilon \qquad \text{for all } j \in \mathbb{N}.$$

Define $a \in \mathbb{A}$ by setting $a_{n_j} := \alpha_{n_j}$ for all $j \in \mathbb{N}$ and a_k arbirtrary for $k \notin \{n_j : j \in \mathbb{N}\}$. But then $\lim_{n \to \infty} x_n^{a_n} = x$ by hypothesis, and hence $\lim_{j \to \infty} x_{n_j}^{a_{n_j}} = x$, contradicting (2.2). $\qquad\qquad\square$

Definition 2.2.   Let μ be a Borel measure on $\mathbb{R}$. If there is $t_0 \in \mathbb{R}$ such that the integral

$$(2.3) \qquad \widetilde{\mu}(t) := \int_{\mathbb{R}} \mathrm{d}\mu(x)\,\mathrm{e}^{-tx} < \infty$$

is finite for all $t \geqslant t_0$, we say that the *(two-sided) Laplace transform* of μ exists for $t \geqslant t_0$.

We will need the following version of a continuity theorem for Laplace transforms.

Theorem 2.3.   *Let $(\mu_n^{a_n})_{n \in \mathbb{N}}$ be a sequence of Borel measures on $\mathbb{R}$ for every $a \equiv (a_n)_{n \in \mathbb{N}} \in \mathbb{A}$. Assume that for every $a \in \mathbb{A}$ there exists $t_a \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ the Laplace transform of $\mu_n^{a_n}$ exists for $t \geqslant t_a$. Suppose further that for every $a \in \mathbb{A}$ and every $t \geqslant t_a$ the limit*

$$(2.4) \qquad \widetilde{\mu}(t) := \lim_{n \to \infty} \widetilde{\mu}_n^{a_n}(t)$$

exists in $\mathbb{R}$ and is independent of $a \in \mathbb{A}$. Then $\widetilde{\mu}$ is the Laplace transform of a Borel measure μ on $\mathbb{R}$ and $\mu_n^{a_n}$ converges vaguely to μ as $n \to \infty$, the convergence being uniform in $a \in \mathbb{A}$. In other words,

$$(2.5) \qquad \lim_{n \to \infty} \int_{\mathbb{R}} \mathrm{d}\mu_n^{a_n}(x)\,f(x) = \int_{\mathbb{R}} \mathrm{d}\mu(x)\,f(x)$$

holds uniformly in $a \in \mathbb{A}$ for every given function f of the form $f = \chi_I g$, where $g \in C(\mathbb{R})$ and χ_I is the indicator function of a (finite) interval $I \subset \mathbb{R}$ whose endpoints are not atoms of the measure μ.

Proof.   Existence of the limiting measure μ and pointwise convergence for every $a \in \mathbb{A}$ of the limit in (2.5) for every given f of the specified form is a standard continuity theorem for Laplace transforms, see for example [5, Thm. 2a in Sect. XIII.1] and replace one-sided by two-sided Laplace transforms there. Uniformity of the convergence in $a \in \mathbb{A}$ then follows from Lemma 2.1 applied to the sequence $\left(\int_{\mathbb{R}} \mathrm{d}\mu_n^{a_n}(x)\,f(x) \right)_{n \in \mathbb{N}}$. $\square$

§ 3.   Proof of Theorem 1.3

We prove Theorem 1.3 on the uniform convergence of the finite-volume spectral shift functions using Theorem 2.3.

Proof of Theorem 1.3.   1. We verify the assumptions of Theorem 2.3. To this end we fix x_0 in the periodicity cell of U and a sequence $(L_n)_{n \in \mathbb{N}}$ of diverging lengths. Thanks to Dirichlet-Neumann bracketing and since both U and V are Kato decomposable, we

have inf spec $H_{1,x_{L_n}}^{(L_n)} \geqslant$ inf spec $H_{1,x_0} > -\infty$ for all $x_{L_n} \in A_{L_n}(x_0)$ and all $n \in \mathbb{N}$. From this we infer that for every $n \in \mathbb{N}$ and for every shift $x_{L_n} \in A_{L_n}(x_0)$ the two-sided Laplace transform

$$
\begin{aligned}
\widetilde{\xi}_{x_{L_n}}^{(n)}(t) &:= \int_{\mathbb{R}} \mathrm{d}E \, \mathrm{e}^{-tE} \, \xi\big(E; H_{1,x_{L_n}}^{(L_n)}, H_0^{(L_n)}\big) \\
&= \frac{1}{t} \, \mathrm{tr}_{\mathrm{L}^2(\Lambda_{L_n})} \Big[\exp\big(-tH_0^{(L_n)}\big) - \exp\big(-tH_{1,x_{L_n}}^{(L_n)}\big) \Big]
\end{aligned}
\tag{3.1}
$$

exists for every $t > 0$. So fix $t > 0$ from now on.

2. The standard Feynman–Kac representation [14] of the heat kernel gives

$$
\begin{aligned}
\varphi_{x_{L_n}}^{(n)}(t) &:= t(2\pi t)^{d/2} \, \widetilde{\xi}_{x_{L_n}}^{(n)}(t) \\
&= \int_{\Lambda_{L_n}} \mathrm{d}x \, \mathbb{E}_{x,x}^{0,t} \Big[\chi_{\Lambda_{L_n}}^t(b) \, \mathrm{e}^{-\int_0^t \mathrm{d}s\, U(b(s))} \Big(1 - \mathrm{e}^{-\int_0^t \mathrm{d}s\, V_{x_{L_n}}(b(s))} \Big) \Big] \\
&= \int_{\Lambda_{L_n}} \mathrm{d}x \, \mathbb{E}_{x,x}^{0,t} \Big[\chi_{\Lambda_{L_n}}^t(b) \, \mathcal{U}_t(b) \, \mathcal{V}_t(b - x_{L_n}) \Big].
\end{aligned}
\tag{3.2}
$$

Here, $\mathbb{E}_{x,y}^{0,t}$ denotes the normalized expectation over all Brownian bridge paths b starting at $x \in \mathbb{R}^d$ at time $s = 0$ and ending at $y \in \mathbb{R}^d$ at time $s = t$. The Dirichlet boundary condition is taken into account by the cut-off functional $\chi_{\Lambda}^t(b)$, which is equal to one if $b(s) \in \Lambda$ for all $s \in [0, t]$, and zero otherwise. Moreover, we have introduced the Brownian functionals $\mathcal{U}_t(b) := \mathrm{e}^{-\int_0^t \mathrm{d}s\, U(b(s))}$ and $\mathcal{V}_t(b) := 1 - \mathrm{e}^{-\int_0^t \mathrm{d}s\, V(b(s))}$.

3. We shift the integration variables $x \to x + x_{L_n}$ and $b \to b + x_{L_n}$. This and the periodicity of U results in

$$
\varphi_{x_{L_n}}^{(n)}(t) = \int_{\mathbb{R}^d} \mathrm{d}x \, \chi_{\Lambda^n}(x) \, \mathbb{E}_{x,x}^{0,t} \big[\chi_{\Lambda^n}^t(b) \, \mathcal{U}_t(b + x_0) \, \mathcal{V}_t(b) \big] = \int_{\mathbb{R}^d} \mathrm{d}x \, \chi_{\Lambda^n}(x) \, F_n(x)
\tag{3.3}
$$

where we have introduced the abbreviations $\Lambda^n := \Lambda_{L_n}(-x_{L_n})$ and

$$
F_n(x) := \mathbb{E}_{x,x}^{0,t} \big[\chi_{\Lambda^n}^t(b) \, \mathcal{U}_t(b + x_0) \, \mathcal{V}_t(b) \big].
\tag{3.4}
$$

4. We evaluate the limit as $n \to \infty$ of $F_n(x)$ in (3.4). We observe that $\Lambda^n \supseteq \Lambda_{\ell+2D(L_n)}$, independently of $x_{L_n} \in A_{L_n}(x_0)$ for every $n \in \mathbb{N}$, whence

$$
\lim_{n \to \infty} \chi_{\Lambda^n}(x) = 1 \qquad \text{and} \qquad \lim_{n \to \infty} \chi_{\Lambda^n}^t(b) = 1
\tag{3.5}
$$

for every $x \in \mathbb{R}^d$, $\mathbb{P}_{0,0}^{0,t}$-a.e. Brownian bridge path b and every choice of shifts

$$
(x_{L_n})_{n \in \mathbb{N}} \in \underset{n \in \mathbb{N}}{\times} A_{L_n}(x_0) =: \mathbb{A}(x_0).
\tag{3.6}
$$

Here, we have exploited the $\mathbb{P}_{0,0}^{0,t}$-a.s continuity of $s \mapsto b(s)$. Next, the expectation of the functiona $\mathcal{U}_t(b)$ is controlled using [4, Eqs. (6.20), (6.21)] that state that

$$(3.7) \qquad \sup_{x \in \mathbb{R}^d} \mathbb{E}_{x,x}^{0,t} \left[e^{- \int_0^t ds\, W(b(s))} \right] < \infty$$

for every $W \in \mathcal{K}_\pm(\mathbb{R}^d)$. This bound (3.7) and dominated convergence yield the existence of

$$(3.8) \qquad F(x) := \lim_{n \to \infty} F_n(x) = \mathbb{E}_{x,x}^{0,t} \left[\mathcal{U}_t(b + x_0)\, \mathcal{V}_t(b) \right]$$

for every $x \in \mathbb{R}^d$ and every $(x_{L_n})_{n \in \mathbb{N}} \in \mathbb{A}(x_0)$.

5. Finally, for every $x \in \mathbb{R}^d$, every $n \in \mathbb{N}$ and every $x_{L_n} \in A_{L_n}(x_0)$ we have the estimate

$$(3.9) \qquad |F_n(x)| \leqslant G(x) := \mathbb{E}_{x,x}^{0,t} \left[\chi_{\Xi_x^t}(b)\, \mathcal{U}_t(b + x_0) \left(1 + e^{- \int_0^t ds\, V(b(s))} \right) \right],$$

where Ξ_x^t denotes the event that $\sup_{s \in [0,t]} |b(s) - x| > \mathrm{dist}\,(x, \mathrm{supp}(V))$. The Cauchy-Schwarz inequality and (3.7) then imply

$$(3.10) \qquad G(x) \leqslant c \left(\mathbb{P}_{x,x}^{0,t} [\Xi_x^t] \right)^{1/2} = c \left(\mathbb{P}_{0,0}^{0,t} \left[\max_{s \in [0,t]} |b(s)| > \mathrm{dist}\,(x, \mathrm{supp}(V)) \right] \right)^{1/2}$$

for an x-independent constant $c \in]0, \infty[$. The probability in the last line is the complement of the distribution function for the maximum of a d-dimensional Bessel bridge [17] and has a Gaussian decay, see, for example, [13, p. 341, Lemma 1] or [7, p. 438]. This yields $G \in \mathrm{L}^1(\mathbb{R}^d)$.

6. We now combine (3.8) and the upper bound (3.9), (3.10) to evaluate the limit of (3.3). Dominated convergence applied to (3.3) gives for all $(x_{L_n})_{n \in \mathbb{N}} \in \mathbb{A}(x_0)$

$$(3.11) \qquad \lim_{n \to \infty} \varphi_{x_{L_n}}^{(n)}(t) = \int_{\mathbb{R}^d} dx \lim_{n \to \infty} F_n(x) = \int_{\mathbb{R}^d} dx\, \mathbb{E}_{x,x}^{0,t} \left[\mathcal{U}_t(b)\, \mathcal{V}_t(b - x_0) \right],$$

where we have used another change of variables. Therefore Theorem 1.3 follows from

$$\lim_{n \to \infty} \widetilde{\xi}_{x_{L_n}}^{(n)}(t) = \frac{1}{t}\, (2\pi t)^{-d/2} \int_{\mathbb{R}^d} dx\, \mathbb{E}_{x,x}^{0,t} \left[\mathcal{U}_t(b)\, \mathcal{V}_t(b - x_0) \right]$$

$$(3.12) \qquad\qquad\quad = \frac{1}{t}\, \mathrm{tr}_{\mathrm{L}^2(\mathbb{R}^d)} \left[\exp(-tH_0) - \exp(-tH_{1,x_0}) \right]$$

$$= \int_{\mathbb{R}} dE\, e^{-tE}\, \xi\left(E; H_{1,x_0}, H_0 \right)$$

for all $(x_{L_n})_{n \in \mathbb{N}} \in \mathbb{A}(x_0)$ and from Theorem 2.3. $\qquad\qquad \square$

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