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Extreme value distributions of noncolliding diffusion processes

By

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Abstract

Noncolliding diffusion processes reported in the present paper are N-particle systems of diffusion processes in one-dimension, which are conditioned so that all particles start from the origin and never collide with each other in a finite time interval $(0, T)$, $0 < T < \infty$. We consider four temporally inhomogeneous processes with duration T, the noncolliding Brownian bridge, the noncolliding Brownian motion, the noncolliding three-dimensional Bessel bridge, and the noncolliding Brownian meander. Their particle distributions at each time $t \in [0, T]$ are related to the eigenvalue distributions of random matrices in Gaussian ensembles and in some two-matrix models. Extreme values of paths in $[0, T]$ are studied for these noncolliding diffusion processes and determinantal and pfaffian representations are given for the distribution functions. The entries of the determinants and pfaffians are expressed using special functions.

§ 1. Introduction

For $N \geq 2$, consider the following region in $\mathbb{R}^N$,

$$\mathbb{W}_N^{\mathrm{A}} = \Big\{ \boldsymbol{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N \Big\}, \tag{1.1}$$

which is called the Weyl chamber of type A_{N-1} in the representation theory [9]. By the Karlin-McGregor formula [10], the density at $\boldsymbol{y} \in \mathbb{W}_N^{\mathrm{A}}$ of an N-dimensional Brownian

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motion at time $t > 0$, which starts from $\boldsymbol{x} \in \mathbb{W}_N^{\mathrm{A}}$ at time 0, and is restricted on the event that it stays in $\mathbb{W}_N^{\mathrm{A}}$ during a time interval $[0, t]$, is given by

$$f_N^{\mathrm{A}}(t, \boldsymbol{y}|\boldsymbol{x}) = \det_{1 \leq i,j \leq N} \Big[p(t, y_i|x_j) \Big],$$

where $p(t, y|x)$ is the heat kernel, $p(t, y|x) = e^{-(x-y)^2/(2t)}/\sqrt{2\pi t}$. It can be regarded as the transition probability density of the absorbing Brownian motion in $\mathbb{W}_N^{\mathrm{A}}$ from $\boldsymbol{x} \in \mathbb{W}_N^{\mathrm{A}}$ to $\boldsymbol{y} \in \mathbb{W}_N^{\mathrm{A}}$ with duration $t > 0$.

Set $0 < T < \infty$, $\mathbf{a} = (a_1, \dots, a_N), \mathbf{b} = (b_1, \dots, b_N) \in \mathbb{W}_N^{\mathrm{A}}$. Then we consider the N-particle system of one-dimensional Brownian motions starting from a configuration $\mathbf{a}$ at time 0 and arriving at the configuration $\mathbf{b}$ at time T, conditioned never to collide with each other in $[0, T]$. The probability density at the configuration $\boldsymbol{x} \in \mathbb{W}_N^{\mathrm{A}}$ at time $t \in [0, T]$ is given by

$$p_{N,T}^{\mathrm{A}}(t, \boldsymbol{x}; \mathbf{a}, \mathbf{b}) = \frac{f_N^{\mathrm{A}}(T-t, \mathbf{b}|\boldsymbol{x}) f_N^{\mathrm{A}}(t, \boldsymbol{x}|\mathbf{a})}{f_N^{\mathrm{A}}(T, \mathbf{b}|\mathbf{a})}. \tag{1.2}$$

We will call the above mentioned process the *noncolliding Brownian motion from* $\mathbf{a}$ *to* $\mathbf{b}$ *with duration* T and write it as $\boldsymbol{X}^{\mathbf{a},\mathbf{b}}(t) = (X_1^{\mathbf{a},\mathbf{b}}(t), \dots, X_N^{\mathbf{a},\mathbf{b}}(t)), t \in [0, T]$.

We can show that the limit of (1.2) in $\mathbf{a} \to \mathbf{0} \equiv (0, 0, \dots, 0)$, $\mathbf{b} \to \mathbf{0}$, is given by

$$\begin{aligned} \lim_{|\mathbf{a}|,|\mathbf{b}| \to 0} p_{N,T}^{\mathrm{A}}(t, \boldsymbol{x}; \mathbf{a}, \mathbf{b}) &= p_{N,T}^{\mathrm{A}}(t, \boldsymbol{x}; \mathbf{0}, \mathbf{0}) \\ &= q_N^{\mathrm{GUE}}(\boldsymbol{x}; \sigma_T(t)^2), \quad t \in [0, T], \quad \boldsymbol{x} \in \mathbb{W}_N^{\mathrm{A}} \end{aligned} \tag{1.3}$$

with $\sigma_T(t) = \sqrt{t(1 - t/T)}$ (Proposition 13 in [15]). Here $q_N^{\mathrm{GUE}}(\boldsymbol{x}; \sigma^2)$ denotes the probability density of eigenvalues $\boldsymbol{x} \in \mathbb{W}_N^{\mathrm{A}}$ of random matrices in the Gaussian unitary ensemble (GUE) with variance σ^2,

$$q_N^{\mathrm{GUE}}(\boldsymbol{x}; \sigma^2) = \frac{\sigma^{-N^2}}{(2\pi)^{N/2} \prod_{k=1}^N \Gamma(k)} e^{-|\boldsymbol{x}|^2/(2\sigma^2)} \prod_{1 \leq i < j \leq N} (x_j - x_i)^2,$$

where $|\boldsymbol{x}|^2 = \sum_{j=1}^N x_j^2$ and $\Gamma(k)$ is the Gamma function. Note that, when $k \in \mathbb{N} \equiv \{1, 2, \dots\}$, $\Gamma(k) = (k-1)!$. We regard (1.3) as the probability density at a configuration $\boldsymbol{x} \in \mathbb{W}_N^{\mathrm{A}}$ at time $t \in [0, T]$ of the *noncolliding Brownian bridges with duration* T denoted by $\boldsymbol{X}^{\mathbf{0},\mathbf{0}}(t) = (X_1^{\mathbf{0},\mathbf{0}}(t), \dots, X_N^{\mathbf{0},\mathbf{0}}(t)), t \in [0, T]$.

When we make the configuration $\mathbf{b}$ at time T be arbitrary in $\mathbb{W}_N^{\mathrm{A}}$, we have another temporally inhomogeneous system of noncolliding Brownian motion, which is denoted by $\boldsymbol{X}^{\mathbf{a},\mathbb{R}}(t) = (X_1^{\mathbf{a},\mathbb{R}}(t), \dots, X_N^{\mathbf{a},\mathbb{R}}(t)), t \in [0, T]$. The probability density at $\boldsymbol{x} \in \mathbb{W}_N^{\mathrm{A}}$ at time $t \in [0, T]$ is given by

$$p_{N,T}^{\mathrm{A}}(t, \boldsymbol{x}; \mathbf{a}, \mathbb{R}) = \frac{\mathcal{N}_N^{\mathrm{A}}(T-t, \boldsymbol{x}) f_N^{\mathrm{A}}(t, \boldsymbol{x}|\mathbf{a})}{\mathcal{N}_N^{\mathrm{A}}(T, \mathbf{a})}, \quad t \in [0, T],$$

where

$$\mathcal{N}_N^{\mathrm{A}}(s,\boldsymbol{x}) = \int_{\mathbb{W}_N^{\mathrm{A}}} d\boldsymbol{y}\, f_N^{\mathrm{A}}(s,\boldsymbol{y}|\boldsymbol{x}), \quad s>0, \quad \boldsymbol{x}\in\mathbb{W}_N^{\mathrm{A}},$$

is the probability that the absorbing Brownian motion in $\mathbb{W}_N^{\mathrm{A}}$ starting from $\boldsymbol{x}\in\mathbb{W}_N^{\mathrm{A}}$ is not yet absorbed at any boundary of the region $\mathbb{W}_N^{\mathrm{A}}$ and is surviving inside of it at time $s>0$. For an even integer n and an antisymmetric $n\times n$ matrix $A=(a_{ij})$ we put

$$\underset{1\le i,j\le n}{\mathrm{Pf}}\left[a_{ij}\right] = \frac{1}{(n/2)!}\sum_{\sigma:\sigma(2k-1)<\sigma(2k),1\le k\le n/2} \mathrm{sgn}(\sigma) a_{\sigma(1)\sigma(2)}a_{\sigma(3)\sigma(4)}\cdots a_{\sigma(n-1)\sigma(n)},$$

where the summation is extended over all permutations σ of $(1,2,\dots,n)$ with restriction $\sigma(2k-1)<\sigma(2k)$, $k=1,2,\dots,n/2$. This expression is known as a pfaffian (see, for example, [24]). By using the de Bruijn identity [5], which will be given as Lemma 3.1 in Section 3, we have the formula [12, 13]

$$\mathcal{N}_N^{\mathrm{A}}(s,\boldsymbol{x}) = \underset{1\le i,j\le \widehat{N}}{\mathrm{Pf}}\left[F_{ij}^{\mathrm{A}}\left(\frac{\boldsymbol{x}}{\sqrt{2s}}\right)\right],$$

where

$$\widehat{N} = \begin{cases} N, & \text{if } N=\text{even},\\ N+1, & \text{if } N=\text{odd},\end{cases}$$

$\boldsymbol{x}/\sqrt{2s}=(x_1/\sqrt{2s},\dots,x_N/\sqrt{2s})$, and

$$F_{ij}^{\mathrm{A}}(\boldsymbol{y}) = \begin{cases} \Psi(y_j-y_i), & \text{if } 1\le i,j\le N,\\ 1, & \text{if } 1\le i\le N, j=N+1,\\ -1, & \text{if } i=N+1, 1\le j\le N,\\ 0, & \text{if } i=j=N+1,\end{cases}$$

with

$$\Psi(u) = \frac{2}{\sqrt{\pi}}\int_0^u e^{-v^2}dv. \tag{1.4}$$

In [12, 13], the process $\boldsymbol{X}^{\mathbf{0},\mathbb{R}}(t) \equiv \lim_{|\mathbf{a}|\to 0}\boldsymbol{X}^{\mathbf{a},\mathbb{R}}(t) = (X_1^{\mathbf{0},\mathbb{R}}(t),\dots,X_N^{\mathbf{0},\mathbb{R}}(t)), t\in[0,T]$ is constructed, where the probability density at $\boldsymbol{x}\in\mathbb{W}_N^{\mathrm{A}}$ at time $t\in[0,T]$ is given by

$$p_{N,T}^{\mathrm{A}}(t,\boldsymbol{x};\mathbf{0},\mathbb{R}) = \frac{T^{N(N-1)/4}t^{-N^2/2}}{2^{N/2}\prod_{k=1}^N\Gamma(k/2)}e^{-|\boldsymbol{x}|^2/(2t)}\prod_{1\le i<j\le N}(x_j-x_i)\mathcal{N}(T-t,\boldsymbol{x}). \tag{1.5}$$

It has been shown that (1.5) exhibits a transition from the eigenvalue distribution of GUE to the eigenvalue distribution of another ensemble of random matrices called the Gaussian orthogonal ensemble (GOE) as $t\to T$ [13, 14]. In other words, there establishes an interesting correspondence [12] between the temporally inhomogeneous system of noncolliding Brownian motion $\boldsymbol{X}^{\mathbf{0},\mathbb{R}}(t), t \in [0,T]$ and the two-matrix model of Pandey and Mehta [21] in the random matrix theory [20].

The equivalence of the distribution of $\boldsymbol{X}^{\mathbf{0},\mathbf{0}}(t)$ and the eigenvalue distribution of random matrices in GUE with variance $\sigma_T(t)^2$ for $t \in [0,T]$ shown by Eq.(1.3), and the correspondence between $\boldsymbol{X}^{\mathbf{0},\mathbb{R}}(t), t \in [0,T]$ and the two-matrix model of Pandey and Mehta are very useful to perform computer simulations of the conditional diffusion processes $\boldsymbol{X}^{\mathbf{0},\mathbf{0}}(t)$ and $\boldsymbol{X}^{\mathbf{0},\mathbb{R}}(t), t \in [0,T]$ [19]. Figure 1 shows samples of paths generated by computer simulations for these two processes.

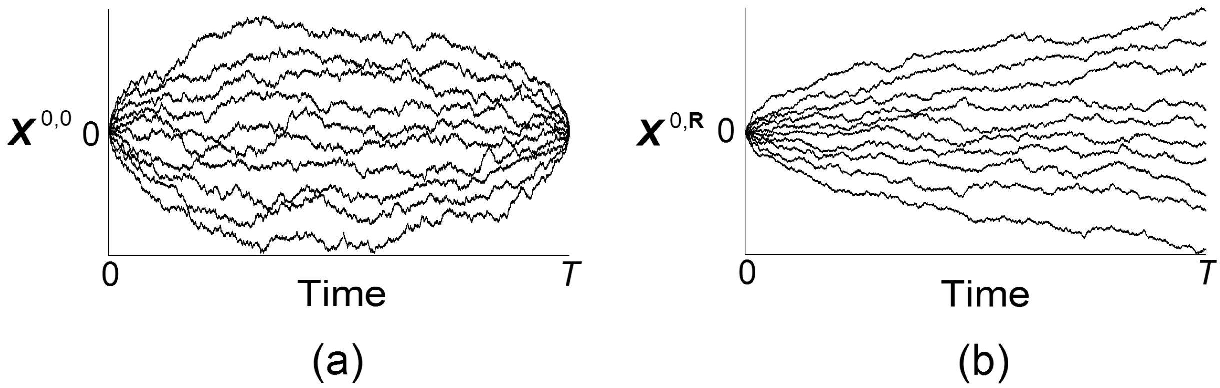

Figure 1. Samples of paths for (a) $\boldsymbol{X}^{\mathbf{0},0}(t)$ and (b) $\boldsymbol{X}^{\mathbf{0},\mathbb{R}}(t), t \in [0,T]$, generated by simulating the corresponding eigenvalue processes of random-matrix models.

In the present paper, we study the distributions of the extreme values defined by

$$
\begin{aligned}
L^{\sharp}(N,T) &= \min_{t\in[0,T]} \boldsymbol{X}^{\sharp}(t) = \min_{t\in[0,T]} X_1^{\sharp}(t), \\
R^{\sharp}(N,T) &= \max_{t\in[0,T]} \boldsymbol{X}^{\sharp}(t) = \max_{t\in[0,T]} X_N^{\sharp}(t),
\end{aligned}
\tag{1.6}
$$

for $\sharp = (\mathbf{a},\mathbf{b}) \in \mathbb{W}_N^{\mathrm{A}} \times \mathbb{W}_N^{\mathrm{A}}$, $\sharp = (\mathbf{0},\mathbf{0})$, and $\sharp = (\mathbf{0},\mathbb{R})$.

In this paper, we also consider the noncolliding processes "with a wall at the origin". Instead of the Weyl chamber of type A_{N-1} given by (1.1), we consider the Weyl chamber of type C_N,

$$
\mathbb{W}_N^{\mathrm{C}} = \Big\{ \boldsymbol{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : 0 < x_1 < x_2 < \cdots < x_N \Big\}.
$$

The density at $\boldsymbol{y} \in \mathbb{W}_N^{\mathrm{C}}$ of an N-dimensional Brownian motion at time $t > 0$, which starts from $\boldsymbol{x} \in \mathbb{W}_N^{\mathrm{C}}$ at time 0, and is restricted on the event that it stays in $\mathbb{W}_N^{\mathrm{C}}$ during a time interval $[0,t]$, is given by the Karlin-McGregor formula as

$$
f_N^{\mathrm{C}}(t, \boldsymbol{y}|\boldsymbol{x}) = \det_{1\le i,j\le N} \Big[p_{\mathrm{abs}}^{(0,\infty)}(t, y_i|x_j) \Big],
\tag{1.7}
$$

where $p_{\rm abs}^{(0,\infty)}(t,y|x)$ is the transition probability density of the one-dimensional absorbing Brownian motion with an absorbing wall at the origin. By the reflection principle of Brownian motion, it is given as

$$\begin{aligned} p_{\rm abs}^{(0,\infty)}(t,y|x) &= p(t,y|x) - p(t,y|-x) \\ &= \frac{1}{\sqrt{2\pi t}} \left\{ e^{-(x-y)^2/(2t)} - e^{-(x+y)^2/(2t)} \right\} \end{aligned}$$

Eq.(1.7) gives the transition probability density of the absorbing Brownian motion in $\mathbb{W}_N^{\rm C}$ from $\boldsymbol{x}$ to $\boldsymbol{y}$ with duration $t > 0$.

For $0 < T < \infty$, $\mathbf{a}, \mathbf{b} \in \mathbb{W}_N^{\rm C}$, we consider the N-particle system of one-dimensional Brownian motions starting from a configuration $\mathbf{a}$ at time 0 and arriving at a configuration $\mathbf{b}$ at time T, which is conditioned so that there is no collision between any pair of particles and that all particles stay positive in a time interval $[0,T]$. The probability density at $\boldsymbol{x} \in \mathbb{W}_N^{\rm C}$ at time $t \in [0,T]$ is given by

$$p_{N,T}^{\rm C}(t,\boldsymbol{x};\mathbf{a},\mathbf{b}) = \frac{f_N^{\rm C}(T-t,\mathbf{b}|\boldsymbol{x}) f_N^{\rm C}(t,\boldsymbol{x}|\mathbf{a})}{f_N^{\rm C}(T,\mathbf{b}|\mathbf{a})}. \tag{1.8}$$

The Brownian motion conditioned to stay positive for a finite time interval $[0,T]$ is called the Brownian meander with duration T [22]. So we call the N-particle system, whose probability density is given by (1.8), the *noncolliding Brownian meander from* $\mathbf{a}$ *to* $\mathbf{b}$ *with duration* T [16] and write it as $\boldsymbol{Y}^{\mathbf{a},\mathbf{b}}(t) = (Y_1^{\mathbf{a},\mathbf{b}}(t),\dots,Y_N^{\mathbf{a},\mathbf{b}}(t)), t \in [0,T]$.

The three-dimensional Bessel bridge with duration T is the conditional Brownian motion such that it starts from the origin, stays positive in $t \in (0,T)$, and first returns to the origin at time T. Then the N-particle system obtained by the limit $\boldsymbol{Y}^{\mathbf{0},\mathbf{0}}(t) \equiv \lim_{|\mathbf{a}|,|\mathbf{b}|\to 0} \boldsymbol{Y}^{\mathbf{a},\mathbf{b}}(t) = (Y_1^{\mathbf{0},\mathbf{0}}(t),\dots,Y_N^{\mathbf{0},\mathbf{0}}(t)), t \in [0,T]$ can be called the *noncolliding three-dimensional Bessel bridge*. It was called the system of nonintersecting Brownian excursions in [25]. The probability density at $\boldsymbol{x} \in \mathbb{W}_N^{\rm C}$ at time $t \in [0,T]$ is given by

$$p_{N,T}^{\rm C}(t,\boldsymbol{x};\mathbf{0},\mathbf{0}) = q_N^{\rm C}(\boldsymbol{x};\sigma_T(t)^2)$$

with $\sigma_T(t) = \sqrt{t(1-t/T)}$ (Proposition 14 in [15]). Here $q_N^{\rm C}(\boldsymbol{x};\sigma^2)$ denotes the probability density of positive eigenvalues $\boldsymbol{x} \in \mathbb{W}_N^{\rm C}$ of random matrices in the ensemble called the class C with variance σ^2 [1],

$$q_N^{\rm C}(\boldsymbol{x};\sigma^2) = \frac{\sigma^{-N(2N+1)}}{(\pi/2)^{N/2}\prod_{\ell=1}^N \Gamma(2\ell)} e^{-|\boldsymbol{x}|^2/(2\sigma^2)} \prod_{1\le i<j\le N} (x_j^2 - x_i^2)^2 \prod_{k=1}^N x_k^2.$$

When we make the configuration $\mathbf{b}$ at time T be arbitrary in $\mathbb{W}_N^{\rm C}$, we have another noncolliding Brownian meander, which is denoted by $\boldsymbol{Y}^{\mathbf{a},\mathbb{R}_+}(t) = (Y_1^{\mathbf{a},\mathbb{R}_+}(t),\dots,Y_N^{\mathbf{a},\mathbb{R}_+}(t))$,

$t \in [0,T]$ with $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. The probability density at $\boldsymbol{x} \in \mathbb{W}_N^{\mathrm{C}}$ at time $t \in [0,T]$ is given by

$$p_{N,T}^{\mathrm{C}}(t, \boldsymbol{x}; \mathbf{a}, \mathbb{R}_+) = \frac{\mathcal{N}_N^{\mathrm{C}}(T-t, \boldsymbol{x}) f_N^{\mathrm{C}}(t, \boldsymbol{x}|\mathbf{a})}{\mathcal{N}_N^{\mathrm{C}}(T, \mathbf{a})},$$

where

$$\mathcal{N}_N^{\mathrm{C}}(s, \boldsymbol{x}) = \int_{\mathbb{W}_N^{\mathrm{C}}} d\boldsymbol{y}\, f_N^{\mathrm{C}}(s, \boldsymbol{y}|\boldsymbol{x}), \quad s > 0, \quad \boldsymbol{x} \in \mathbb{W}_N^{\mathrm{C}},$$

is the probability that the absorbing Brownian motion in $\mathbb{W}_N^{\mathrm{C}}$ starting from $\boldsymbol{x} \in \mathbb{W}_N^{\mathrm{C}}$ is not yet absorbed at any boundary of $\mathbb{W}_N^{\mathrm{C}}$ and is surviving inside of it at time $s > 0$. We can prove the formula [12, 13]

$$\mathcal{N}_N^{\mathrm{C}}(s, \boldsymbol{x}) = \mathop{\mathrm{Pf}}_{1 \le i,j \le \widehat{N}} \left[F_{ij}^{\mathrm{C}} \left(\frac{\boldsymbol{x}}{\sqrt{2s}} \right) \right],$$

where $\boldsymbol{x}/\sqrt{2s} = (x_1/\sqrt{2s}, \dots, x_N/\sqrt{2s})$,

$$F_{ij}^{\mathrm{C}}(\boldsymbol{y}) = \begin{cases} \Psi(y_i, y_j), & \text{if } 1 \le i, j \le N, \\ \Psi(y_i), & \text{if } 1 \le i \le N, j = N+1, \\ -\Psi(y_j), & \text{if } i = N+1, 1 \le j \le N, \\ 0, & \text{if } i = j = N+1, \end{cases}$$

with (1.4) and

$$\Psi(u_1, u_2) = \frac{2}{\pi} \left[\int_0^{u_1} dv_1 \int_{u_1 - u_2}^{u_2 - u_1} dv_2\, e^{-v_1^2 - (v_1 - v_2)^2} - \int_{u_1}^{u_2} dv_1 \int_{u_2 - u_1}^{u_1 + u_2} dv_2\, e^{-v_1^2 - (v_1 - v_2)^2} \right].$$

The process $\boldsymbol{Y}^{\mathbf{0}, \mathbb{R}_+}(t) \equiv \lim_{|\mathbf{a}| \to 0} \boldsymbol{Y}^{\mathbf{a}, \mathbb{R}_+}(t) = (Y_1^{\mathbf{0}, \mathbb{R}_+}(t), \dots, Y_N^{\mathbf{0}, \mathbb{R}_+}(t)), t \in [0,T]$ is studied in [13, 18], where the probability density at $\boldsymbol{x} \in \mathbb{W}_N^{\mathrm{C}}$ at time $t \in [0,T]$ is given by

$$\begin{aligned} (1.9) \quad & p_{N,T}^{\mathrm{C}}(t, \boldsymbol{x}; \mathbf{0}, \mathbb{R}_+) \\ &= \frac{T^{N^2/2} t^{-N(2N+1)/2}}{\prod_{\ell=1}^N \Gamma(\ell)} e^{-|\boldsymbol{x}|^2/(2t)} \prod_{1 \le i < j \le N} (x_j^2 - x_i^2) \prod_{k=1}^N x_k \, \mathcal{N}^{\mathrm{C}}(T-t, \boldsymbol{x}). \end{aligned}$$

It has been shown that (1.9) exhibits a transition from the eigenvalue distribution of random matrices in the ensemble called the class C to the eigenvalue distribution of the class CI as $t \to T$ [18, 15].

We can perform computer simulations of a matrix-valued Brownian bridge such that its distribution is the same as the distribution of random matrices in the class C with variance $\sigma_T(t) = \sqrt{t(1-t/T)}, t \in [0,T]$, and a matrix-valued Brownian motion, whose distribution at time $t \in [0,T]$ changes continuously from the class C distribution to the class CI distribution of random matrices as $t \to T$. By numerically calculating eigenvalues of these two matrix-valued diffusion processes in $t \in [0,T]$, we can draw samples of paths of noncolliding diffusion particles for $\boldsymbol{Y}^{\mathbf{0},\mathbf{0}}(t)$ and $\boldsymbol{Y}^{\mathbf{0},\mathbb{R}_+}(t), t \in [0,T]$, as shown in Figure 2 [19].

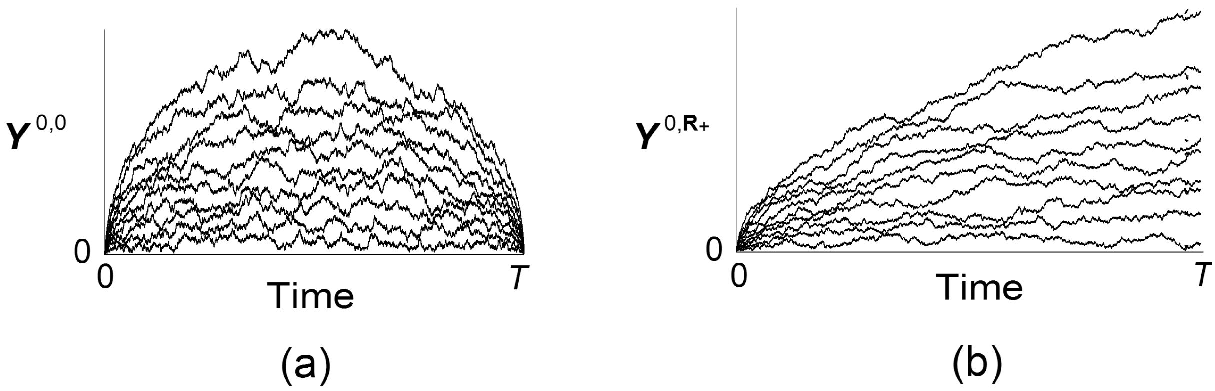

Figure 2. Samples of paths for (a) $\boldsymbol{Y}^{\mathbf{0},\mathbf{0}}(t)$ and (b) $\boldsymbol{Y}^{\mathbf{0},\mathbb{R}_+}(t), t \in [0,T]$.

For $\boldsymbol{Y}^\sharp$ with $\sharp = (\mathbf{a},\mathbf{b}) \in \mathbb{W}_N^{\rm C} \times \mathbb{W}_N^{\rm C}$, $\sharp = (\mathbf{0},\mathbf{0})$, and $\sharp = (\mathbf{0},\mathbb{R}_+)$, we will report the distributions of the maximum value

$$H^\sharp(N,T) = \max_{t\in[0,T]} \boldsymbol{Y}^\sharp(t) = \max_{t\in[0,T]} Y_N^\sharp(t). \tag{1.10}$$

§ 2. Results

For $k \in \mathbb{N}_0 \equiv \{0,1,2,\dots\}$, let $H_k(x)$ be the k-th Hermite polynomial

$$H_k(x) = k! \sum_{m=0}^{[k/2]} \frac{(-1)^m (2x)^{k-2m}}{m!(k-2m)!},$$

where $[a]$ denotes the largest integer that is not greater than a. We define the function of $u,v \in \mathbb{R}$ with an index $k \in \mathbb{N}_0$,

$$\Theta_k(u,v) = \sum_{n\in\mathbb{Z}} H_k(un+v) e^{-(un+v)^2}. \tag{2.1}$$

If we consider the following version of Jacobi's theta function

$$\vartheta(x,y) = \sum_{n\in\mathbb{Z}} q^{n^2} z^{2n} = \sum_{n\in\mathbb{Z}} e^{2\pi\sqrt{-1}xn + \pi\sqrt{-1}yn^2},$$

$\operatorname{Im} y > 0$, where we have set $z = e^{\pi\sqrt{-1}x}$ and $q = e^{\pi\sqrt{-1}y}$, the reciprocal relation

$$\vartheta(x, y) = \vartheta\left(\frac{x}{y}, -\frac{1}{y}\right) e^{-\pi\sqrt{-1}x^2/y} \left(\frac{\sqrt{-1}}{y}\right)^{1/2}$$

holds (see Sec.10.12 in [2], Sec.A.3.1 in [8]). Combining this functional equation with the fact

$$e^{2ux-u^2} = \sum_{k=0}^{\infty} H_k(x)\frac{u^k}{k!},$$

the following identities are derived for $k \in \mathbb{N}_0, \eta, \xi \in \mathbb{R}$ (Lemma 3 in [19]);

$$\sum_{n\in\mathbb{Z}} n^k \exp\left(-\frac{\pi}{\eta^2}n^2 + \frac{2\pi\sqrt{-1}\xi}{\eta^2}n\right) = \frac{(-1)^{k/2}\eta^{k+1}}{2^k\pi^{k/2}}\Theta_k\left(\sqrt{\pi}\eta, \frac{\sqrt{\pi}\xi}{\eta}\right). \tag{2.2}$$

The following two theorems are our main results.

Theorem 2.1. *For $\ell, r > 0$,*

$$\begin{aligned}
&\mathbf{P}\Big(-\ell < L^{\mathbf{0},\mathbf{0}}(N,T), R^{\mathbf{0},\mathbf{0}}(N,T) < r\Big) \\
&= \frac{(-1)^N}{2^{N(N-1)/2}\prod_{k=1}^N \Gamma(k)} \\
&\times \det_{1\le i,j\le N}\left[\Theta_{i+j-2}\left(\frac{2(\ell+r)}{\sqrt{2T}}, \frac{2\ell}{\sqrt{2T}}\right) + (-1)^j\Theta_{i+j-2}\left(\frac{2(\ell+r)}{\sqrt{2T}}, 0\right)\right],
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&\mathbf{P}\Big(-\ell < L^{\mathbf{0},\mathbb{R}}(N,T), R^{\mathbf{0},\mathbb{R}}(N,T) < r\Big) \\
&= \frac{(-1)^{N(N+1)/2}}{2^{N(N-1)/4}\prod_{k=1}^N \Gamma(k/2)} \operatorname*{Pf}_{1\le i,j\le \widehat{N}}\left[G_{ij}^{\mathrm{A}}\left(-\frac{\ell}{\sqrt{T}}, \frac{r}{\sqrt{T}}\right)\right],
\end{aligned} \tag{2.4}$$

where

$$G_{ij}^{\mathrm{A}}\left(-\frac{\ell}{\sqrt{T}}, \frac{r}{\sqrt{T}}\right) = \begin{cases} I_{ij}^{\mathrm{A}}, & \text{if } 1\le i,j\le N, \\ I_i^{\mathrm{A}}, & \text{if } 1 \le i \le N, j = N+1, \\ -I_j^{\mathrm{A}}, & \text{if } i = N+1, 1\le j\le N, \\ 0, & \text{if } i=j=N+1, \end{cases}$$

with

$$\begin{aligned}
z_i^{\mathrm{A}}(x) &= \Theta_{i-1}\left(\frac{2(\ell+r)}{\sqrt{2T}}, \frac{2\ell}{\sqrt{2T}} + x\right) + (-1)^i\Theta_{i-1}\left(\frac{2(\ell+r)}{\sqrt{2T}}, x\right) \\
I_i^{\mathrm{A}} &= \int_{-\ell/\sqrt{2T}}^{r/\sqrt{2T}} z_i^{\mathrm{A}}(x)dx, \\
I_{ij}^{\mathrm{A}} &= \int_{-\ell/\sqrt{2T}<x_1<x_2<r/\sqrt{2T}} \det\begin{bmatrix} z_i^{\mathrm{A}}(x_1) & z_i^{\mathrm{A}}(x_2) \\ z_j^{\mathrm{A}}(x_1) & z_j^{\mathrm{A}}(x_2)\end{bmatrix} dx_1 dx_2.
\end{aligned}$$

Theorem 2.2. *For $h > 0$,*

$$\mathbf{P}\Big(H^{\mathbf{0},\mathbf{0}}(N,T) < h\Big) = \frac{(-1)^N}{2^{N^2}\prod_{k=1}^{N}\Gamma(2k)} \det_{1\le i,j\le N}\left[\Theta_{2(i+j-1)}\left(\frac{2h}{\sqrt{2T}},0\right)\right], \tag{2.5}$$

and

$$\mathbf{P}\Big(H^{\mathbf{0},\mathbb{R}_+}(N,T) < h\Big) = \frac{1}{2^{N(N-1)/2}\prod_{k=1}^{N}\Gamma(k)} \operatorname*{Pf}_{1\le i,j\le \widehat{N}}\left[G^{\mathrm{C}}_{ij}\left(\frac{h}{\sqrt{T}}\right)\right], \tag{2.6}$$

where

$$G^{\mathrm{C}}_{ij}\left(\frac{h}{\sqrt{T}}\right) = \begin{cases} I^{\mathrm{C}}_{ij}, & \text{if } \ 1\le i,j\le N, \\ I^{\mathrm{C}}_{i}, & \text{if } \ 1\le i\le N, j=N+1, \\ -I^{\mathrm{C}}_{j}, & \text{if } \ i=N+1, 1\le j\le N, \\ 0, & \text{if } \ i=j=N+1, \end{cases}$$

with

$$\begin{aligned} z^{\mathrm{C}}_i(x) &= \Theta_{2i-1}\left(\frac{2h}{\sqrt{2T}},x\right) \\ I^{\mathrm{C}}_i &= \int_0^{h/\sqrt{2T}} z^{\mathrm{C}}_i(x)dx, \\ I^{\mathrm{C}}_{ij} &= \int_{0<x_1<x_2<h/\sqrt{2T}} \det\begin{bmatrix} z^{\mathrm{C}}_i(x_1) & z^{\mathrm{C}}_i(x_2) \\ z^{\mathrm{C}}_j(x_1) & z^{\mathrm{C}}_j(x_2)\end{bmatrix} dx_1dx_2. \end{aligned}$$

Remarks.

(1) By the scaling property of Brownian motion, we have the equalities in distribution

$$L^{\sharp}(N,T) \stackrel{\mathrm{d}}{=} \sqrt{T}L^{\sharp}(N,1), \quad R^{\sharp}(N,T) \stackrel{\mathrm{d}}{=} \sqrt{T}R^{\sharp}(N,1),$$

for $\sharp = (\mathbf{0},\mathbf{0}), (\mathbf{0},\mathbb{R})$, and

$$H^{\sharp}(N,T) = \sqrt{T}H^{\sharp}(N,1)$$

for $\sharp = (\mathbf{0},\mathbf{0}), (\mathbf{0},\mathbb{R}_+)$, $\forall T > 0$. As a matter of fact, the probability distributions (2.3) and (2.4) in Theorem 2.1 are functions of $\ell/\sqrt{T}$ and $r/\sqrt{T}$, and (2.5) and (2.6) are of $h/\sqrt{T}$, respectively.

(2) For $N = 1$, (2.5) gives

$$\begin{aligned} \mathbf{P}\Big(H^{\mathbf{0},\mathbf{0}}(1,T) < h\Big) &= -\frac{1}{2}\sum_{n\in\mathbb{Z}} H_2\left(\frac{2h}{\sqrt{2T}}n\right) e^{-2h^2n^2/T} \\ &= \sum_{n\in Z}\left(1-\frac{4h^2n^2}{T}\right) e^{-2h^2n^2/T}, \end{aligned}$$

since $H_2(x) = 4x^2 - 2$. This is a classical result for the height distribution of the three-dimensional Bessel bridge. It should be remarked that this result for distribution is equivalent with the fact on moments

$$\mathbf{E}\Big[\Big(H^{\mathbf{0},\mathbf{0}}(1,T)\Big)^m\Big] = 2\left(\frac{\pi T}{2}\right)^{m/2} \xi(m), \quad m \in \mathbb{C}$$

with

$$\xi(m) = \frac{1}{2} m(m-1)\pi^{-m/2}\Gamma\left(\frac{m}{2}\right)\zeta(m),$$

where $\zeta(m)$ is the Riemann zeta function

$$\zeta(m) = \sum_{n\in\mathbb{N}} \frac{1}{n^m}, \quad \mathrm{Re}\, m > 1.$$

See [3] for interesting relations of the probability laws of one-dimensional conditional Brownian motions to the Jacobi theta functions and the Riemann zeta function.

(3) In [11], $N = 2$ case of the formula (2.5) of Theorem 2.2 and moments calculated from it have been intensively studied, which are related to the double Dirichlet series.

(4) The formula (2.3) of Theorem 2.1 gives the joint distribution of two extreme values $L^{\mathbf{0},\mathbf{0}}(N,T)$ and $R^{\mathbf{0},\mathbf{0}}(N,T)$ of the process $\boldsymbol{X}^{\mathbf{0},\mathbf{0}}(t), t \in [0,T]$. The distributions of single variable $R^{\mathbf{0},\mathbf{0}}(N,T)$ was studied in [7] and [23]. Feierl [7] also reported the distribution of the width (or "range") defined by

$$\begin{aligned} W^{\mathbf{0},\mathbf{0}}(N,T) &= R^{\mathbf{0},\mathbf{0}}(N,T) - L^{\mathbf{0},\mathbf{0}}(N,T) \\ &= \max_{t\in[0,T]} \Big(X_N^{\mathbf{0},\mathbf{0}}(t) - X_1^{\mathbf{0},\mathbf{0}}(t)\Big). \end{aligned}$$

(5) A part of results recently reported by Borodin *et al.* [4] may be stated as follows in the present notations; if N is even, for each fixed $T \geq 0$,

$$\max_{t\in[0,T/2]} \frac{1}{\sqrt{2(1-t/T)}} X_N^{\mathbf{0},\mathbf{0}}(t) \stackrel{\mathrm{d}}{=} Y_{N/2}^{\mathbf{0},\mathbf{0}}(T/2),$$

where $Y_{N/2}^{\mathbf{0},\mathbf{0}}(T/2)$ in the RHS denotes the position of the top particle at time $T/2$ of the $N/2$-particle system $\boldsymbol{Y}^{\mathbf{0},\mathbf{0}}(t) = (Y_1^{\mathbf{0},\mathbf{0}}(t), \dots, Y_{N/2}^{\mathbf{0},\mathbf{0}}(t)), t \in [0,T]$. The distribution of the top path $Y_{N/2}^{\mathbf{0},\mathbf{0}}(t), t \in [0,T]$ was studied by Tracy and Widom [25].

(6) The distribution of the "maximum height" $H^{\mathbf{0},\mathbf{0}}(N,T)$ of the process $\boldsymbol{Y}^{\mathbf{0},\mathbf{0}}(t), t \in [0,T]$ has been studied by Feierl [6] and by Schehr *et al.* [23] in different context and by different methods. See [19] for comparison.

§ 3. Proof of Theorem 2.1

Let $\mathbf{1}\{\omega\}$ be the indicator function of a condition ω. Assume that $-\ell < a_1, -\ell < b_1, a_N < r, b_N < r$. By definition of the process and (1.6), for $(\mathbf{a}, \mathbf{b}) \in \mathbb{W}_N^{\mathrm{A}} \times \mathbb{W}_N^{\mathrm{A}}$

$$
\begin{aligned}
(3.1) \qquad & \mathbf{P}\Big(-\ell < L^{\mathbf{a},\mathbf{b}}(N,T), R^{\mathbf{a},\mathbf{b}}(N,T) < r \Big) \\
&= \mathbf{E}\Big(\mathbf{1}\{-\ell < X_1^{\mathbf{a},\mathbf{b}}(t), X_N^{\mathbf{a},\mathbf{b}}(t) < r, \ \forall t \in [0,T]\} \Big) \\
&= \mathbf{E}\Big(\mathbf{1}\{-\ell < X_i^{\mathbf{a},\mathbf{b}}(t) < r, \ 1 \le \forall i \le N, \ \forall t \in [0,T]\} \Big).
\end{aligned}
$$

For $\ell, r > 0$, we introduce the following restricted region in $\mathbb{W}_N^{\mathrm{A}}$,

$$
\mathbb{W}_N^{\mathrm{A}}(-\ell, r) = \Big\{ \boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{W}_N^{\mathrm{A}} : -\ell < x_1, x_N < r \Big\}.
$$

Then we consider the absorbing Brownian motion in this region starting from $\mathbf{a} \in \mathbb{W}_N^{\mathrm{A}}(-\ell, r)$. Let the transition probability density of a one-dimensional Brownian motion in an interval $(-\ell, r)$, where two absorbing walls are put at $x = -\ell$ and $x = r$, be $p_{\mathrm{abs}}^{(-\ell,r)}(t, \cdot\,|\,\cdot), t \ge 0$. Then, by the Karlin-McGregor formula, the probability density that the absorbing Brownian motion in $\mathbb{W}_N^{\mathrm{A}}(-\ell, r)$ arrives at $\mathbf{b} \in \mathbb{W}_N^{\mathrm{A}}(-\ell, r)$ at time $T > 0$, avoided from any absorption at boundary of the region $\mathbb{W}_N^{\mathrm{A}}(-\ell, r)$, is given by

$$
f_N^{\mathrm{A}(-\ell,r)}(T, \mathbf{b}|\mathbf{a}) = \det_{1 \le i,j \le N} \Big[p_{\mathrm{abs}}^{(-\ell,r)}(T, b_i | a_j) \Big].
$$

The probability density $p_{\mathrm{abs}}^{(-\ell,r)}(t, y|x)$ is obtained by solving the diffusion equation $\partial u(t,y)/\partial t = (1/2)\partial^2 u(t,y)/\partial y^2, -\ell \le y \le r$ with the Dirichlet boundary condition $u(\cdot\,, -\ell) = u(\cdot\,, r) = 0$ and with the initial condition $u(0, y) = \delta_x(y), -\ell < x, y < r$. The solution is given by

$$
p_{\mathrm{abs}}^{(-\ell,r)}(t, y|x) = \frac{2}{\ell + r} \sum_{n \in \mathbb{N}} e^{-\pi^2 t n^2 / \{2(\ell+r)^2\}} \sin\left(\pi \frac{\ell + x}{\ell + r} n \right) \sin\left(\pi \frac{\ell + y}{\ell + r} n \right),
$$

$t > 0, x, y \in (-\ell, r)$.

Then the following equality will be established,

$$
(3.2) \qquad \mathbf{E}\Big(\mathbf{1}\{-\ell < X_i^{\mathbf{a},\mathbf{b}}(t) < r, \ 1 \le \forall i \le N, \ \forall t \in [0,T]\} \Big) = \frac{f_N^{\mathrm{A}(-\ell,\mathrm{r})}(T, \mathbf{b}|\mathbf{a})}{f_N^{\mathrm{A}}(T, \mathbf{b}|\mathbf{a})}.
$$

And then we have the equalities

$$
\begin{aligned}
(3.3) \qquad & \mathbf{P}\Big(-\ell < L^{\mathbf{a},\mathbb{R}}(N,T), R^{\mathbf{a},\mathbb{R}}(N,T) < r \Big) \\
&= \mathbf{E}\Big(\mathbf{1}\{-\ell < X_i^{\mathbf{a},\mathbb{R}}(t) < r, \ 1 \le \forall i \le N, \ \forall t \in [0,T]\} \Big) \\
&= \frac{\displaystyle\int_{\mathbb{W}_N^{\mathrm{A}}(-\ell,r)} d\mathbf{b}\, f_N^{\mathrm{A}(-\ell,\mathrm{r})}(T, \mathbf{b}|\mathbf{a})}{\mathcal{N}_N^{\mathrm{A}}(T, \mathbf{a})}.
\end{aligned}
$$

Now we want to take the limit $\mathbf{a}, \mathbf{b} \to \mathbf{0}$ in (3.2) and $\mathbf{a} \to \mathbf{0}$ in (3.3) in order to prove Theorem 2.1. The following asymptotics in $|\mathbf{a}| \to 0$ for the denominators of (3.2) and (3.3) are given in [15, 17]; as $|\mathbf{a}| \to 0$,

$$f_N^{\mathrm{A}}(T, \mathbf{b}|\mathbf{a}) = \frac{T^{-N^2/2}}{(2\pi)^{N/2}\prod_{k=1}^{N}\Gamma(k)} e^{-|\mathbf{b}|^2/(2T)} \times \prod_{1 \le i < j \le N} \Big\{(a_j - a_i)(b_j - b_i)\Big\} \times \Big\{1 + \mathcal{O}(|\mathbf{a}|)\Big\}, \tag{3.4}$$

$$\mathcal{N}_N^{\mathrm{A}}(T, \mathbf{a}) = \frac{1}{\pi^{N/2}} \prod_{k=1}^{N} \frac{\Gamma(k/2)}{\Gamma(k)} T^{-N(N-1)/4} \times \prod_{1 \le i < j \le N} (a_j - a_i) \times \Big\{1 + \mathcal{O}(|\mathbf{a}|)\Big\}. \tag{3.5}$$

By multilinearity of determinants, we have

$$\begin{aligned} &f_N^{\mathrm{A}(-\ell,r)}(T, \mathbf{b}|\mathbf{a}) \\ &= \det_{1 \le i,j \le N}\left[\frac{2}{\ell + r}\sum_{n_i \in \mathbb{N}} e^{-\pi^2 T n_i^2/\{2(\ell+r)^2\}} \sin\left(\pi \frac{\ell + a_i}{\ell + r} n_i\right) \sin\left(\pi \frac{\ell + b_j}{\ell + r} n_i\right)\right] \\ &= \frac{1}{N!}\left(\frac{2}{\ell + r}\right)^N \sum_{\mathbf{n} \in \mathbb{N}^N} e^{-\pi^2 T |\mathbf{n}|^2/\{2(\ell+r)^2\}} \\ &\quad \times \det_{1 \le i,j \le N}\left[\sin\left(\pi \frac{\ell + a_i}{\ell + r} n_j\right)\right] \det_{1 \le \alpha,\beta \le N}\left[\sin\left(\pi \frac{\ell + b_\alpha}{\ell + r} n_\beta\right)\right], \end{aligned} \tag{3.6}$$

where $|\mathbf{n}|^2 = \sum_{i=1}^{N} n_i^2$. We will use the series expansion

$$\sin\left(\pi \frac{\ell + a}{\ell + r} n\right) = \sum_{p \in \mathbb{N}_0} c_p(n)(an)^p \tag{3.7}$$

with

$$c_p(n) = \frac{(-1)^{(p-1)/2}}{2p!}\left(\frac{\pi}{\ell + r}\right)^p \left\{\exp\left(\frac{\pi\sqrt{-1}\ell}{\ell + r} n\right) + (-1)^{p+1} \exp\left(-\frac{\pi\sqrt{-1}\ell}{\ell + r} n\right)\right\}, \tag{3.8}$$

$p \in \mathbb{N}_0$. We will also use the property of the Vandermonde determinant

$$\det_{1 \le i,j \le N}\left[a_i^{j-1}\right] = \prod_{1 \le i < j \le N} (a_j - a_i).$$

Then

$$
\begin{aligned}
(3.9)\quad & f_N^{\mathrm{A}(-\ell,r)}(T,\mathbf{b}|\mathbf{a}) \\
&= \frac{1}{N!}\left(\frac{2}{\ell+r}\right)^N \sum_{\mathbf{n}\in\mathbb{N}^N} e^{-\pi^2 T|\mathbf{n}|^2/\{2(\ell+r)^2\}} \\
&\quad \times \sum_{0\le p_1<\cdots<p_N} \det_{1\le i,j\le N}\left[a_i^{p_j}\right] \det_{1\le\gamma,\delta\le N}\left[c_{p_\gamma}(n_\delta) n_\delta^{p_\gamma}\right] \det_{1\le\alpha,\beta\le N}\left[\sin\left(\pi\frac{\ell+b_\alpha}{\ell+r}n_\beta\right)\right] \\
&= \frac{1}{N!}\left(\frac{2}{\ell+r}\right)^N \sum_{\mathbf{n}\in\mathbb{N}^N} e^{-\pi^2 T|\mathbf{n}|^2/\{2(\ell+r)^2\}} \det_{1\le\gamma,\delta\le N}\left[c_{\gamma-1}(n_\delta) n_\delta^{\gamma-1}\right] \\
&\quad \times \det_{1\le\alpha,\beta\le N}\left[\sin\left(\pi\frac{\ell+b_\alpha}{\ell+r}n_\beta\right)\right] \prod_{1\le i<j\le N}(a_j-a_i)\times\Big\{1+\mathcal{O}(|\mathbf{a}|)\Big\} \\
&= \left(\frac{2}{\ell+r}\right)^N \det_{1\le\alpha,\beta\le N}\left[\sum_{n\in\mathbb{N}} e^{-\pi^2 T n^2/\{2(\ell+r)^2\}} c_{\alpha-1}(n) n^{\alpha-1}\sin\left(\pi\frac{\ell+b_\beta}{\ell+r}n\right)\right] \\
&\qquad \times \prod_{1\le i<j\le N}(a_j-a_i)\times\Big\{1+\mathcal{O}(|\mathbf{a}|)\Big\}.
\end{aligned}
$$

By (3.8), we have

$$
\begin{aligned}
& \sum_{n\in\mathbb{N}} e^{-\pi^2 T n^2/\{2(\ell+r)^2\}} c_{\alpha-1}(n) n^{\alpha-1}\sin\left(\pi\frac{\ell+b_\beta}{\ell+r}n\right) \\
&= \frac{(-1)^{(\alpha-2)/2}}{2(\alpha-1)!}\left(\frac{\pi}{\ell+r}\right)^{\alpha-1} \\
&\quad \times \Bigg[\sum_{n\in\mathbb{N}} n^{\alpha-1}\exp\left(-\frac{\pi^2 T}{2(\ell+r)^2}n^2+\frac{\pi\sqrt{-1}\ell}{\ell+r}n\right)\sin\left(\pi\frac{\ell+b_\beta}{\ell+r}n\right) \\
&\qquad + \sum_{n\in\mathbb{N}} (-n)^{\alpha-1}\exp\left(-\frac{\pi^2 T}{2(\ell+r)^2}(-n)^2+\frac{\pi\sqrt{-1}\ell}{\ell+r}(-n)\right)\sin\left(\pi\frac{\ell+b_\beta}{\ell+r}(-n)\right)\Bigg] \\
&= \frac{(-1)^{(\alpha-2)/2}}{2(\alpha-1)!}\left(\frac{\pi}{\ell+r}\right)^{\alpha-1} \\
&\quad \times \sum_{n\in\mathbb{Z}} n^{\alpha-1}\exp\left(-\frac{\pi^2 T}{2(\ell+r)^2}n^2+\frac{\pi\sqrt{-1}\ell}{\ell+r}n\right)\sin\left(\pi\frac{\ell+b_\beta}{\ell+r}n\right). \\
&= \frac{(-1)^{(\alpha-3)/2}}{4(\alpha-1)!}\left(\frac{\pi}{\ell+r}\right)^{\alpha-1} \Bigg\{\sum_{n\in\mathbb{Z}} n^{\alpha-1}\exp\left(-\frac{\pi^2 T}{2(\ell+r)^2}n^2+\frac{\pi\sqrt{-1}(2\ell+b_\beta)}{\ell+r}n\right) \\
&\qquad + (-1)^\alpha \sum_{n\in\mathbb{Z}} n^{\alpha-1}\exp\left(-\frac{\pi^2 T}{2(\ell+r)^2}n^2+\frac{\pi\sqrt{-1}b_\beta}{\ell+r}n\right)\Bigg\}.
\end{aligned}
$$

By the identity (2.2), the above is equal to

$$\frac{(-1)^{\alpha}(\ell+r)}{2^{(\alpha+2)/2}\pi^{1/2}(\alpha-1)!}T^{-\alpha/2}$$
$$\times\left\{\Theta_{\alpha-1}\left(\frac{2(\ell+r)}{\sqrt{2T}},\frac{2\ell+b_{\beta}}{\sqrt{2T}}\right)+(-1)^{\alpha}\Theta_{\alpha-1}\left(\frac{2(\ell+r)}{\sqrt{2T}},\frac{b_{\beta}}{\sqrt{2T}}\right)\right\}.$$

Then we have the following asymptotics in $|\mathbf{a}|\to 0$,

$$\begin{aligned}(3.10)\quad & f_N^{\mathrm{A}(-\ell,r)}(T,\mathbf{b}|\mathbf{a})\\ &=\frac{(-1)^{N(N+1)/2}}{2^{N(N+1)/4}\pi^{N/2}\prod_{k=1}^{N}\Gamma(k)}T^{-N(N+1)/4}\\ &\quad\times\det_{1\le\alpha,\beta\le N}\left[\Theta_{\alpha-1}\left(\frac{2(\ell+r)}{\sqrt{2T}},\frac{2\ell+b_{\beta}}{\sqrt{2T}}\right)+(-1)^{\alpha}\Theta_{\alpha-1}\left(\frac{2(\ell+r)}{\sqrt{2T}},\frac{b_{\beta}}{\sqrt{2T}}\right)\right]\\ &\quad\quad\times\prod_{1\le i<j\le N}(a_j-a_i)\times\Big\{1+\mathcal{O}(|\mathbf{a}|)\Big\}.\end{aligned}$$

From (3.6) to (3.9), we performed the series expansion (3.7) in only one of the two determinants in (3.6). If we perform the series expansions in both of them, we have the following estimate,

$$\begin{aligned}& f_N^{\mathrm{A}(-\ell,r)}(T,\mathbf{b}|\mathbf{a})\\ &=\frac{1}{N!}\left(\frac{2}{\ell+r}\right)^{N}\sum_{\mathbf{n}\in\mathbb{N}^N}e^{-\pi^2T|\mathbf{n}|^2/\{2(\ell+r)^2\}}\\ &\quad\times\sum_{0\le p_1<\cdots<p_N}\det_{1\le i,j\le N}\left[a_i^{p_j}\right]\det_{1\le\alpha,\beta\le N}\left[c_{p_\alpha}(n_\beta)n_\beta^{p_\alpha}\right]\\ &\quad\times\sum_{0\le q_1<\cdots<q_N}\det_{1\le i',j'\le N}\left[b_{i'}^{q_{j'}}\right]\det_{1\le\alpha',\beta'\le N}\left[c_{q_{\alpha'}}(n_{\beta'})n_{\beta'}^{q_{\alpha'}}\right]\\ &=\left(\frac{2}{\ell+r}\right)^{N}\det_{1\le\alpha,\beta\le N}\left[\sum_{n\in\mathbb{N}}e^{-\pi^2Tn^2/\{2(\ell+r)^2\}}c_{\alpha-1}(n)c_{\beta-1}(n)n^{\alpha+\beta-2}\right]\\ &\quad\times\prod_{1\le i<j\le N}\Big\{(a_j-a_i)(b_j-b_i)\Big\}\times\Big\{1+\mathcal{O}(|\mathbf{a}|,|\mathbf{b}|)\Big\}.\end{aligned}$$

By (3.8), we see

$$
\begin{aligned}
&\sum_{n\in\mathbb{N}} e^{-\pi^2 T n^2/\{2(\ell+r)^2\}} c_{\alpha-1}(n) c_{\beta-1}(n) n^{\alpha+\beta-2} \\
&= \frac{(-1)^{(\alpha+\beta)/2}}{4(\alpha-1)!(\beta-1)!}\left(\frac{\pi}{\ell+r}\right)^{\alpha+\beta-2} \sum_{n\in\mathbb{N}} e^{-\pi^2 T n^2/\{2(\ell+r)^2\}} n^{\alpha+\beta-2} \\
&\quad \times \left\{\exp\left(\frac{2\pi\sqrt{-1}\ell}{\ell+r}n\right) + (-1)^\alpha + (-1)^\beta + (-1)^{\alpha+\beta}\exp\left(-\frac{2\pi\sqrt{-1}\ell}{\ell+r}n\right)\right\} \\
&= \frac{(-1)^{(\alpha+\beta)/2}}{4(\alpha-1)!(\beta-1)!}\left(\frac{\pi}{\ell+r}\right)^{\alpha+\beta-2} \\
&\quad \times \left\{\sum_{n\in\mathbb{Z}} \exp\left(-\frac{\pi^2 T}{2(\ell+r)^2}n^2 + \frac{2\pi\sqrt{-1}\ell}{\ell+r}n\right) n^{\alpha+\beta-2}\right. \\
&\qquad\qquad \left. + (-1)^\beta \sum_{n\in\mathbb{Z}} \exp\left(-\frac{\pi^2 T}{2(\ell+r)^2}n^2\right) n^{\alpha+\beta-2}\right\}.
\end{aligned}
$$

By the identities (2.2), the above is written as

$$
\begin{aligned}
&\frac{(-1)^{\alpha+\beta-1}(\ell+r)}{2^{(\alpha+\beta+1)/2}\pi^{1/2}(\alpha-1)!(\beta-1)!} T^{-(\alpha+\beta-1)/2} \\
&\times \left\{\Theta_{\alpha+\beta-2}\left(\frac{2(\ell+r)}{\sqrt{2T}}, \frac{2\ell}{\sqrt{2T}}\right) + (-1)^\beta \Theta_{\alpha+\beta-2}\left(\frac{2(\ell+r)}{\sqrt{2T}}, 0\right)\right\}.
\end{aligned}
$$

Therefore, we have the following asymptotics in $|\mathbf{a}| \to 0$ and $|\mathbf{b}| \to 0$,

$$
\begin{aligned}
(3.11)\quad & f_N^{\mathrm{A}(-\ell,r)}(T, \mathbf{b}|\mathbf{a}) \\
&= \frac{(-1)^N}{2^{N^2/2}\pi^{N/2}\prod_{k=1}^N \Gamma(k)^2} T^{-N^2/2} \\
&\quad \times \det_{1\le\alpha,\beta\le N}\left[\Theta_{\alpha+\beta-2}\left(\frac{2(\ell+r)}{\sqrt{2T}}, \frac{2\ell}{\sqrt{2T}}\right) + (-1)^\beta \Theta_{\alpha+\beta-2}\left(\frac{2(\ell+r)}{\sqrt{2T}}, 0\right)\right] \\
&\qquad \times \prod_{1\le i<j\le N}\Big\{(a_j-a_i)(b_j-b_i)\Big\} \times \Big\{1 + \mathcal{O}(|\mathbf{a}|, |\mathbf{b}|)\Big\}.
\end{aligned}
$$

By the equalities (3.1) and (3.2), and using the asymptotics formulas (3.4) and (3.11), we can take the limit

$$
\lim_{|\mathbf{a}|,|\mathbf{b}|\to 0} \mathbf{P}\Big(-\ell < L^{\mathbf{a},\mathbf{b}}(N,T), R^{\mathbf{a},\mathbf{b}}(N,T) < r\Big) = \lim_{|\mathbf{a}|,|\mathbf{b}|\to 0} \frac{f_N^{\mathrm{A}(-\ell,r)}(T, \mathbf{b}|\mathbf{a})}{f_N^{\mathrm{A}}(T, \mathbf{b}|\mathbf{a})},
$$

and then the formula (2.3) is obtained.

By the equalities (3.3) and using the asymptotics formulas (3.5) and (3.10), we have

$$
\begin{aligned}
(3.12)\qquad &\mathbf{P}\Big(-\ell < L^{\mathbf{0},\mathbb{R}}(N,T), R^{\mathbf{0},\mathbb{R}}(N,T) < r\Big)\\
&= \lim_{|\mathbf{a}|\to 0} \frac{\displaystyle\int_{\mathbb{W}_N^{\mathrm{A}}(-\ell,r)} d\mathbf{b}\, f_N^{\mathrm{A}(-\ell,r)}(T,\mathbf{b}|\mathbf{a})}{\mathcal{N}_N^{\mathrm{A}}(T,\mathbf{a})}\\
&= \frac{(-1)^{N(N+1)/2}}{2^{N(N+1)/4}\prod_{k=1}^{N}\Gamma(k/2)} T^{-N/2}\\
&\quad\times \int_{\mathbb{W}_N^{\mathrm{A}}(-\ell,r)} d\mathbf{b} \det_{1\le i,j\le N}\Bigg[\Theta_{i-1}\left(\frac{2(\ell+r)}{\sqrt{2T}}, \frac{2\ell}{\sqrt{2T}} + \frac{b_j}{\sqrt{2T}}\right)\\
&\qquad\qquad +(-1)^i\Theta_{i-1}\left(\frac{2(\ell+r)}{\sqrt{2T}}, \frac{b_j}{\sqrt{2T}}\right)\Bigg]\\
&= \frac{(-1)^{N(N+1)/2}}{2^{N(N-1)/4}\prod_{k=1}^{N}\Gamma(k/2)}\\
&\quad\times \int_{\mathbb{W}_N^{\mathrm{A}}(-\ell/\sqrt{2T},r/\sqrt{2T})} d\boldsymbol{x} \det_{1\le i,j\le N}\Bigg[\Theta_{i-1}\left(\frac{2(\ell+r)}{\sqrt{2T}}, \frac{2\ell}{\sqrt{2T}} + x_j\right)\\
&\qquad\qquad +(-1)^i\Theta_{i-1}\left(\frac{2(\ell+r)}{\sqrt{2T}}, x_j\right)\Bigg].
\end{aligned}
$$

If we apply the following lemma, which is known as the de Bruijn identity [5], the formula (2.4) is obtained.

Lemma 3.1. *Let $z_i(x), 1 \le i \le \widehat{N}$ be an integrable piecewise continuous function on a region $\Lambda \subset \mathbb{R}$. Let $\mathbb{W}_N^{\mathrm{A}}(\Lambda) = \Big\{\boldsymbol{x} \in \mathbb{W}_N^{\mathrm{A}} : x_i \in \Lambda, 1 \le i \le N\Big\}$. Then*

$$
(3.13)\qquad \int_{\mathbb{W}_N^{\mathrm{A}}(\Lambda)} d\boldsymbol{x} \det_{1\le i,j\le N}\Big[z_i(x_j)\Big] = \operatorname*{Pf}_{1\le i,j\le \widehat{N}}\Big[Z_{ij}\Big],
$$

where

$$
\begin{aligned}
I_i &= \int_{\Lambda} z_i(x)dx,\\
I_{ij} &= \int_{(x_1,x_2)\in\Lambda^2: x_1<x_2} \det\begin{bmatrix} z_i(x_1) & z_i(x_2)\\ z_j(x_1) & z_j(x_2)\end{bmatrix} dx_1 dx_2,
\end{aligned}
$$

and

$$
Z_{ij} = \begin{cases} I_{ij}, & \text{if } 1 \le i,j \le N,\\ I_i, & \text{if } 1 \le i \le N, j = N+1,\\ -I_j, & \text{if } i = N+1, 1 \le j \le N,\\ 0, & \text{if } i = j = N+1.\end{cases}
$$

The proof of Theorem 2.1 was then completed.

§ 4. Proof of Theorem 2.2

For $h > 0$, consider the restricted region in $\mathbb{W}_N^{\mathrm{C}}$,

$$\mathbb{W}_N^{\mathrm{C}}(h) = \Big\{ \boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{W}_N^{\mathrm{C}} : x_N < h \Big\}.$$

The transition probability density of a one-dimensional Brownian motion in an interval $(0, h)$, where two absorbing walls are put at $x = 0$ and $x = h > 0$, is given by

$$p_{\mathrm{abs}}^{(0,h)}(t, y|x) = \frac{2}{h} \sum_{n \in \mathbb{N}} e^{-\pi^2 t n^2/(2h^2)} \sin\left(\frac{\pi x n}{h}\right) \sin\left(\frac{\pi y n}{h}\right),$$

$t > 0, x, y \in \mathbb{W}_N^{\mathrm{C}}(h)$. For $\mathbf{a}, \mathbf{b} \in \mathbb{W}_N^{\mathrm{C}}(h)$, let

$$f_N^{\mathrm{C}(h)}(T, \mathbf{b}|\mathbf{a}) = \det_{1 \le i,j \le N} \Big[p_{\mathrm{abs}}^{(0,h)}(T, b_i|a_j) \Big].$$

By the Karlin-McGregor formula, it gives the probability density that the absorbing Brownian motion in $\mathbb{W}_N^{\mathrm{C}}(h)$ starting from $\mathbf{a} \in \mathbb{W}_N^{\mathrm{C}}(h)$ at time 0 arrives at $\mathbf{b} \in \mathbb{W}_N^{\mathrm{C}}(h)$ at time T.

Then we have the equalities, for $h > 0$

$$\mathbf{P}\Big(H^{\mathbf{a},\mathbf{b}}(N, T) < h\Big) = \frac{f_N^{\mathrm{C}(h)}(T, \mathbf{b}|\mathbf{a})}{f_N^{\mathrm{C}}(T, \mathbf{b}|\mathbf{a})}, \quad \mathbf{a}, \mathbf{b} \in \mathbb{W}_N^{\mathrm{C}}(h), \tag{4.1}$$

$$\mathbf{P}\Big(H^{\mathbf{a},\mathbb{R}_+}(N, T) < h\Big) = \frac{\displaystyle\int_{\mathbb{W}_N^{\mathrm{C}}(h)} d\mathbf{b}\, f_N^{\mathrm{C}(h)}(T, \mathbf{b}|\mathbf{a})}{\mathcal{N}_N^{\mathrm{C}}(T, \mathbf{a})}, \quad \mathbf{a} \in \mathbb{W}_N^{\mathrm{C}}(h). \tag{4.2}$$

The following asymptotics in $|\mathbf{a}| \to 0$ were reported in [13, 15, 19]; as $|\mathbf{a}| \to 0$,

$$\begin{aligned} f_N^{\mathrm{C}}(T, \mathbf{b}|\mathbf{a}) &= \frac{2^{N/2}}{\pi^{N/2} \prod_{m=1}^N \Gamma(2m)} T^{-N(2N+1)/2} e^{-|\mathbf{b}|^2/(2T)} \\ &\quad \times \prod_{1 \le i < j \le N} \Big\{ (a_j^2 - a_i^2)(b_j^2 - b_i^2) \Big\} \prod_{k=1}^N \Big\{ a_k b_k \Big\} \times \Big\{ 1 + \mathcal{O}(|\mathbf{a}|) \Big\}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \mathcal{N}_N^{\mathrm{C}}(T, \mathbf{b}|\mathbf{a}) &= \left(\frac{2}{\pi}\right)^{N/2} \prod_{m=1}^N \frac{\Gamma(m)}{\Gamma(2m)} T^{-N^2/2} \\ &\quad \times \prod_{1 \le i < j \le N} (a_j^2 - a_i^2) \prod_{k=1}^N a_k \times \Big\{ 1 + \mathcal{O}(|\mathbf{a}|) \Big\}, \end{aligned} \tag{4.4}$$

By multilinearity of determinants and by performing the series expansion of a sine function in a determinant,

$$
\begin{aligned}
(4.5)\qquad & f_N^{\mathrm{C}(h)}(T, \mathbf{b}|\mathbf{a}) \\
&= \frac{1}{N!}\left(\frac{2}{h}\right)^N \sum_{\mathbf{n}\in\mathbb{N}^N} e^{-\pi^2 T|\mathbf{n}|^2/(2h^2)} \\
&\quad \times \det_{1\le i,j\le N}\left[\sin\left(\frac{\pi a_i}{h} n_j\right)\right] \det_{1\le \alpha,\beta\le N}\left[\sin\left(\frac{\pi b_\alpha}{h} n_\beta\right)\right] \\
&= \frac{(-1)^{N(N-1)/2} 2^N \pi^{N^2}}{h^{N(N+1)} \prod_{m=1}^N \Gamma(2m)} \\
&\quad \times \det_{1\le \alpha,\beta\le N}\left[\sum_{n\in\mathbb{N}} e^{-\pi^2 T n^2/(2h^2)} n^{2\alpha-1} \sin\left(\frac{\pi b_\beta n}{h}\right)\right] \\
&\quad \times \prod_{1\le i<j\le N}(a_j^2 - a_i^2) \prod_{k=1}^N a_k \times \Big\{1 + \mathcal{O}(|\mathbf{a}|)\Big\}.
\end{aligned}
$$

Here we note that

$$
\begin{aligned}
& \sum_{n\in\mathbb{N}} e^{-\pi^2 T n^2/(2h^2)} n^{2\alpha-1} \sin\left(\frac{\pi b_\beta n}{h}\right) \\
&\quad = \frac{1}{2\sqrt{-1}} \sum_{n\in\mathbb{Z}} n^{2\alpha-1} \exp\left(-\frac{\pi^2 T}{2h^2} n^2 + \frac{\pi\sqrt{-1} b_\beta}{h} n\right) \\
&\quad = \frac{(-1)^{\alpha-1} h^{2\alpha}}{2^\alpha \pi^{(4\alpha-1)/2}} T^{-\alpha} \Theta_{2\alpha-1}\left(\frac{2h}{\sqrt{2T}}, \frac{b_\beta}{\sqrt{2T}}\right),
\end{aligned}
$$

where we have used the identity (2.2). Then (4.5) is written as

$$
\begin{aligned}
(4.6)\qquad & f_N^{\mathrm{C}(h)}(T, \mathbf{b}|\mathbf{a}) \\
&= \frac{T^{-N(N+1)/2}}{2^{N(N-1)/2} \pi^{N/2} \prod_{m=1}^N \Gamma(2m)} \det_{1\le \alpha,\beta\le N}\left[\Theta_{2\alpha-1}\left(\frac{2h}{\sqrt{2T}}, \frac{b_\beta}{\sqrt{2T}}\right)\right] \\
&\quad \times \prod_{1\le i<j\le N}(a_j^2 - a_i^2) \prod_{k=1}^N a_k \times \Big\{1 + \mathcal{O}(|\mathbf{a}|)\Big\}.
\end{aligned}
$$

On the other hands, in $|\mathbf{b}| \to 0$, (4.5) has the asymptotics

$$
\begin{aligned}
&f_N^{\mathrm{C}(h)}(T, \mathbf{b}|\mathbf{a}) \\
&= \frac{1}{N!}\left(\frac{2}{h}\right)^N \frac{1}{\prod_{i=1}^N \Gamma(2m)^2} \sum_{\mathbf{n}\in\mathbb{N}^N} e^{-\pi^2 T|\mathbf{n}|^2/(2h^2)} \\
&\times \det_{1\le\alpha,\beta\le N}\left[n_\alpha^{2\beta-1}\right] \det_{1\le\gamma,\delta\le N}\left[n_\gamma^{2\delta-1}\right] \\
&\times \prod_{1\le i<j\le N}\Big\{(a_j^2-a_i^2)(b_j^2-b_i^2)\Big\} \prod_{k=1}^N \Big\{a_k b_k\Big\} \times \Big\{1+\mathcal{O}(|\mathbf{a}|,|\mathbf{b}|)\Big\} \\
&= \frac{2^N \pi^{2N^2}}{h^{N(2N+1)}\prod_{m=1}^N \Gamma(2m)^2} \det_{1\le\alpha,\beta\le N}\left[\sum_{n\in\mathbb{N}} n^{2\alpha+2\beta-2} e^{-\pi^2 T n^2/(2h^2)}\right] \\
&\times \prod_{1\le i<j\le N}\Big\{(a_j^2-a_i^2)(b_j^2-b_i^2)\Big\} \prod_{k=1}^N \Big\{a_k b_k\Big\} \times \Big\{1+\mathcal{O}(|\mathbf{a}|,|\mathbf{b}|)\Big\}.
\end{aligned}
$$

By using the identities (2.2), the above is written as follows,

$$
\begin{aligned}
&f_N^{\mathrm{C}(h)}(T, \mathbf{b}|\mathbf{a}) \\
&= \frac{(-1)^N}{2^{N(2N-1)/2}\pi^{N/2}\prod_{m=1}^N \Gamma(2m)^2} T^{-N(2N+1)/2} \\
&\times \det_{1\le\alpha,\beta\le N}\left[\Theta_{2(\alpha+\beta-1)}\left(\frac{2h}{\sqrt{2T}}, 0\right)\right] \\
&\times \prod_{1\le i<j\le N}\Big\{(a_j^2-a_i^2)(b_j^2-b_i^2)\Big\} \prod_{k=1}^N \Big\{a_k b_k\Big\} \times \Big\{1+\mathcal{O}(|\mathbf{a}|,|\mathbf{b}|)\Big\}.
\end{aligned}
\tag{4.7}
$$

Using the asymptotics formulas (4.3) and (4.7), we can take the limit $|\mathbf{a}|, |\mathbf{b}| \to 0$ in (4.1). Then (2.5) is obtained.

By the equality (4.2), and using the asymptotics (4.4) and (4.6), we have

$$
\begin{aligned}
&\mathbf{P}\Big(H^{\mathbf{0},\mathbb{R}_+}(N,T) < h\Big) \\
&= \lim_{|\mathbf{a}|\to 0} \frac{\int_{\mathbb{W}_N^{\mathrm{C}}(h)} d\mathbf{b}\, f_N^{\mathrm{C}(h)}(T,\mathbf{b}|\mathbf{a})}{\mathcal{N}_N^{\mathrm{C}}(T,\mathbf{a})} \\
&= \frac{T^{-N/2}}{2^{N^2/2}\prod_{k=1}^N \Gamma(k)} \int_{\mathbb{W}_N^{\mathrm{C}}(h)} d\mathbf{b} \det_{1\le i,j\le N}\left[\Theta_{2i-1}\left(\frac{2h}{\sqrt{2T}}, \frac{b_j}{\sqrt{2T}}\right)\right]. \\
&= \frac{1}{2^{N(N-1)/2}\prod_{k=1}^N \Gamma(k)} \int_{\mathbb{W}_N^{\mathrm{C}}(h/\sqrt{2T})} d\boldsymbol{x} \det_{1\le i,j\le N}\left[\Theta_{2i-1}\left(\frac{2h}{\sqrt{2T}}, x_j\right)\right].
\end{aligned}
$$

Then, we use the de Bruijn identity given in Lemma 3.1 and the formula (2.6) is obtained. The proof of Theorem 2.2 was then completed.

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References

[1] Altland, A. and Zirnbauer, M. R., Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structure, *Phys. Rev. B*, **55** (1997), 1142–1161.

[2] Andrews, G. E., Askey, R. and Roy, R., *Special Functions*, Cambridge University Press, Cambridge, 1999.

[3] Biane, P., Pitman, J. and Yor, M., Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions, *Bull. Amer. Math. Soc.*, **38** (2001), 435–465.

[4] Borodin, A., Ferrari, P.L., Prähofer, M., Sasamoto, T. and Warren, J., Maximum of Dyson Brownian motion and non-colliding systems with a boundary, *Elect. Commun. in Probab.* **14** (2009), 486-494.

[5] de Bruijn, N. G., On some multiple integrals involving determinants, *J. Indian Math. Soc.* **19** (1955), 133–151.

[6] Feierl, T., The height of watermelons with wall (extended abstract), *Proceedings of the 2007 Conference on Analysis of Algorithms*, Discrete Mathematics and Theoretical Computer Science Proceedings, (2007).

[7] Feierl, T., The height and range of watermelons without wall (extended abstract), *Proceedings of the International Workshop on Combinatorial Algorithms*, edited by J. Fiala, J. Kratochvíl, and M. Miller, *Lecture Notes in Computer Science*, Springer-Verlag, Berlin, vol. 5874 (2009), 242–253.

[8] Fulmek, M., Asymptotics of the average height of 2-watermelons with a wall, *Electron. J. Comb.*, **14** (2007), #R64/1–20.

[9] Fulton, W. and Harris, J., *Representation Theory, A First Course*, Springer, New York, 1991.

[10] Karlin, S. and McGregor, J., Coincidence probabilities, *Pacific J. Math.*, **9** (1959), 1141–1164.

[11] Katori, M, Izumi, M. and Kobayashi, N., Two Bessel bridges conditioned never to collide, double Dirichlet series, and Jacobi theta function, *J. Stat. Phys.*, **131** (2008), 1067–1083.

[12] Katori, M. and Tanemura, H., Scaling limit of vicious walks and two-matrix model, *Phys. Rev.* E **66** (2002), 011105/1–12.

[13] Katori, M. and Tanemura, H., Functional central limit theorems for vicious walkers, *Stoch. Stoch. Rep.*, **75** (2003), 369–390.

[14] Katori, M. and Tanemura, H., Noncolliding Brownian motions and Harish-Chandra formula, *Elect. Comm. in Probab.*, **8** (2003), 112–121.

[15] Katori, M. and Tanemura, H., Symmetry of matrix-valued stochastic processes and non-colliding diffusion particle systems, *J. Math. Phys.*, **45** (2004), 3058–3085.

[16] Katori, M. and Tanemura, H., Infinite systems of non-colliding generalized meanders and Riemann–Liouville differintegrals, *Probab. Theory Relat. Fields*, **138** (2007), 113–156.

[17] Katori, M. and Tanemura, H., Noncolliding Brownian motion and determinantal processes, *J. Stat. Phys.*, **129** (2007), 1233–1277.

[18] Katori, M.,Tanemura, H., Nagao, T. and Komatsuda, N., Vicious walk with a wall, noncolliding meanders, chiral and Bogoliubov-de Gennes random matrices, *Phys. Rev.* E **68** (2003), 021112/1–16.

[19] Kobayashi, N., Izumi, M. and Katori, M., Maximum distributions of bridges of noncolliding Brownian paths, *Phys. Rev.* E, **78** (2008), 051102/1–15.

[20] Mehta, M. L., *Random Matrices*, 3rd ed., Elsevier Academic Press, London, 2004.

[21] Pandey, A. and Mehta, M. L., Gaussian ensembles of random Hermitian intermediate between orthogonal and unitary ones, *Commun. Math. Phys.* **87**, (1983) 449–468.

[22] Revuz, D. and Yor, M., *Continuous Martingales and Brownian Motion*, 3rd ed., Springer, New York, 1998.

[23] Schehr, G., Majumdar, S. N., Comtet, A. and Randon-Furling, J., Exact distribution of the maximum height of p vicious walkers, *Phys. Rev. Lett.* **101**, (2008) 150601/1–4.

[24] Stembridge, J. R., Nonintersecting paths, pfaffians, and the plane partitions, *Adv. in Math.* **83** (1990), 96–131.

[25] Tracy, C. A. and Widom, H., Nonintersecting Brownian excursions, *Ann. Appl. Probab.* **17** (2007), 953–979.