RIMS Kôkyûroku Bessatsu
B27 (2011), 67–78

On limit behavior of eigenvalues spacing for 1-D random Schrödinger operators

By

Shinichi KOTANI*

Abstract

A limit distribution for spacing of random eigenvalues is obtained. The eigenvalues arise from 1-D Schrödinger operators with decaying random potentials.

§ 1. Background

Let

$$H_l = -\frac{d^2}{dt^2} + q \quad \text{on} \quad (0, l) \text{ with Dirichlet boundary condition.}$$

where q is a random potential. Let $\{X_t\}_{t \geq 0}$ be a Brownian motion on a compact Riemannian manifold M and F be a smooth non- constant function on M. For a positive decreasing function $a(t)$, a typical random decaying potential q is given by

$$q(t) = a(t) F(X_t).$$

Assume here for simplicity

$$a(t) \sim ct^{-\alpha} \text{ as } t \to \infty \text{ with some constant } c > 0.$$

Case 1. If $\alpha = 0$, Goldseid-Molchanov-Pastur([1]) proved that the operator H_∞ has only point spectrum distributing densely on $[0, \infty)$.
Case 2. Kotani-Ushiroya([2]) obtained the followings:
(i) $0 < \alpha < \frac{1}{2} \Longrightarrow H_\infty$ has only point spectrum distributing densely on $[0, \infty)$.

Received July 8, 2010. Revised December 5, 2010.
2000 Mathematics Subject Classification(s): 60F05, 34L20:
Key Words: *random eigenvalues, spacing*:
Supported by Grant-in-Aid for Scientific Research(C) 22540163
*Kwanseigakuin University, Sanda 669-1337, Japan.
e-mail: kotani@kwansei.ac.jp

(ii) $\alpha = \frac{1}{2} \Longrightarrow H_\infty$ has only point spectrum distributing densely on $[0, E_0]$ and purely singular continuous spectrum on $[E_0, \infty)$, where E_0 can be determined exactly from F and $a(t)$.
(iii) $\alpha > \frac{1}{2}$ and $\displaystyle\int_M F(x)dx = 0 \Longrightarrow H_\infty$ has purely absolutely continuous spectrum on $[0,\infty)$.

Motivated by condensed matter physics, Molchanov investigated finer structure of the spectrum in Case 1. Let $\{E_1(l) < E_2(l) \le \cdots\}$ be the eigenvalues of H_l and for an interval I set

$$N_l(I) = \#\{k;\ E_k(l) \in I\}.$$

Molchanov ([3]): For each $k = 0, 1, 2, \cdots$ and $E > 0$, $a > 0$

$$P\left(N_l\left(\left(E - \frac{a}{2l}, E + \frac{a}{2l}\right)\right) = k\right) \overset{l\to\infty}{\to} e^{-an(E)} \frac{(an(E))^k}{k!} \tag{1.1}$$

holds, where

$$n(E) = \frac{dN(E)}{dE}.$$

To investigate the spacing between two neighboring eigenvalues let

$$\Delta_n = n\left(E_{n+1}(nt) - E_n(nt)\right), \qquad t = N(E)^{-1}.$$

Molchanov's result (1.1) implies in particular that, in Case 1

$$\lim_{n\to\infty} P(\Delta_n \le x) = 1 - e^{-n(E)x/N(E)}$$

holds. A similar problem can be considered in the case of random matrices $\{X(\omega)\}$, where $X(\omega)$ is a sample of real symmetric $n \times n$ matrix whose elements $\{x_{ij}(\omega)\}_{1\le i\le j\le n}$ are independent and distributed by an identical Gaussian law. Then it is known that we have a different limit distribution.

For random Schrödinger operators in higher dimensions we can ask the same question. In discrete case, the result corresponding to Molchanov's one was established by Minami ([4]) in point spectrum region. In continuous case, Combes-Germinet-Klein ([5]) obtained recently a similar result.

The purpose of this note is to investigate the limit behavior of Δ_n in Case 2. In the third section we try to obtain a kind of central limit theorem, however the proof has not been completed yet. We would like to postpone making the argument rigorous in future.

§ 2. First limit theorem (Law of large numbers)

For $E > 0$, setting $\kappa = \sqrt{E}$, we have the equivalence:

$$
\begin{aligned}
& -x_t'' + a(t)F\left(X_t(\omega)\right)x_t = Ex_t \\
(2.1) \qquad & \Longleftrightarrow \\
& \begin{pmatrix} x_t \\ y_t \end{pmatrix}' = \begin{pmatrix} 0 & \kappa \\ \kappa^{-1}a(t)F\left(X_t(\omega)\right) - \kappa & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}.
\end{aligned}
$$

Since we are considering the Dirichlet boundary condition, we assume

$$x_0 = 0, \ y_0 = 1.$$

We transform the variables $\{x_t, y_t\}$ to

$$\begin{pmatrix} \widetilde{x}_t \\ \widetilde{y}_t \end{pmatrix} = \begin{pmatrix} \cos\kappa t & -\sin\kappa t \\ \sin\kappa t & \cos\kappa t \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix},$$

and set

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = r_t \begin{pmatrix} \sin\theta_t \\ \cos\theta_t \end{pmatrix}, \ \begin{pmatrix} \widetilde{x}_t \\ \widetilde{y}_t \end{pmatrix} = \widetilde{r}_t \begin{pmatrix} \sin\widetilde{\theta}_t \\ \cos\widetilde{\theta}_t \end{pmatrix}, \quad \theta_t = \kappa t + \widetilde{\theta}_t.$$

Then the equation (2.1) is equivalent to

$$(2.2) \qquad \frac{d}{dt}\begin{pmatrix} \log\widetilde{r}_t^2 \\ \widetilde{\theta}_t \end{pmatrix} = \kappa^{-1}a(t)F\left(X_t\right) \begin{pmatrix} \sin 2\left(\kappa t + \widetilde{\theta}_t\right) \\ -\sin^2\left(\kappa t + \widetilde{\theta}_t\right) \end{pmatrix},$$

and it holds that

$$
(2.3) \qquad
\begin{aligned}
\widetilde{\theta}_t &= -\frac{1}{2\kappa}\int_0^t a(s)F\left(X_s\right)ds + \frac{1}{2\kappa}\operatorname{Re}\int_0^t a(s)F\left(X_s\right)e^{2i\left(\kappa s + \widetilde{\theta}_s\right)}ds, \\
\widetilde{r}_t^2 &= \exp\left(\frac{1}{\kappa}\operatorname{Im}\int_0^t a(s)F\left(X_s\right)e^{2i\left(\kappa s + \widetilde{\theta}_s\right)}ds\right), \\
\frac{\partial\theta_t}{\partial\kappa} &= \int_0^t \frac{\widetilde{r}_s^2}{\widetilde{r}_t^2}ds + \frac{1}{2\kappa^2}\int_0^t \frac{\widetilde{r}_s^2}{\widetilde{r}_t^2}a(s)F\left(X_s\right)\left(1 - \operatorname{Re}e^{2i\left(\kappa s + \widetilde{\theta}_s\right)}\right)ds.
\end{aligned}
$$

We remark here that Sturm oscillation theorem implies for $n = 1, 2, \cdots$

$$(2.4) \qquad \theta_t(\kappa) = n\pi \Longleftrightarrow \kappa = \sqrt{E_n(t)}.$$

Our method to investigate the problem is to use the identities (2.3) and the relation (2.4).

First we study a non-random estimate of $E_n(nt)$ depending only on F and $a(s)$. Set

$$\kappa_+ = \max_{t\geq 0, x\in M} (F(x)a(t)), \quad \kappa_- = \min_{t\geq 0, x\in M} (F(x)a(t)).$$

Then the mini-max principle for eigenvalues shows

$$\left(\frac{n\pi}{l}\right)^2 + \kappa_- \leq E_n(l) \leq \left(\frac{n\pi}{l}\right)^2 + \kappa_+.$$

Hence, if $\left(\frac{\pi}{t}\right)^2 + \kappa_- \geq 0$

$$\text{(2.5)} \qquad \sqrt{\left(\frac{\pi}{t}\right)^2 + \kappa_-} \leq \sqrt{E_n(nt)} \leq \sqrt{\left(\frac{\pi}{t}\right)^2 + \kappa_+}$$

Set

$$\kappa_0 = \frac{\pi}{\sqrt{(-\kappa_-)\vee 0}}, \quad \text{where } \frac{1}{0} = \infty$$

Then, if $t < \kappa_0$ is valid, then it holds necessarily that $E_n^{(nt)} > 0$. On the other hand, from (2.3) it follows that

$$\text{(2.6)} \qquad \left|\widetilde{\theta}_t\right| \leq \frac{\|F\|_\infty}{\kappa}\int_0^t a(s)ds,$$

hence

$$\left|\widetilde{\theta}_{nt}\left(\sqrt{E_n(nt)}\right)\right| \leq \frac{\|F\|_\infty}{\sqrt{E_n(nt)}}\int_0^{nt} a(s)ds \leq \frac{t}{\pi}\|F\|_\infty\int_0^{nt} a(s)ds$$

holds. Then the identity

$$n\pi = \theta_{nt}\left(\sqrt{E_n(nt)}\right) = nt\sqrt{E_n(nt)} + \widetilde{\theta}_{nt}\left(\sqrt{E_n(nt)}\right)$$

shows

$$\left|\sqrt{E_n(nt)} - \frac{\pi}{t}\right| = \frac{\left|\widetilde{\theta}_{nt}\left(\sqrt{E_n(nt)}\right)\right|}{nt} \leq \frac{\|F\|_\infty}{n\pi}\int_0^{nt} a(s)ds \to 0 \text{ as } n\to\infty.$$

and gives

Proposition 2.1. *As long as t remains on any compact set of $(0,\kappa_0)$, $\sqrt{E_n(nt)}$ converges to π/t as $n\to\infty$ uniformly with respect to t.*

In order to investigate the behavior of the difference

$$\Delta_n = n\left(E_{n+1}(nt) - E_n(nt)\right),$$

we have to know the asymptotic behavior of

$$Y_t(\kappa,\beta)=\int_0^t a(s)F(X_s)\,e^{\beta i\left(\kappa s+\widetilde{\theta}_s\right)}ds \quad \text{for } \beta\in\mathbb{R}.$$

We impose a condition on $a(t)$:

$$\int_0^\infty a(s)^2 s^\delta ds<\infty \quad \text{for some } \delta>0. \tag{2.7}$$

Lemma 2.2. *Assume $a(t)$ satisfies* (2.7) *and F fulfils*

$$\int_M F(x)dx=0.$$

Let K be a compact set of $(0,\infty)$. Then, for any $\beta\in\mathbb{R}$

$$\lim_{t\to\infty} Y_t(\kappa,\beta)=Y_\infty(\kappa,\beta) \ \textit{uniformly on } K \quad \textit{a.s.}.$$

Moreover, for any $\epsilon<\delta/2$ and $\beta\in\mathbb{R}$

$$\sup_{t\geq 0,\kappa,\kappa_1\in K}\frac{|Y_t(\kappa,\beta)-Y_t(\kappa_1,\beta)|}{|\kappa-\kappa_1|^\epsilon}<\infty \quad \textit{a.s.}$$

is valid.

Proof. The first statement is (2) of Lemma 2.2 of ([2]). The necessary estimates for the second statement are given in the sublemma of ([2]). They proved the estimates

$$E\left(\sup_{t\geq 0}|Y_t(\kappa,\beta)-Y_t(\kappa_1,\beta)|^{2p}\right)\leq \textit{const.}\,|\kappa-\kappa_1|^{p\eta} \quad \text{if } \kappa,\kappa_1\in K.$$

Then Kolmogorov theorem implies the conclusion of Lemma. □

Remark. In the proof of the sublemma, they estimated

$$E\left(\sup_{s\in(t_0,t)}|J(s)|^{2p}\right)$$

by decomposing it into the 5 terms in page 254. However the third term should be

$$\textit{const.}\left\{|\lambda-\lambda'|^\eta\int_{t_0}^t\left(1+|s|^\eta\right)a(s)^2ds\right\}^p.$$

Therefore the statement of the sublemma must be

$$E\left(\sup_{t\in[0,\infty)}|I(t,\omega,\lambda,\lambda')|^{2p}\right)\leq \textit{const.}\,|\lambda-\lambda'|^{p\eta}$$

instead of

$$E\left(\sup_{t\in[0,\infty)}|I(t,\omega,\lambda,\lambda')|^{2p}\right)\leq \textit{const.}\,|\lambda-\lambda'|^{2p\eta}.$$

Then we can improve Proposition 2.1:

Theorem 2.3. *Assume the condition* (2.7) *and* F *has mean* 0. *Then, we have*

$$n\left(\sqrt{E_n(nt)}-\frac{\pi}{t}\right)\underset{n\to\infty}{\to}-\frac{1}{t}\widetilde{\theta}_\infty\left(\frac{\pi}{t}\right)=\pi\int_0^\infty a(s)F(X_s)\sin^2\left(\frac{\pi s}{t}+\widetilde{\theta}_s\left(\frac{\pi}{t}\right)\right)ds \quad a.s..$$

The convergence is compact uniform with respect to $t\in(0,\kappa_0)$.

Remark. We denote $\widetilde{\theta}_t$ by $\widetilde{\theta}_t(\kappa)$ if it is necessary to make the $\kappa-$dependence explicit.

Setting

$$\Lambda_n(t)=n\left(\sqrt{E_{n+1}(nt)}-\sqrt{E_n(nt)}\right),$$

which is an equivalent quantity with Δ_n, we can conclude

Theorem 2.4. *Suppose* $a(t)$ *satisfies the condition* (2.7) *and* F *has mean* 0. *Then, if* $t<\kappa_0$

$$\Lambda_n(t)=\frac{\pi}{t}+O\left(n^{-\epsilon}\right)\quad a.s.\quad for\ any\ \epsilon<\delta/2.$$

This holds compact uniformly in $t\in(0,\kappa_0)$.

Proof. From (2.3) we have

$$\theta_t(\kappa)=\kappa t-\frac{1}{2\kappa}\int_0^t a(s)F(X_s)\,ds+\frac{1}{2\kappa}\operatorname{Re}\int_0^t a(s)F(X_s)\,e^{2i\left(\kappa s+\widetilde{\theta}_s\right)}ds.$$

Hence

$$\theta_t(\kappa)-\theta_t(\kappa_1)=(\kappa-\kappa_1)\,t-\frac{1}{2}\left(\frac{1}{\kappa}-\frac{1}{\kappa_1}\right)\int_0^t a(s)F(X_s)\,ds$$

$$+\frac{1}{2}\left(\frac{1}{\kappa}-\frac{1}{\kappa_1}\right)\operatorname{Re}Y_t(\kappa,2)+\frac{1}{2\kappa_1}\operatorname{Re}\left(Y_t(\kappa,2)-Y_t(\kappa_1,2)\right) \tag{2.8}$$

is valid. Now we assume t moves in a compact set K of $(0,\kappa_0)$ and set

$$l_-=\min_{t\in\widetilde{K}}\sqrt{\left(\frac{\pi}{t}\right)^2+\kappa_-},\quad l_+=\max_{t\in\widetilde{K}}\sqrt{\left(\frac{\pi}{t}\right)^2+\kappa_+},$$

where $\widetilde{K}$ is a compact set such that $K\subset\left(\widetilde{K}\right)^\circ\subset\widetilde{K}\subset(0,\kappa_0)$. Then the inequality (2.5) assures that $\sqrt{E_n(nt)}$ and $\sqrt{E_{n+1}(nt)}$ move in $[l_-,l_+]$. Therefore, setting $\kappa=\sqrt{E_{n+1}(nt)}$, $\kappa_1=\sqrt{E_n(nt)}$ and $t\to nt$, we have

$$\pi=t\Lambda_n(t)-\frac{1}{2}\left(\frac{1}{\kappa}-\frac{1}{\kappa_1}\right)\int_0^{nt}a(s)F(X_s)\,ds+\frac{1}{2}\left(\frac{1}{\kappa}-\frac{1}{\kappa_1}\right)\operatorname{Re}Y_{nt}(\kappa,2)$$

$$+\frac{1}{2\kappa_1}\operatorname{Re}\left(Y_{nt}(\kappa,2)-Y_{nt}(\kappa_1,2)\right). \tag{2.9}$$

Then Lemma2.2 implies

$$\frac{1}{2}\left(\frac{1}{\kappa}-\frac{1}{\kappa_1}\right)\int_0^{nt} a(s)F\left(X_s\right)ds = O\left(\frac{\Lambda_n\left(t\right)}{n}\right),$$
$$\frac{1}{2}\left(\frac{1}{\kappa}-\frac{1}{\kappa_1}\right)\operatorname{Re}Y_{nt}(\kappa,2) = O\left(\frac{\Lambda_n\left(t\right)}{n}\right),$$
$$\frac{1}{2\kappa_1}\operatorname{Re}\left(Y_{nt}(\kappa,2)-Y_{nt}(\kappa_1,2)\right) = O\left(\frac{\Lambda_n\left(t\right)}{n}\right)^{\epsilon}. \tag{2.10}$$

A priori we know $\Lambda_n\left(t\right)/n$ is bounded, hence from (2.9) the boundedness of $\Lambda_n\left(t\right)$ follows. Substituting this consequence to the estimates (2.10), we see

$$\Lambda_n\left(t\right)-\frac{\pi}{t} = O\left(n^{-\epsilon}\right).$$

□

§ 3. Second limit theorem

In this section we assume

$$a(t) = t^{-\alpha} \text{ for } t \geq 1 \text{ with } \alpha \in \left(\frac{1}{2},1\right) \quad \text{and} \quad \int_M F(x)dx = 0.$$

To investigate the asymptotic behavior of the error term we consider a sequence of stochastic processes with parameters c_1 and c_2

$$\Theta_t^{(n)}\left(c_1,c_2\right) = n^{\alpha-\frac{1}{2}}\left(\widetilde{\theta}_{nt}\left(\kappa+\frac{c_1}{n}\right)-\widetilde{\theta}_{nt}\left(\kappa+\frac{c_2}{n}\right)\right).$$

Define

$$g(x) = \left(L+2i\kappa\right)^{-1}F(x), \quad \text{where } L = \frac{1}{2}\Delta.$$

Then, Ito's formula implies

$$\int_0^t e^{2i\kappa s}F\left(X_s\right)ds = e^{2i\kappa t}g\left(X_t\right)+M_t, \text{with a martingale } M_t = M_t\left(\kappa\right).$$

Then we have

$$\int_0^t a(s)F\left(X_s\right)e^{2i\left(\kappa s+\widetilde{\theta}_s\right)}ds = \int_0^t a(s)e^{2i\widetilde{\theta}_s}d\left(e^{2i\kappa s}g\left(X_s\right)\right)+\int_0^t a(s)e^{2i\widetilde{\theta}_s}dM_s$$
$$= a(0)g\left(X_0\right)-a(t)e^{2i\left(\widetilde{\theta}_t+\kappa t\right)}g\left(X_t\right)-\int_0^t a'(s)e^{2i\left(\kappa s+\widetilde{\theta}_s\right)}g\left(X_s\right)ds$$
$$-2i\int_0^t a(s)e^{2i\left(\kappa s+\widetilde{\theta}_s\right)}\widetilde{\theta}'_s g\left(X_s\right)ds+\int_0^t a(s)e^{2i\widetilde{\theta}_s}dM_s.$$

Therefore, substituting (2.3) into the above identity leads us to

$$\begin{aligned}(3.1)\qquad \int_0^t a(s)F\left(X_s\right)e^{2i\left(\kappa s+\widetilde{\theta}_s\right)}ds &= -a(0)g\left(X_0\right)+a(t)e^{2i\left(\widetilde{\theta}_t+\kappa t\right)}g\left(X_t\right)\\ &\quad -\frac{i}{2\kappa}\int_0^t a(s)^2F\left(X_s\right)g\left(X_s\right)ds+I_t\left(\kappa\right)+J_t\left(\kappa\right)+N_t\left(\kappa\right),\end{aligned}$$

where

(3.2)

$$\begin{aligned}I_t\left(\kappa\right) &= -\int_0^t a'(s)e^{2i\left(\kappa s+\widetilde{\theta}_s\right)}g\left(X_s\right)ds\\ J_t\left(\kappa\right) &= -\frac{i}{2\kappa}\int_0^t a(s)^2e^{4i\left(\kappa s+\widetilde{\theta}_s\right)}F\left(X_s\right)g\left(X_s\right)ds+\frac{i}{\kappa}\int_0^t a(s)^2e^{2i\left(\kappa s+\widetilde{\theta}_s\right)}F\left(X_s\right)g\left(X_s\right)ds\\ N_t\left(\kappa\right) &= \int_0^t a(s)e^{2i\widetilde{\theta}_s}dM_s.\end{aligned}$$

Setting $\kappa_1=c_1/n+\kappa,\kappa_2=c_2/n+\kappa$ we have

$$\begin{aligned}\Theta_t^{(n)}\left(c_1,c_2\right) &= -n^{\alpha-\frac{1}{2}}\left(\frac{1}{2\kappa_1}-\frac{1}{2\kappa_2}\right)\int_0^{nt}a(s)F\left(X_s\right)ds\\ &\quad +n^{\alpha-\frac{1}{2}}\operatorname{Re}\int_0^{nt}a(s)F\left(X_s\right)\left(\frac{e^{2i\left(\kappa_1 s+\widetilde{\theta}_s(\kappa_1)\right)}}{2\kappa_1}-\frac{e^{2i\left(\kappa_2 s+\widetilde{\theta}_s(\kappa_2)\right)}}{2\kappa_2}\right)ds\\ &= -n^{\alpha-\frac{1}{2}}\left(\frac{1}{2\kappa_1}-\frac{1}{2\kappa_2}\right)\int_0^{nt}a(s)F\left(X_s\right)ds\\ &\quad +n^{\alpha-\frac{1}{2}}\left(\frac{1}{2\kappa_1}-\frac{1}{2\kappa_2}\right)\operatorname{Re}\int_0^{nt}a(s)F\left(X_s\right)e^{2i\left(\kappa_1 s+\widetilde{\theta}_s(\kappa_1)\right)}ds\\ &\quad +\frac{n^{\alpha-\frac{1}{2}}}{2\kappa_2}\operatorname{Re}\left(\int_0^{nt}a(s)F\left(X_s\right)\left(e^{2i\left(\kappa_1 s+\widetilde{\theta}_s(\kappa_1)\right)}-e^{2i\left(\kappa_2 s+\widetilde{\theta}_s(\kappa_2)\right)}\right)ds\right).\end{aligned}$$

Since, $\kappa_1-\kappa_2=\left(c_1-c_2\right)/n$

$$\left|n^{\alpha-\frac{1}{2}}\left(\frac{1}{2\kappa_1}-\frac{1}{2\kappa_2}\right)\int_0^{nt}a(s)F\left(X_s\right)ds\right|\leq Cn^{\alpha-\frac{1}{2}-1+1-\alpha}=Cn^{-\frac{1}{2}},$$

$$\left|n^{\alpha-\frac{1}{2}}\left(\frac{1}{2\kappa_1}-\frac{1}{2\kappa_2}\right)\operatorname{Re}\int_0^{t}a(s)F\left(X_s\right)e^{2i\left(\kappa_1 s+\widetilde{\theta}_s(\kappa_1)\right)}ds\right|\leq Cn^{\alpha-\frac{1}{2}-1+1-\alpha}=Cn^{-\frac{1}{2}},$$

hold with some constant C. Hence

$$\Theta_t^{(n)}\left(c_1,c_2\right)=\frac{n^{\alpha-\frac{1}{2}}}{2\kappa}\operatorname{Re}\left(\int_0^{nt}a(s)F\left(X_s\right)\left(e^{2i\left(\kappa_1 s+\widetilde{\theta}_s(\kappa_1)\right)}-e^{2i\left(\kappa_2 s+\widetilde{\theta}_s(\kappa_2)\right)}\right)ds\right)+\eta_t^{(n)}$$

$$\left|\eta_t^{(n)}\right|\leq const.n^{-\frac{1}{2}}$$

is valid. Denote g corresponding to κ by g_κ. Then there exists a constant C' such that

$$\left|g_{\kappa_1}(x) - g_{\kappa_2}(x)\right| \le C' \left|\kappa_1 - \kappa_2\right| = const.n^{-1}.$$

We use the inequality

$$\left|e^{2i\kappa_1 s} - e^{2i\kappa_2 s}\right| \le 2\left|\kappa_1 - \kappa_2\right| s = const.n^{-1}.$$

Then we have

$$n^{\alpha-\frac{1}{2}}\int_1^{nt} \left|a'(s)\right| sn^{-1} ds \le const.n^{-\frac{1}{2}}, \quad n^{\alpha-\frac{1}{2}}\int_1^{nt} a(s)^2 sn^{-1} ds \le const.n^{\frac{1}{2}-\alpha},$$
$$n^{\alpha-\frac{1}{2}}\int_1^{nt} \left|a'(s)\right| ds \le const.n^{-\frac{1}{2}}, \quad n^{\alpha-\frac{1}{2}}\int_1^{nt} a(s)^2 ds \le const.n^{\frac{1}{2}-\alpha}$$

hence

$$\Theta_t^{(n)}(c_1, c_2) = n^{\alpha-\frac{1}{2}} \operatorname{Re}\left(S_t^{(n)}(\kappa_1) - S_t^{(n)}(\kappa_2)\right) + \xi_t^{(n)} \tag{3.3}$$
$$\left|\xi_t^{(n)}\right| \le const.n^{\frac{1}{2}-\alpha},$$

where

$$S_t^{(n)}(\kappa) = \int_0^{nt} a(s) e^{2i\widetilde{\theta}_s(\kappa)} d\frac{M_s(\kappa)}{2\kappa}.$$

Set

$$T_t^{(n)}(c_1, c_2) = n^{\alpha-\frac{1}{2}} \operatorname{Re}\left(S_t^{(n)}(\kappa_1) - S_t^{(n)}(\kappa_2)\right).$$

Now we compute $\left\langle T^{(n)}(c_1, c_2)\right\rangle_t$ and obtain the limit of $T_t^{(n)}(c_1, c_2)$ when $n \to \infty$. Introduce the notation:

$$\begin{aligned}[g_1, g_2](x) &= L(g_1 g_2)(x) - g_1(x)(Lg_2)(x) - g_2(x)(Lg_1)(x) \\ &= (\nabla g_1, \nabla g_2)(x).\end{aligned}$$

To simplify the notations we set

$$g_\kappa(s, x) = \frac{1}{2\kappa} e^{2i\widetilde{\theta}_s(\kappa)} e^{2i\kappa s} g_\kappa(x).$$

Then

$$\begin{aligned}\left\langle S^{(n)}(\kappa), S^{(n)}(\kappa')\right\rangle_t &= \int_0^{nt} a(s)^2 \left[g_\kappa(s), g_{\kappa'}(s)\right](X_s) ds, \\ \left\langle S^{(n)}(\kappa), \overline{S^{(n)}(\kappa')}\right\rangle_t &= \int_0^{nt} a(s)^2 \left[g_\kappa(s), \overline{g_{\kappa'}(s)}\right](X_s) ds.\end{aligned}$$

and

$$\begin{aligned}
&n^{1-2\alpha}\left\langle T^{(n)}\left(c_1,c_2\right),T^{(n)}\left(c_1',c_2'\right)\right\rangle_t\\
&=\left\langle \mathrm{Re}\left(S^{(n)}\left(\kappa_1\right)-S^{(n)}\left(\kappa_2\right)\right),\mathrm{Re}\left(S^{(n)}\left(\kappa_1'\right)-S^{(n)}\left(\kappa_2'\right)\right)\right\rangle_t\\
&=\int_0^{nt}a(s)^2\left[\mathrm{Re}\left(g_{\kappa_1}\left(s,\cdot\right)-g_{\kappa_2}\left(s,\cdot\right)\right),\mathrm{Re}\left(g_{\kappa_1'}\left(s,\cdot\right)-g_{\kappa_2'}\left(s,\cdot\right)\right)\right]\left(X_s\right)ds
\end{aligned}$$

hold, where $\kappa_1'=c_1'/n+\kappa,\kappa_2'=c_2'/n+\kappa$. Hence

$$\begin{aligned}
&\left\langle T^{(n)}\left(c_1,c_2\right),T^{(n)}\left(c_1',c_2'\right)\right\rangle_t\\
&=n^{2\alpha-1}\int_0^{nt}a(s)^2\left[\mathrm{Re}\left(g_{\kappa_1}\left(s,\cdot\right)-g_{\kappa_2}\left(s,\cdot\right)\right),\mathrm{Re}\left(g_{\kappa_1'}\left(s,\cdot\right)-g_{\kappa_2'}\left(s,\cdot\right)\right)\right]\left(X_s\right)ds
\end{aligned}$$

is valid. Now applying Lemma 2.2 for $\beta=0,2$ and (2.3) imply

$$\sup_{s\geq 0}\frac{\left|\widetilde{\theta}_s\left(\kappa_1\right)-\widetilde{\theta}_s\left(\kappa\right)\right|}{\left|\kappa_1-\kappa\right|^{\epsilon}}<\infty\quad \text{a.s.},$$

which shows

$$\sup_{s\geq 0}\frac{\left|e^{2i\widetilde{\theta}_s\left(\kappa_1\right)}-e^{2i\widetilde{\theta}_s\left(\kappa\right)}\right|}{\left|\kappa_1-\kappa\right|^{\epsilon}}<\infty\quad \text{a.s.}$$

for any $\epsilon<\alpha-\frac{1}{2}$. Hence we see

$$\begin{aligned}
&\left\langle T^{(n)}\left(c_1,c_2\right),T^{(n)}\left(c_1',c_2'\right)\right\rangle_t\\
&=n^{2\alpha-1}\int_0^{nt}a(s)^2\left[\mathrm{Re}\left(e^{2ic_1s/n}-e^{2ic_2s/n}\right)g_\kappa\left(s,\cdot\right),\mathrm{Re}\left(e^{2ic_1's/n}-e^{2ic_2's/n}\right)g_\kappa\left(s,\cdot\right)\right]\left(X_s\right)ds\\
&+O\left(n^{-\epsilon}\right)\quad \text{a.s.}.
\end{aligned}$$

Because of the strong averaging property of $\{X_t\}$, the main term is

$$\frac{n^{2\alpha-1}}{8\kappa^2}\left\langle\left[g_\kappa,\overline{g_\kappa}\right]\right\rangle\int_0^{nt}a(s)^2\,\mathrm{Re}\left(e^{2ic_1s/n}-e^{2ic_2s/n}\right)\overline{\left(e^{2ic_1's/n}-e^{2ic_2's/n}\right)}ds=l_t\left(\left(c_1,c_2\right),\left(c_1',c_2'\right)\right),$$

where

$$l_t\left(\left(c_1,c_2\right),\left(c_1',c_2'\right)\right)=\frac{\left\langle\left[g_\kappa,\overline{g_\kappa}\right]\right\rangle}{8\kappa^2}\int_0^t s^{-2\alpha}\,\mathrm{Re}\left(e^{2ic_1s}-e^{2ic_2s}\right)\overline{\left(e^{2ic_1's}-e^{2ic_2's}\right)}ds,$$

and $\langle g\rangle$ denote the integral of g by the Riemannian volume element dx. To give a complete proof we have to use the martingale representation if the term $e^{2i\widetilde{\theta}_s(\kappa)}e^{2i\kappa s}$ appears. The detail of the proof will be given elsewhere. We remark

$$\left\langle\left[g_\kappa,\overline{g_\kappa}\right]\right\rangle=\left(-\Delta g_\kappa,g_\kappa\right)=-2\,\mathrm{Re}\left(\left(L+2i\kappa\right)^{-1}F,F\right).$$

Let $\{Z(t, c_1, c_2)\}_{t\geq 0,\ c_1,\ c_2\geq 0}$ be a Gaussian system with covariance

$$l_{t\wedge t'}\left((c_1, c_2), (c_1', c_2')\right).$$

Since $T_t^{(n)}(c_1, c_2)$ is a sequence of martingales, the martingale central limit theorem implies

Lemma 3.1. *For fixed* $\kappa > 0$, *as* $n \to \infty$ $\left\{\Theta_t^{(n)}(c_1, c_2)\right\}_{t\geq 0,\ c_1,\ c_2\geq 0}$ *converges weakly to*

$$\{Z(t, c_1, c_2)\}_{t\geq 0,\ c_1,\ c_2\geq 0}.$$

From now on we make a heuristic argument to guess a result. Set

$$\kappa + \frac{c_2}{n} = \sqrt{E_n(nt)}, \quad \kappa + \frac{c_1}{n} = \sqrt{E_{n+1}(nt)} \ \text{ with } \kappa = \frac{\pi}{t}.$$

Then we can describe the difference by $\Theta_t^{(n)}$ as follows:

$$\begin{aligned}
& n^{\alpha-\frac{1}{2}}\left(\Lambda_n(t) - \frac{\pi}{t}\right) \\
&= n^{\alpha-\frac{1}{2}}\left(\Lambda_n(t) - \frac{1}{t}\left(\theta_{nt}\left(\sqrt{E_{n+1}(nt)}\right) - \theta_{nt}\left(\sqrt{E_n(nt)}\right)\right)\right) \\
&= -\frac{n^{\alpha-\frac{1}{2}}}{t}\left(\widetilde{\theta}_{nt}\left(\kappa + \frac{c_1}{n}\right) - \widetilde{\theta}_{nt}\left(\kappa + \frac{c_2}{n}\right)\right) \\
&= -\frac{1}{t}\Theta_t^{(n)}(c_1, c_2).
\end{aligned}$$

From Theorems 2.3,2.4 it follows that

$$\begin{aligned}
& c_2 \underset{n\to\infty}{\to} -\frac{1}{t}\widetilde{\theta}_\infty\left(\frac{\pi}{t}\right) = \pi\int_0^\infty a(s)F(X_s)\sin^2\left(\frac{\pi s}{t} + \widetilde{\theta}_s\left(\frac{\pi}{t}\right)\right) ds, \\
& c_1 - c_2 \underset{n\to\infty}{\to} \frac{\pi}{t} \quad \text{a.s..}
\end{aligned}$$

Hence, to obtain the conclusion we have to investigate the limiting behavior of joint distributions:

$$\widetilde{\theta}_{nt}(\kappa), \ \Theta_t^{(n)}(c_1, c_2).$$

The complete proof will be given elsewhere.

References

[1] Goldseid,I.Ja. - Molchanov,S.A. - Pastur,L.A., A random homogeneous Schrödinger operator has a pure point spectrum (Russian), *Funct. Anal. Appl.* , **11**(1977), 1-10.

[2] Kotani,S. - Ushiroya,N., One-dimensional Schrödinger operators with random decaying potentials, *Comm. Math. Phys.,* **115.**(1988), 247-266.

[3] Molchanov,S.A., The local structure of the spectrum of the one-dimensional Schrödinger operator, *Comm. Math. Phys.,* **78**.(1981), 429-446.

[4] Minami,N., Local fluctuation of the spectrum of a multidimensional Anderson tight binding model, *Comm. Math. Phys.*, **177.**(1996), 709-725.

[5] Combes,J.M. - Germinet,F. - Klein,A., Poisson Statistics for Eigenvalues of Continuum Random Schrödinger Operators, *preprint.*