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Energy level statistics: a formulation and some examples

By

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Abstract

A formulation of energy level statistics for random (and non-random) operators is given based on the notions of *unfolding of spectra* and of *asymptotic ergodicity of unfolded spectra*. Two concrete examples are discussed under this formulation. As a related question, we also discuss the almost sure limit of the empirical distribution for the spacings between order statistics.

§ 1. Introduction: a formulation of energy level statistics

§ 1.1. Statistics on the ensemble and statistics along individual spectra

Energy level statistics, or spectral statistics, is a group of questions in mathematical physics in which one asks about properties of statistical fluctuation of energy levels (eigenvalues of a quantum Hamiltonian) rather than the asymptotic or average distribution of levels. In order to obtain results with some universality, one needs to observe a large number of energy levels, and for this purpose, one usually considers a family of Hamiltonians depending on a parameter and see what would happen when this parameter tends to a limit. The parameter may, e.g., be the Planck's constant $\hbar$, which one let $\hbar \to 0$ to observe the semi-classical limit, or the size L of a spatial domain, which one let $L \to \infty$ to observe the thermodynamic limit. Moreover, in order to make the question mathematically tractable, one often introduce some randomness into the Hamiltonian, and try to obtain some results on *level statistics on the ensemble* in the following sense:

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Let $\{\mathcal{H}^h_\omega\}_h$ be a family of random operators, self-adjoint in some separable Hilbert space, with discrete spectrum $\{E^h_j(\omega); j = 1, 2, \ldots\}$, where $h > 0$ is a parameter, and ω is taken from a probability space $(\Omega, \mathcal{F}, \mathbf{Q})$ (the ensemble of Hamiltonians). Then if as $h \to 0$ (or $h \to \infty$), the point process $\xi^h_\omega(dx) := \sum_j \delta_{E^h_j(\omega)}(dx)$ converges in law to a limiting point process $\xi(dx)$, this ξ would characterize the fluctuation of the spectrum of $\{\mathcal{H}^h_\omega\}$. A well known result in this direction is Molchanov's theorem ([9]), in which $\mathcal{H}^L_\omega$ is a rescaled random one-dimensional Schrödinger operator, $h = L$ is the length of the interval on which the operator is restricted, and ξ is a Poisson point process. Later, Molchanov's result was extended by the present author ([7]) to the multidimensional lattice Anderson model H_ω: Namely, let $H_\omega = -\Delta + V_\omega$ be a random operator acting in $\ell^2(\mathbf{Z}^d)$, where Δ is the discretized Laplacian $(\Delta u)(x) = \sum_{y:|y-x|=1} u(y)$, and the random potential $V_\omega = \{V_\omega(x)\}_{x\in\mathbf{Z}^d}$ consists of independent, identically distributed random variables with a common bounded density $\rho(v)$. Further, let H^L_ω be the restriction $\chi_\Lambda H_\omega \chi_\Lambda$ of H_ω to the hypercube $\Lambda = [0, L]^d \cap \mathbf{Z}^d$, $L = 1, 2, \ldots$ and set $\mathcal{H}^L_\omega = L^d(H^L_\omega - E)$. Then it was shown in [7] that if the bound $\|\rho\|_\infty$ of $\rho(v)$ is small enough (namely if the disorder is large enough), so that the Anderson localization holds throughout the spectrum of H_ω (see [1], [2]), then the point process $\xi^L_\omega(dx)$ converges in law to the Poisson point process ξ with intensity measure $n(E)dx$, provided E is in the interior of $\sigma(H_\omega)$ and that the density of states $n(E)$ exists and is positive for E.

The level statistics on the ensemble describes, for example, the following situation: Fix a cube Λ of large side length, and count the number $\xi^L_\omega(I)$ of eigenvalues of $\mathcal{H}^L_\omega$ within a fixed interval I. If you pick a large number of realizations $\omega_1, \ldots, \omega_n$ independently from Ω, then the empirical distribution (visualized as a suitable histogram) of $\xi^L_{\omega_1}(I), \ldots, \xi^L_{\omega_n}(I)$ is close to the Poisson distribution. However, the situation usually encountered in physics literature is something different, something which may be called *level statistics along individual spectra*, in which one observes a large number of consecutive eigenvalues $\{E^h_j\}$ of a typical realization of $\mathcal{H}^h_\omega$, and asks, for example, about the empirical distribution of the level spacings $\{E^h_{j+1}(\omega) - E^h_j(\omega)\}$, with the hope of obtaining something universal in the limit $h \to 0$ or $h \to \infty$. But in this case, we need to formulate our question more carefully.

§ 1.2. "unfoldability" and "unfolding" of spectra

Let $\{E^h_j\}$ be a one-parameter family of a finite or infinite increasing sequences, which we regard as the discrete spectrum of a one-parameter family of self-adjoint operators $\{\mathcal{H}^h\}$.

Definition 1.1. We shall say that a family of increasing sequences $\{E^h_j\}$ is *unfoldable* as $h \to 0$ (or $h \to \infty$) if there exist $\alpha > 0$ and $\nu(E)$, a non-negative, non-decreasing function of E with $\lim_{E\to-\infty}\nu(E)=0$, such that

$$N^h(E) := \sharp\{j;\ E_j^h \le E\} \sim \nu(E)h^{-\alpha}\ ,\quad h\to 0 \tag{1.1}$$

(or $\sim \nu(E)h^{\alpha}$ when we let $h\to\infty$).

Example 1.2 (Ergodic lattice Anderson model)**.** Let $H_\omega = -\Delta + V_\omega$ be the lattice Anderson model introduced in the previous subsection, where the random potential $V_\omega = \{V_\omega(x)\}_{x\in\mathbf{Z}^d}$ is only assumed to be a $\mathbf{Z}^d$-ergodic random field. Let H_ω^L be as before and let $\{E_j^L(\omega)\}$ be its spectrum. Then it is well known that there exists a continuous non-decreasing function $\mathcal{N}(E)$, called the integrated density of states (IDS), with $\lim_{E\to-\infty}\mathcal{N}(E)=0$ and $\lim_{E\to+\infty}\mathcal{N}(E)=1$, such that with probability one,

$$N_\omega^L(E) := \sharp\{j:\ E_j^L(\omega)\le E\} \sim L^d\mathcal{N}(E)\ ,\quad L\to\infty \tag{1.2}$$

holds for all $E\in\mathbf{R}$. Namely $\{E_j^L(\omega)\}$ is unfoldable with probability one with $\nu(E)=\mathcal{N}(E)$, $h=L\to\infty$, and $\alpha=d$.

Example 1.3 (One-dimensional Schrödinger operator with δ-potentials [6])**.** Let $\{E_j^\hbar\}$ $(\hbar>0)$ be the spectrum of the operator

$$H_v^\hbar := -\hbar^2\frac{d^2}{dx^2} + v\sum_{s=1}^{n}\delta(x-x_s)\ ,\quad 0\le x\le 1\ , \tag{1.3}$$

with Dirichlet boundary condition at $x=0,1$. Here $v>0$ and $0=:x_0<x_1<\cdots<x_n<x_{n+1}:=1$. Then for each $E>0$, one has

$$N^\hbar(E) := \sharp\{j;\ E_j^\hbar\le E\} \sim \frac{1}{\pi}\sqrt{E}\hbar^{-1}\ ,\quad \hbar\to 0\ , \tag{1.4}$$

so that $\{E_j^\hbar\}$ is unfoldable with $h=\hbar\to 0$, $\alpha=1$, and $\nu(E)=\frac{1}{\pi}(E\vee 0)^{1/2}$.

Returning to the general situation, let $\{E_j^h\}$ be an unfoldable family of sequences in the sense of Definition 1.1. For each $t\in(0,\nu(+\infty))$, define

$$\nu^{-1}(t) := \inf\{E;\ \nu(E)>t\}\ . \tag{1.5}$$

Then we see that $\nu(\nu^{-1}(t))=t$ and that $\nu(E)\le t$ if and only if $E\le\nu^{-1}(t)$. Now let us call $e_j^h := \nu(E_j^h)$ the *unfolded levels*. Then for each $t\in(0,\nu(+\infty))$, we have the asymptotic relation

$$\sharp\{j\ge 1;\ e_j^h\le t\} = \sharp\{j;\ E_j^h\le\nu^{-1}(t)\} \sim \nu(\nu^{-1}(t))h^{-\alpha} = th^{-\alpha}\ ,\quad h\to 0\ , \tag{1.6}$$

or $\sim h^\alpha$ when we let $h\to\infty$. Hence if we further let $x_j^L := Le_j^{L^{-1/\alpha}}$ (or $=Le_j^{L^{1/\alpha}}$ when we let $h\to\infty$), then $\{x_j^L\}$ has *asymptotic uniform distribution* (AUD) in the sense that

$$\sharp\{j;\ x_j^L\le\gamma L\} \sim \gamma L\ ,\quad L\to\infty \tag{1.7}$$

for each $\gamma \in (0, \nu(+\infty))$.

For the ergodic Anderson model of Example 1.2, we have $x_j^L(\omega) = L^d \mathcal{N}(E_j^L(\omega))$, and for the Schrödinger operator of Example 1.3, we have $x_j^L = (L/\pi)\sqrt{E_j(L^{-1})}$.

Remark. In [8], the present author called the unfolding described above *the unfolding of the first kind.*

§ 1.3. Asymptotic ergodicity of AUD sequences and the energy level statistics along individual spectra

Let $\{x_j^L\}$ $(L > 0)$ be a family of sequences which has asymptotically uniform distribution in the sense that there exist constants $0 < A \leq \infty$ and $0 < \lambda < \infty$ such that for any $\gamma \in [0, A)$ one has

$$\sharp\{j;\ x_j^L \leq \gamma L\} \sim \lambda \gamma L\ , \quad (L \to \infty). \tag{1.8}$$

(In the example of the previous subsection, $A = \nu(+\infty)$ and $\lambda = 1$.) Let μ be the uniform distribution on the interval $(0, B)$, where we have set $B = 1 \wedge \nu(+\infty)$, and consider the point process

$$\Xi_t^L(dx) := \sum_j \delta_{x_j(L) - Lt}(dx)\ , \quad t \in (0, B). \tag{1.9}$$

Definition 1.4. We shall say that an AUD family of sequences $\{x_j^L\}_{L>0}$ is asymptotically ergodic, if the probability law of $\Xi_\cdot^L$ under μ converges weakly to the probability law $\mathbf{P}$ of some stationary point process ξ on $\mathbf{R}$.

We shall say that energy level statistics along the individual spectrum is possible, if the AUD family of sequences $\{x_j^L\}_{L>0}$ which is made through unfolding from the spectra of a family of operators $\{\mathcal{H}^h\}$ is asymptotically ergodic.

If the asymptotic ergodicity holds in the above sense, then in particular for any $c > 0$ and $k = 0, 1, 2, \ldots$, the limit

$$\begin{aligned}
\pi_k(c) &:= \lim_{L\to\infty} \frac{1}{BL} \int_0^{BL} 1_{\{(t,t+c] \text{ contains exactly } k \text{ points from } \{x_j^L\}_j\}} dt \\
&= \lim_{L\to\infty} \mu(\{t \in (0, B];\ \Xi_t^L((0, c]) = k\}) \\
&= \mathbf{P}(\xi((0, c]) = k)
\end{aligned}$$

exists.

Now suppose that $\pi_0(c)$ is differentiable with respect to $c > 0$. Then by Proposition 4.4 of [8], the limit

$$\rho(c) := \lim_{L\to\infty} \frac{\sharp\{j;\ x_j^L \in (0, L],\ x_{j+1}^L - x_j^L > c\}}{\sharp\{j;\ x_j^L \in (0, L]\}} \tag{1.10}$$

exists for any $c > 0$, and is given by $\rho(c) = -(d/dc)\pi_0(c)$. In particular, when the limiting process in Definition 1.4 is the stationary Poisson point process, then $\pi_0(c) = e^{-c}$, and it holds that the limit of the empirical distribution of the level spacing is well defined and coincides with the exponential distribution.

§ 2. One-dimensional Schrödinger operator with δ-potentials

In this section, we consider the Schrödinger operator $H_v^\hbar$ defined in Example 1.3, and shall prove that by slightly refining results in [6], the energy level statistics along individual spectra is possible for this operator. Namely we shall show

Theorem 2.1. *Suppose that the numbers $y_j := x_{j+1} - x_j$, $j = 0, 1, \ldots, n$, are rationally independent. Then if we define $x_j^L = (L/\pi)\sqrt{E_j(L^{-1})}$, where $E_j(\hbar)$'s are eigenvalues of $H_v^\hbar$, then the one-parameter family $\{x_j^L\}_{L>0}$ of sequences is asymptotically ergodic, and the probability law $\mathbf{P}$ of the limiting point process ξ is characterized by the Laplace functional*

$$\mathcal{L}(\phi) = \mathbf{E}_{\mathbf{P}}\Big[\exp\{-\xi(\phi)\}\Big] = \prod_{s=0}^{n} \int_0^1 \exp\Big\{-\sum_{m\in\mathbf{Z}} \phi\Big(-\frac{\theta_s - m}{y_s}\Big)\Big\} d\theta_s \ , \tag{2.1}$$

$\phi \in C_0^+(\mathbf{R})$, where $C_0(\mathbf{R})$ is the totality of compactly supported, continuous functions, and $C_0^+(\mathbf{R}) := \{\phi \in C_0(\mathbf{R});\ \phi \geq 0\}$.

Proof. As in [6], we begin with treating the special case $v = \infty$, where $H_\infty^\hbar$ is a direct sum of $n+1$ Dirichlet Laplacians, each on $[x_{i-1}, x_i]$. In this case, the sequence $\{x_j(L)\}$ does not depend on L, and is the rearrangement in ascending order of the countable discrete set $\cup_{s=1}^{n+1}\{j/(x_s - x_{s-1});\ j \geq 1\}$. Hence the point process $\Xi_t^{\infty,L}$ is given by

$$\Xi_t^{\infty,L}(dx) = \sum_{s=1}^{n+1}\sum_{j=1}^{\infty} \delta_{\frac{j}{x_s - x_{s-1}} - Lt}(dx) \ . \tag{2.2}$$

The Laplace functional of this point process is defined for $\phi \in C_0^+(\mathbf{R})$ by

$$\mathcal{L}_L^\infty(\phi) = \int_0^1 \exp\Big\{-\Xi_t^{\infty,L}(\phi)\Big\} d\mu(t), \tag{2.3}$$

where μ is the uniform distribution on $(0,1)$. If we set $y_s = x_{s+1} - x_s$, $s = 0, 1, \ldots, n$, we can compute $\mathcal{L}_L^\infty(\phi)$ as follows:

$$\begin{aligned}
\mathcal{L}_L^\infty(\phi) &= \int_0^1 \exp\Big\{-\sum_{s=0}^{n}\sum_{m=1}^{\infty} \phi\Big(\frac{m}{y_s} - Lt\Big)\Big\} dt \\
&= \frac{1}{L}\int_0^L \exp\Big\{-\sum_{s=0}^{n}\sum_{m=1}^{\infty} \phi\Big(-\frac{y_s t - m}{y_s}\Big)\Big\} dt \ .
\end{aligned}$$

Now define a function Φ on $\mathbf{T}^{n+1}$, where $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, by

$$\Phi(z_0, z_1, \ldots, z_n) = \prod_{s=0}^{n} \exp\Big\{-\sum_{m\in\mathbf{Z}} \phi\Big(-\frac{z_s - m}{y_s}\Big)\Big\} . \tag{2.4}$$

Then $\lim_{L\to\infty} \mathcal{L}_L^{\infty}(\phi)$ exists and equals to

$$\lim_{L\to\infty} \frac{1}{L}\int_0^L \Phi(y_0 t, y_1 t, \ldots, y_n t)dt \ , \tag{2.5}$$

if the latter exists. But if $y_0, y_1, \ldots, y_n$ are rationally independent, then the flow $S_t(\theta_0, \ldots, \theta_n) = (\theta_0 + y_0 t, \ldots, \theta_n + y_n t)$ on $\mathbf{T}^{n+1}$ is uniquely ergodic, its invariant distribution being the normalized Lebesgue measure on $\mathbf{T}^{n+1}$. Hence we get

$$\lim_{L\to\infty} \mathcal{L}_L^{\infty}(\phi) = \int\!\!\cdots\!\!\int_{\mathbf{T}^{n+1}} \Phi(\theta_0, \theta_1, \ldots, \theta_n)d\theta_0 d\theta_1 \cdots d\theta_n \ . \tag{2.6}$$

This being true for all $\phi \in C_0^+(\mathbf{R})$, we see that the probability law of $\Xi_t^{\infty,L}$ under $\mu(dt)$ converges as $L \to \infty$ weakly to the probability law $\mathbf{P}$ of a point process, whose Laplace functional is

$$\mathcal{L}_{\mathbf{P}}(\phi) = \prod_{s=0}^{n} \int_0^1 \exp\Big\{-\sum_{m\in\mathbf{Z}} \phi\Big(-\frac{\theta_s - m}{y_s}\Big)\Big\}d\theta_s \ . \tag{2.7}$$

To prove the assertion in the general case of $0 < v < \infty$, first note that the equation (2.6) is valid if $\phi \in C_0(\mathbf{R})$ is replaced by an arbitrary piecewise constant function $\psi(x)$ with compact support. It then suffices to show for such a function that

$$\lim_{L\to\infty} \mathcal{L}_L^{v}(\psi) := \lim_{L\to\infty} \int_0^1 \exp\Big\{-\Xi_t^{v,L}(\psi)\Big\}dt = \lim_{L\to\infty} \mathcal{L}_L^{\infty}(\psi) \ , \tag{2.8}$$

where

$$\Xi_t^{v,L}(dx) := \sum_{j=1}^{\infty} \delta_{x_j^L - Lt}(dx) \ . \tag{2.9}$$

For this purpose, it is sufficient to verify that for any bounded interval $I = (a, b)$, one has

$$\lim_{L\to\infty} \mu(\{t \in (0,1);\ \Xi_t^{v,L}(I) - \Xi_t^{\infty,L}(I) \neq 0\}) = 0 \ . \tag{2.10}$$

To show this, let us write for any positive integer K,

$$\begin{aligned}
&\mu(\{t \in (0,1);\ \Xi_t^{v,L}(I) - \Xi_t^{\infty,L}(I) \neq 0\}) \\
&\le \mu(\{t \in (0,1);\ \Xi_t^{\infty,L}(I) \ge K\}) + \sum_{k=0}^{K-1} \mu(\{t \in (0,1);\ \Xi_t^{\infty,L}(I) = k,\ \Xi_t^{v,L}(I) \neq k\}) \ .
\end{aligned}$$

The assertion which has just been proved for $v = \infty$implies in particular that for any $\varepsilon > 0$, we can choose a sufficiently large K such that

$$\limsup_{L\to\infty} \mu(\{t \in (0,1);\ \Xi_t^{\infty,L}(I) \geq K\}) \leq \varepsilon\ . \tag{2.11}$$

On the other hand, the argument in the proof of Theorem 1 in [6] shows

$$\lim_{L\to\infty} \mu(\{t \in (0,1);\ \Xi_t^{\infty,L}(I) = k,\ \Xi_t^{v,L}(I) \neq k\}) = 0 \tag{2.12}$$

for any $k = 0, 1, \ldots$. This completes the proof. □

Let us show that when n is large, the limiting point process obtained in Theorem 2.1 is close to the Poisson point process for "typical"choice of $x_1, x_2, \ldots, x_n$. More precisely, let $X_1, X_2, \ldots$ be a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{Q})$ all of which are distributed uniformly on $(0,1)$. For each n, let $X_1^{(n)}(\omega) \leq X_2^{(n)}(\omega) \leq \cdots \leq X_n^{(n)}(\omega)$ be the rearrangement of $(X_1, \ldots, X_n)$ in the ascending order. Here the inequalities are strict with probability one, and if we let $Y_s^{(n)}(\omega) := X_{s+1}^{(n)}(\omega) - X_s^{(n)}(\omega)$, $s = 0, 1, \ldots, n$, with $X_0^{(n)} = 0$ and $X_{n+1}^{(n)} = 1$, then the numbers $Y_s^{(n)}(\omega)$, $s = 0, 1, \ldots, n$, are rationally independent with probability one.

Theorem 2.2. *Let $\mathcal{L}_\omega^n(\phi)$, $\phi \in C_0(\mathbf{R})$, be the Laplace functional for the limiting point process ξ_ω^n obtained in Theorem 2.1 with δ-potentials placed at $X_1^{(n)}(\omega), \ldots, X_n^{(n)}$. Then for $\mathbf{Q}$-almost every ω, ξ_ω^n converges weakly to the stationary Poisson point process with intensity measure dx. Namely we have $\mathbf{Q}$-almost surely*

$$\lim_{n\to\infty} \mathcal{L}_\omega^n(\phi) = \exp\Big\{-\int_{-\infty}^{+\infty} (1 - e^{-\phi(x)})dx\Big\} =: \mathcal{L}_0(\phi) \tag{2.13}$$

for any $\phi \in C_0^+(\mathbf{R})$.

Proof. As was verified in [6], for any $\delta \in (0,1)$, and for $\mathbf{Q}$- almost every $\omega \in \Omega$, one can choose a sufficiently large $N_\delta(\omega)$ so that for all $n > N_\delta(\omega)$, one has

$$\max_{0\leq s\leq n} Y_s^{(n)}(\omega) \leq n^{-\delta}\ . \tag{2.14}$$

Now by periodicity, one can write

$$\mathcal{L}_\omega^n(\phi) = \prod_{s=0}^{n} \int_{-1/2}^{1/2} \exp\Big\{-\sum_{m\in\mathbf{Z}} \phi\Big(-\frac{\theta_s - m}{Y_s^{(n)}(\omega)}\Big)\Big\} d\theta_s\ . \tag{2.15}$$

Set $\phi_m^s(\theta) := \phi(-(\theta_s - m)/Y_s^{(m)}(\omega))$. If $\max_{0\leq s\leq n} Y_s^{(n)}(\omega)$ is small, then the supports of $\phi_0^s(\cdot)$, $s = 0, 1, \ldots, n$, are contained in $(-1/2, 1/2)$, so that for each s, supports of

$\phi_m^s(\cdot)$, $m \in \mathbf{Z}\setminus\{0\}$, are disjoint from each other and from $(-1/2,1/2)$. This allows us to write

$$\mathcal{L}_\omega^n(\phi) = \prod_{s=0}^n \int_{-1/2}^{1/2} \exp\Big\{-\phi_0^s(\theta_s)\Big\} d\theta_s \ . \tag{2.16}$$

Since the radius of the support of $\phi_0^s(\cdot)$ is $\mathcal{O}(Y_s^{(n)}(\omega))$, we can compute, for n large, as follows:

$$\begin{aligned}
\log \mathcal{L}_\omega^n(\phi) &= \sum_{s=0}^n \log\Big[1 - \int_{-1/2}^{1/2}\Big\{1-\exp\Big(-\phi\Big(\frac{\theta_s}{Y_s^{(n)}(\omega)}\Big)\Big)\Big\}d\theta_s\Big] \\
&= -\sum_{s=0}^n \int_{-1/2}^{1/2}\Big\{1-\exp\Big(-\phi\Big(\frac{\theta_s}{Y_s^{(n)}(\omega)}\Big)\Big)\Big\}d\theta_s + \mathcal{O}\Big(\max_{0\le s\le n} Y_s^{(n)}(\omega)\Big) \\
&= -\sum_{s=0}^n Y_s^{(n)}(\omega)\int_{-\infty}^{\infty}(1-e^{-\phi(z)})dz + \mathcal{O}\Big(\max_{0\le s\le n} Y_s^{(n)}(\omega)\Big) \ .
\end{aligned}$$

Thus we have $\lim_{n\to\infty}\log\mathcal{L}_\omega^n(\phi) = -\int_{-\infty}^{\infty}(1-e^{-\phi(z)})dz$ with probability one, and the proof is finished. □

§ 3. Lattice Anderson model

In this section, we shall consider the energy level statistics along individual spectra of the lattice Anderson model of Example 1.2. Since our results are still partial, the discussion in this section will be sketchy.

Let us assume that the following conditions hold:

(C.1) The random potential $\{V_\omega(x)\}_{x\in\mathbf{Z}^d}$ consists of independent, identically distributed random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{Q})$. The distribution of each $V_\omega(x)$ has a common bounded density $\rho(v)$.

(C.2) The integrated density of states $\mathcal{N}(E)$ is C^1 in E, and the density of states $n(E) := d\mathcal{N}(E)/dE$ is strictly positive for all $E \in (E_0, E_1)$, where

$$E_0 := \inf\{E;\ \mathcal{N}(E) > 0\}\ ; \quad E_1 := \sup\{E;\ \mathcal{N}(E) < 1\}\ . \tag{3.1}$$

(C.3) For some $s\in(0,1)$, $C>0$ and $m>0$,

$$\mathbf{E}\Big[|(H_\omega^D - z)^{-1}(x,y)|^s\Big] \le Ce^{-m|x-y|} \tag{3.2}$$

holds for all hypercube D, all $x, y \in D$ and all $z \in \mathbf{C}\setminus\mathbf{R}$.

Note that, according to [1] and [2], given $s \in (0,1)$, **(C.3)** holds as far as $\|\rho\|_\infty$ is sufficiently small.

It was proved in [7] that under these conditions, the energy level statistics for H_ω^Λ on the ensemble Ω is possible in the sense that the point process

$$\xi_\omega(L,E)(dx) = \sum_j \delta_{\xi_j^L(E;\omega)}(dx) \ , \tag{3.3}$$

where $\xi_j^L(E;\omega) = L^d(E_j^L(\omega) - E)$, which is associated to the random operator $\mathcal{H}_\omega^L = |\Lambda|(H_\omega^L - E)$, converges weakly to the Poisson point process on $\mathbf{R}$ with intensity measure $n(E)dx$, for any $E \in (E_0, E_1)$.

Now let us observe the spectrum $\{E_j^L(\omega)\}$ of H_ω^Λ under the unfolding defined in §1. Namely consider

$$x_j^L(\omega) := L^d \mathcal{N}(E_j^L(\omega)) \ . \tag{3.4}$$

Then the one-parameter family of random sequences $\{x_j^L(\omega)\}$ has asymptotic uniform distribution with probability one, namely

$$\sharp\{j;\ x_j^L(\omega) \le \gamma L^d\} \sim \gamma L^d \ , \quad L \to \infty \tag{3.5}$$

holds for any $\gamma \in (0,1)$. For $(\omega,t) \in \Omega \times (0,1)$, define the point process

$$\Xi_{(\omega,t)}^L(dx) = \sum_j \delta_{x_j^L(\omega) - L^d t}(dx) \ . \tag{3.6}$$

We conjecture that the following assertion should be true:

Conjecture. For $\mathbf{Q}$-almost every $\omega \in \Omega$, the probability law of the point process $\Xi_{(\omega,\cdot)}^L(dx)$ under the uniform distribution $\mu(dt) = dt$ on $(0,1)$ converges to the law $\mathbf{P}$ of the Poisson point process with intensity measure dx.

To approach this conjecture, we observe the Laplace functional

$$\mathcal{L}_\omega^L(\phi) := \int_0^1 \exp\Big\{-\Xi_{(\omega,t)}^L(\phi)\Big\} dt \ , \tag{3.7}$$

defined for $\phi \in C_0^+(\mathbf{R})$, where

$$\Xi_{(\omega,t)}^L(\phi) = \sum_j \phi(x_j^L(\omega) - L^d t) \ , \tag{3.8}$$

and wish to prove that

$$\lim_{L\to\infty} \mathcal{L}_\omega^L(\phi) = \exp\Big[-\int_{-\infty}^{\infty} (1 - e^{-\phi(x)}) dx\Big] =: \mathcal{L}_0(\phi) \tag{3.9}$$

holds for $\mathbf{Q}$-almost every $\omega \in \Omega$. A recent result of F. Klopp [4] enables us to prove a weaker version of this conjecture in the case of space dimension one:

Theorem 3.1. *In addition to conditions* ***(C.1)*** *to* ***(C.3)****, assume $d = 1$, and that $\rho(v)$ is compactly supported. Then we have, for any $\phi \in C_0^+(\mathbf{R})$,*

$$\lim_{L\to\infty} \mathbf{E}[\{\mathcal{L}_\omega^L(\phi) - \mathcal{L}_0(\phi)\}^2] = 0 \ . \tag{3.10}$$

Remark. By a standard argument using a countable dense subclass of $C_0^+(\mathbf{R})$, it is obvious from Theorem 3.1 that we can choose a subsequence $L_k \to \infty$ along which the law of $\Xi_{(\omega,\cdot)}^{L_k}(dx)$ under μ converges to $\mathbf{P}$ for $\mathbf{Q}$-almost every $\omega \in \Omega$.

Remark. If, by improving the estimate, we could prove

$$\sum_{L=1}^{\infty} \mathbf{E}\Big[\{\mathcal{L}_\omega^L(\phi) - \mathcal{L}_0(\phi)\}^2\Big] < \infty \ , \tag{3.11}$$

then this would give the proof of our conjecture. The remark following (4.14) suggests however that this is hopeless. Nevertheless, the author believes that an argument similar to the last part of the proof of Theorem 4.1 will enable us to accomplish the proof of our conjecture.

The rest of this section is devoted to the proof of Theorem 3.1. We shall postpone the restriction to the one-dimensional case until the necessity arises.

To prove Theorem 3.1, it suffices to show that

(a) $\lim_{L\to\infty} \mathbf{E}[\mathcal{L}_\omega^L(\phi)] = \mathcal{L}_0(\phi)$ and

(b) $\lim_{L\to\infty} \mathbf{E}[\mathcal{L}_\omega^L(\phi)^2] = \mathcal{L}_0(\phi)^2$

holds for any $\phi \in C_0^+(\mathbf{R})$.

Now let us make the change of variable $t = \mathcal{N}(E)$ in the definition of $\mathcal{L}_\omega^L(\phi)$:

$$\mathcal{L}_\omega^L(\phi) = \int_{E_0}^{E_1} \exp\Big\{-\Xi_{(\omega,\mathcal{N}(E))}(\phi)\Big\} n(E)dE \ . \tag{3.12}$$

For each $E \in (E_0, E_1)$, define a function

$$\phi_{L,E}(x) = \phi(L^d(\mathcal{N}(E + L^{-d}x) - \mathcal{N}(E))) \ , \tag{3.13}$$

then we have

$$\lim_{L\to\infty} \phi_{L,E}(x) = \phi(n(E)x) =: \phi_E(x) \tag{3.14}$$

and

$$\Xi_{(\omega,\mathcal{N}(E))}^L(\phi) = \sum_j \phi_{L,E}(\xi_j^L(E;\omega)) = \xi_\omega(L,E)(\phi_{L,E}) \ . \tag{3.15}$$

If supp$\phi \subset [\alpha, \beta]$ with $-\infty < \alpha < \beta < \infty$, then $\phi_{L,E}(x) \neq 0$ only if

$$L^d\{\mathcal{N}^{-1}(\mathcal{N}(E) + L^{-d}\alpha) - E\} \leq x \leq L^d\{\mathcal{N}^{-1}(\mathcal{N}(E) + L^{-d}\beta) - E\} \ . \tag{3.16}$$

Left and right hand side of these inequalities converge to $\alpha/n(E)$ and $\beta/n(E)$ respectively. Hence $\phi_{L,E}(\cdot)$ is compactly supported, and for sufficiently large L,

$$\text{supp}\phi_{L.E} \subset \left[\frac{\alpha}{n(E)} - 1, \ \frac{\beta}{n(E)} + 1\right] \ . \tag{3.17}$$

Since $\mathbf{E}[\xi_\omega(L, E)(dx)] \leq \|\rho\|_\infty dx$ (see (2.23) of [7]), we have, as $L \to \infty$,

$$\begin{aligned}\mathbf{E}\Big[|\Xi^L_{(\omega,\mathcal{N}(E))}(\phi) - \xi_\omega(L, E)(\phi_E)|\Big] &= \mathbf{E}\Big[|\xi_\omega(L, E)(\phi_{L,E}) - \xi_\omega(L, E)(\phi_E)|\Big] \\ &\leq \|\rho\|_\infty \int_{-1+\alpha/n(E)}^{1+\beta/n(E)} |\phi_{L,E}(x) - \phi_E(x)|dx \longrightarrow 0 \ .\end{aligned}$$

This implies

$$\lim_{L\to\infty} \mathbf{E}\Big[\Big|\exp\Big\{-\Xi^L_{(\omega,\mathcal{N}(E))}(\phi))\Big\} - \exp\Big\{-\xi_\omega(L, E)(\phi_E)\Big\}\Big|\Big] = 0 \ . \tag{3.18}$$

But since we already know that the point process $\xi_\omega(L, E)(dx)$ converges weakly to the Poisson point process with intensity measure $n(E)dx$, we have

$$\lim_{L\to\infty} \mathbf{E}\Big[\exp\Big\{-\xi_\omega(L, E)(\phi_E)\Big\}\Big] = \exp\Big\{-\int_\infty^\infty (1 - e^{-\phi(n(E)x)})n(E)dx\Big\} = \mathcal{L}_0(\phi) \ . \tag{3.19}$$

We can now complete the proof of **(a)** by dominated convergence theorem:

$$\begin{aligned}\lim_{L\to\infty} \mathbf{E}[\mathcal{L}^L_\omega(\phi)] &= \lim_{L\to\infty} \int_{E_0}^{E_1} \mathbf{E}\Big[\exp\Big\{-\Xi^L_{(\omega,\mathcal{N}(E))}(\phi)\Big\}\Big]n(E)dE \\ &= \lim_{L\to\infty} \int_{E_0}^{E_1} \mathbf{E}\Big[\exp\{-\xi_\omega(L, E)(\phi_E)\}n(E)dE \\ &= \int_{E_0}^{E_1} \mathcal{L}_0(\phi)n(E)dE = \mathcal{L}_0(\phi) \ .\end{aligned}$$

This being true for every $\phi \in C_0^+(\mathbf{R})$, we get the following result at the same time:

Proposition 3.2. *Under the probability measure $\mathbf{Q} \times \mu$, the point process $\Xi^L_{(\cdot,\cdot)}$ converges weakly to the Poisson point process with intensity measure dx.*

We proceed to the proof of **(b)**. Again by the change of variable,

$$\mathbf{E}[\mathcal{L}^L_\omega(\phi)^2] = \int_{E_0}^{E_1} dE \int_{E_0}^{E_1} dE'\mathbf{E}\Big[\exp\Big\{-\Xi^L_{(\omega,\mathcal{N}(E))}(\phi)\Big\} \exp\Big\{-\Xi^L_{(\omega,\mathcal{N}(E'))}(\phi)\Big\}\Big] \ . \tag{3.20}$$

Since we have

$$\begin{aligned}\lim_{L\to\infty} \quad & \mathbf{E}\Big[\Big|\exp\Big\{-\Xi^{L}_{(\omega,\mathcal{N}(E))}(\phi)\Big\}\exp\Big\{-\Xi^{L}_{(\omega,\mathcal{N}(E'))}(\phi)\Big\} \\ & - \exp\{-\xi_\omega(L,E)(\phi_E)\}\exp\{-\xi_\omega(L,E')(\phi_{E'})\}\Big|\Big] = 0\end{aligned}$$

as before, it suffices for our purpose to prove that, for any $E \neq E'$ and for any $\varphi, \psi \in C_0^+(\mathbf{R})$,

$$\begin{aligned}(3.21) \qquad \lim_{L\to\infty} \quad & \mathbf{E}[\exp\{-\xi_\omega(L,E)(\varphi) - \xi_\omega(L,E')(\psi)\}] \\ & = \exp\Big[-n(E)\int_{-\infty}^{\infty}(1-e^{-\varphi(x)})dx - n(E')\int_{-\infty}^{\infty}(1-e^{-\psi(x)})dx\Big] \;.\end{aligned}$$

Note that this will show the asymptotic independence as $L \to \infty$ of the point processes $\xi_\omega(L,E)$ and $\xi_\omega(L,E')$ for $E \neq E'$. (See [4].)

Again from $\mathbf{E}[\xi_\omega(L,E)(dx)] \leq \|\rho\|_\infty dx$, we see that the right hand side of (3.21) is equi-continuous in φ and ψ with respect to L^1-topology. Hence it suffices to prove (3.21) with φ and ψ replaced by arbitrary functions f and g from the function class

$$\begin{aligned}(3.22) \qquad \mathcal{A} := \Big\{ & f;\ f(x) = \sum_{j=1}^{n} \frac{a_j\tau}{(x-\sigma_j)^2+\tau^2} \\ & \text{for some } n \geq 1,\ \tau > 0,\ a_j > 0,\ \sigma_j \in \mathbf{R},\ \text{for } j = 1,\dots,n\Big\} \;,\end{aligned}$$

because any $\varphi \in C_0^+(\mathbf{R})$ can be approximated by elements of $\mathcal{A}$. (See the argument in *Step 1* of [7].)

Now divide the cube $\Lambda = [0,L]^d$ into (nearly) equal cubes C_p, $p = 1,2,\dots,(N_L)^d$, with $N_L = L^\alpha$ ($\alpha \in (0,1)$), and let $\{E_j^{C_p}(\omega)\}$ be the spectrum of the operator $H_\omega^{C_p} = \chi_{C_p} H_\omega \chi_{C_p}$. By the argument of *Step 3* of [7], we have

$$(3.23) \qquad \Big| L^{-d}\Im\mathrm{Tr}(H_\omega^{\Lambda} - z)^{-1} - L^{-d}\sum_{p=1}^{N_L^d}\Im\mathrm{Tr}(H_\omega^{C_p} - z)^{-1}\Big| \longrightarrow 0$$

in probability $\mathbf{Q}$, uniformly in $z \in \mathbf{C}\setminus\mathbf{R}$.

If for any hypercube D, $\{E_j^D(\omega)\}$ is the spectrum of the operator $H_\omega^D = \chi_D H_\omega \chi_D$, then for

$$f(x) = \sum_{j=1}^{n}\frac{a_j\tau}{(x-\sigma_j)^2+\tau^2} \in \mathcal{A} \;,$$

one has

$$(3.24) \qquad \eta_\omega^D(L,E)(f) := \sum_j f(L^d(E_j^D(\omega)-E)) = L^{-d}\sum_{j=1}^{n} a_j \Im(\mathrm{Tr}G^D(E+L^{-d}\zeta_j)) \;,$$

with $\zeta_j = \sigma_j + i\tau$. Combined with (3.23), this shows

$$|\xi_\omega(L,E)(f) - \eta_\omega(L,E)(f)| \longrightarrow 0 \tag{3.25}$$

in probability $\mathbf{Q}$ as $L \to \infty$, where we have set $\eta_\omega(L,E)(dx) = \sum_p \eta_\omega^{C_p}(L,E)(dx)$. Hence we obtain for any $E \neq E'$ and any $f, g \in \mathcal{A}$,

$$\begin{aligned} &\lim_{L\to\infty} \mathbf{E}[\exp\{-\xi_\omega(L,E)(f) - \xi_\omega(L,E')(g)\}] \\ &= \lim_{L\to\infty} \mathbf{E}[\exp\{-\eta_\omega(L,E)(f) - \eta_\omega(L,E')(g)\}] \ . \end{aligned} \tag{3.26}$$

In studying the right hand side of (3.26), we can again replace $f, g \in \mathcal{A}$ by $\varphi, \psi \in C_0(\mathbf{R})$ because of $\mathbf{E}[\eta_\omega(L,E)(dx)] \leq \|\rho\|_\infty dx$. (See (2.45) in [7].) By stochastic independence of $H_\omega^{C_p}$ for different p, one can write

$$\begin{aligned} &\lim_{L\to\infty} \mathbf{E}[\exp\{-\eta_\omega(L,E)(\varphi) - \eta_\omega(L,E')(\psi)\}] \\ &= \lim_{L\to\infty} \prod_p \mathbf{E}[\exp\{-\eta_\omega^{C_p}(L,E)(\varphi) - \eta_\omega^{C_p}(L,E')(\psi)\}] \end{aligned} \tag{3.27}$$

For $1 \leq p \leq N_L^d$, consider the event

$$B_p := \{\omega \in \Omega;\ \eta_\omega^{C_p}(L,E)(\mathrm{supp}\varphi) \leq 1,\ \eta_\omega^{C_p}(L,E)(\mathrm{supp}\psi) \leq 1\} \ . \tag{3.28}$$

Since for each bounded interval I one has

$$\mathbf{Q}(\eta_\omega^{C_p}(L,E)(I) \geq 2) = \mathcal{O}(N_L^{-2d}) \tag{3.29}$$

uniformly in p and E (*Step 6* in [7] and [3]), we have $\mathbf{Q}(B_p^c) = \mathcal{O}(N_L^{-2d})$.

But on the event B_p, one has

$$\{\eta_\omega^{C_p}(L,E)(\varphi)\}^n = \eta_\omega^{C_p}(L,E)(\varphi^n) \ , \tag{3.30}$$

and hence

$$\begin{aligned} &\exp\{-\eta_\omega^{C_p}(L,E)(\varphi) - \eta_\omega^{C_p}(L,E')(\psi)\} \\ &= \Big\{\sum_{n\geq 0} \frac{(-1)^n}{n!}(\eta_\omega^{C_p}(L,E)(\varphi))^n\Big\}\Big\{\sum_{k\geq 0} \frac{(-1)^k}{k!}(\eta_\omega^{C_p}(L,E)(\psi))^k\Big\} \\ &= \Big\{\sum_{n\geq 0} \frac{(-1)^n}{n!}\eta_\omega^{C_p}(L,E)(\varphi^n)\Big\}\Big\{\sum_{k\geq 0} \frac{(-1)^k}{k!}\eta_\omega^{C_p}(L,E)(\psi^k)\Big\} \\ &= \{1 - \eta_\omega^{C_p}(L,E)(1-e^{-\varphi})\}\{1 - \eta_\omega^{C_p}(L,E')(1-e^{-\psi})\} \\ &= 1 - \eta_\omega^{C_p}(L,E)(1-e^{-\varphi}) - \eta_\omega^{C_p}(L,E')(1-e^{-\psi}) \\ &\quad + \eta_\omega^{C_p}(L,E)(1-e^{-\varphi})\eta_\omega^{C_p}(L,E')(1-e^{-\psi}) \ . \end{aligned}$$

At this stage, we assume that the following "decorrelation estimate "is valid: for any $E \neq E'$ and any finite intervals I and J,

$$\mathbf{Q}(A_p) = o(N_L^{-d}) \ , \quad L \to \infty \ , \tag{3.31}$$

where the event A_p is defined by

$$A_p = \{\eta_\omega^{C_p}(L,E)(I) \geq 1 \text{ and } \eta_\omega^{C_p}(L,E')(J) \geq 1\} \ . \tag{3.32}$$

Then we can compute, by taking such I and J that $I \supset \mathrm{supp}\varphi$, $J \supset \mathrm{supp}\psi$,

$$\begin{aligned}
&\lim_{L\to\infty} \prod_{1\leq p\leq N_L^d} \mathbf{E}[\exp(-\eta_\omega^{C_p}(L,E)(\varphi) - \eta_\omega^{C_p}(L,E')(\psi))] \\
&= \lim_{L\to\infty} \prod_{1\leq p\leq N_L^d} \Big\{\mathbf{E}[1_{B_p\setminus A_p} \exp(-\eta_\omega^{C_p}(L,E)(\varphi) - \eta_\omega^{C_p}(L,E')(\psi))] + o(N_L^{-d}\Big\} \\
&= \lim_{L\to\infty} \prod_{1\leq p\leq N_L^d} \Big\{\mathbf{E}[1_{B_p\setminus A_p}(1 - \eta_\omega^{C_p}(L,E)(1-e^{-\varphi}) - \eta_\omega^{C_p}(L,E')(1-e^{-\psi})] + o(N_L^{-d})\Big\} \\
&= \lim_{L\to\infty} \prod_{1\leq p\leq N_L^d} \Big\{\mathbf{E}[1 - \eta_\omega^{C_p}(L,E)(1-e^{-\varphi}) - \eta_\omega^{C_p}(L,E')(1-e^{-\psi})] + o(N_L^{-d})\Big\} \ .
\end{aligned}$$

By the results of *Step 4* in [7], we have

$$\mathbf{E}[\eta_\omega^{C_p}(L,E)(1-e^{-\varphi})] \sim N_L^{-d} \int_{-\infty}^{\infty} (1-e^{-\varphi(x)}) n(E) dx \tag{3.33}$$

uniformly in p. Hence the right hand side of the above equality reduces to

$$\begin{aligned}
&\lim_{L\to\infty} \prod_{1\leq p\leq N_L^d} \Big[1 - \frac{1}{N_L^d}\Big\{n(E)\int_{-\infty}^{\infty}(1-e^{-\varphi(x)})dx + n(E')\int_{-\infty}^{\infty}(1-e^{-\psi(x)})dx\Big\} + o(N_L^{-d})\Big] \\
&= \exp\Big[-n(E)\int_{-\infty}^{\infty}(1-e^{-\varphi(x)})dx - n(E')\int_{-\infty}^{\infty}(1-e^{-\psi(x)})dx\Big] \ .
\end{aligned}$$

The present author had been unable to prove the decorrelation estimate (3.31). Recently, F. Klopp obtained it for the case $d = 1$. His result, translated into our notation, reads as follows:

Lemma 3.3 (Lemma 1.1 of [4]). *Assume $d = 1$ and that $\rho(\cdot)$ is compactly supported, and pick $\beta \in (1/2, 1)$, $\alpha \in (0,1)$ and $E \neq E'$. Then there exists a constant $C > 0$ such that for L large*

$$\mathbf{Q}(A_p) \leq C N_L^{-2} \exp\Big[\Big(\frac{1}{1-\alpha}\Big)^\beta (\log N_L)^\beta\Big] \ .$$

In particular, one has

$$\mathbf{Q}(A_p) = \mathcal{O}(N_L^{-2d+\varepsilon}) \ , \quad L \to \infty$$

for any $\varepsilon > 0$.

Now restricting ourselves to the case $d = 1$, we complete the proof of Theorem 3.1.

Remark. After finishing this work, our conjecture was proved by F. Klopp by a different method ([5]).

§ 4. Appendix: Empirical distribution of the spacings between order statistics

Under conditions **(C.1)** and **(C.3)**, one has Anderson localization throughout the energy spectrum of H_ω (see [1], [2]). The main effect of Anderson localization on the Poisson nature of energy level statistics is that the spectrum of H_ω^Λ is well approximated by independent superposition of sparse random spectra of subsystems. In relation to this, the following question was posed to the present author by F. Germinet and F. Klopp during the workshop.

Let $X_1(\omega), X_2(\omega), \ldots$ be independent random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{Q})$, each of which is uniformly distributed on $(0, 1)$, and for each n, let $X_1^{(n)} \le \cdots \le X_n^{(n)}$ be the rearrangement of $X_1(\omega), \ldots, X_n(\omega)$ in the ascending order (order statistics). The question is: does the empirical distribution of the spacings between $nX_j^{(n)}(\omega)$ and $nX_{j+1}^{(n)}(\omega)$ converges almost surely to the unit exponential distribution $e^{-c}dc$? The answer is yes. Precisely stated, we have, setting $X_0^{(n)} = 0$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} 1_{\{X_j^{(n)}(\omega) - X_{j-1}^{(n)}(\omega) > c/n\}} = e^{-c} \tag{4.1}$$

for each $c > 0$ and for $\mathbf{Q}$-almost every $\omega \in \Omega$. Since we were unable to find a convenient reference for this seemingly very old result, we shall provide a proof here. (The result seems to be known since 1950's. See the note at the end of [11]. See also [10] for related topics.)

Instead of directly proving (4.1), we shall consider the following more general question. By the strong law of large number, we have, for any $t \in \mathbf{R}$,

$$\sharp\{j;\ X_j^{(n)}(\omega) \le t\} = \sum_{j=1}^{n} 1_{(-\infty,t]}(X_j(\omega)) \sim n((0 \vee t) \wedge 1) \tag{4.2}$$

as $n \to \infty$ $\mathbf{Q}$-almost surely. Hence the family of sequences $\{X_j^{(n)}(\omega)\}_{1\le j\le n}$ is unfoldable with $h = n \to \infty$, $\alpha = 1$ and $\nu(E) = (0 \vee E) \wedge 1$. In this case, the unfolding is trivial, and we see that the family of sequences $\{nX_j^{(n)}(\omega)\}_{1\le j\le n}$ has asymptotic uniform distribution. From the argument at the end of §1, (4.1) is a direct corollary of the following

Theorem 4.1. *For $\mathbf{Q}$-almost every $\omega \in \Omega$, the family of sequences $\{x_j^n(\omega)\}_{n\geq 1}$ is asymptotically ergodic in the sense of Definition 1.4, and the limiting probability law $\mathbf{P}$ is that of the Poisson point process with intensity measure dx.*

Proof. For each $\omega \in \Omega$, consider the point process

$$\Xi^{(n)}_{(\omega,t)}(dx) = \sum_{j=1}^{n} \delta_{nX_j^{(n)}(\omega)-nt}(dx) = \sum_{j=1}^{n} \delta_{n(X_j(\omega)-t)}(dx) \tag{4.3}$$

defined for $t \in (0,1)$, and observed under the uniform distribution μ on $(0,1)$. Let

$$\mathcal{L}^{(n)}_\omega(\phi) = \int_0^1 \exp\Big\{-\Xi^{(n)}_{(\omega,t)}(\phi)\Big\}dt \ , \quad \phi \in C_0(\mathbf{R}) \tag{4.4}$$

be the Laplace functional of the point process $\Xi^n_{(\omega,\cdot)}$. Although it suffices for our purpose to show $\mathcal{L}^{(n)}_\omega(\phi) \to \mathcal{L}_0(\phi)$ for non-negative $\phi \in C_0(\mathbf{R})$ only, we do not impose this restriction for the moment.

Now for each $\phi \in C_0(\mathbf{R})$, we shall prove the following two estimates:

(c) $\mathbf{E_Q}\Big[\mathcal{L}^{(n)}_\omega(\phi)\Big] = \mathcal{L}_0(\phi) + \mathcal{O}(n^{-1})$;

(d) $\mathbf{E_Q}\Big[\mathcal{L}^{(n)}_\omega(\phi)^2\Big] = \mathcal{L}_0(\phi)^2 + \mathcal{O}(n^{-1})$,

where $\mathcal{L}_0(\phi) = \exp\Big[-\int_{-\infty}^{\infty}(1-e^{-\phi(x)})dx\Big]$ is the Laplace functional for the Poisson point process with intensity measure dx.

Proof of (c): It is easy to see from the independence of $X_j(\omega)$'s that

$$\begin{aligned}\mathbf{E}\Big[\mathcal{L}^{(n)}_\omega(\phi)\Big] &= \int_0^1 \Big\{\int_0^1 \exp\Big(-\phi(n(x-s))\Big)dx\Big\}^n ds \\ &= \int_0^1 \Big\{1 - \frac{1}{n}\int_0^n (1-e^{-\phi(y-ns)})dy\Big\}^n ds \ .\end{aligned} \tag{4.5}$$

Let $\inf(\mathrm{supp}\phi) = a$ and $\sup(\mathrm{supp}\phi) = b$. Then for s satisfying $[ns+a, ns+b] \subset [0,n]$, one has

$$\frac{1}{n}\int_0^n (1-e^{-\phi(y-ns)})dy = \frac{1}{n}\int_{-\infty}^{\infty}(1-e^{-\phi(x)})dx \ , \tag{4.6}$$

while

$$|\{s \in [0,1];\ [ns+a, ns+b] \not\subset [0,n]\}| = \mathcal{O}(n^{-1}) \ . \tag{4.7}$$

Hence, from the boundedness of ϕ, we have

$$\begin{aligned}\mathbf{E_Q}\Big[\mathcal{L}^{(n)}_\omega(\phi)\Big] &= \Big\{1 - \frac{1}{n}\int_{-\infty}^{\infty}(1-e^{-\phi(y-ns)})dy\Big\}^n + \mathcal{O}(n^{-1}) \\ &= \mathcal{L}_0(\phi) + \mathcal{O}(n^{-1}) \ .\end{aligned} \tag{4.8}$$

Proof of (d): Some simple computations give

$$\mathbf{E_Q}\Big[\mathcal{L}_\omega^{(n)}(\phi)^2\Big] = \int_0^1\int_0^1 \Big[1-\frac{1}{n}\int_0^n \Big\{1-e^{-\phi(y-ns)-\phi(y-nt)}\Big\}dy\Big]^n dsdt\ . \tag{4.9}$$

Now define two subsets A_n, B_n of $[0,1]^2$ by

$$A_n = \{(s,t)\in[0,1]^2;\ (ns+a,ns+b)\cap(nt+a,nt+b)\neq\emptyset\} \tag{4.10}$$

and

$$B_n = \{(s,t)\in[0,1]^2;\ (ns+a,ns+b)\subset[0,n],\ (nt+a,nt+b)\subset[0,n]\}\ , \tag{4.11}$$

then $|A_n| = \mathcal{O}(n^{-1})$ and $|B_n^c| = \mathcal{O}(n^{-1})$. Moreover, when $(s,t)\in B_n\setminus A_n$, one has

$$\frac{1}{n}\int_0^n\Big\{1-e^{-\phi(y-ns)-\phi(y-nt)}\Big\}dy = \frac{2}{n}\int_{-\infty}^{\infty}(1-e^{-\phi(x)})dx\ . \tag{4.12}$$

Hence again from the boundedness of ϕ, we can conclude

$$\begin{aligned}\mathbf{E_Q}\Big[\mathcal{L}_\omega^{(n)}(\phi)^2\Big] &= \Big[1-\frac{2}{n}\int_{-\infty}^{\infty}(1-e^{-\phi(x)})dx\Big]^n + \mathcal{O}(n^{-1})\\ &= \exp\Big\{-2\int_{-\infty}^{\infty}(1-e^{-\phi(x)})dx\Big\} + \mathcal{O}(n^{-1})\\ &= \mathcal{L}_0(\phi)^2 + \mathcal{O}(n^{-1})\ .\end{aligned} \tag{4.13}$$

Combining the estimates (c) and (d) just proved, we obtain

$$\mathbf{E_Q}\Big[\Big\{\mathcal{L}_\omega^{(n)}(\phi)-\mathcal{L}(\phi)\Big\}^2\Big] = \mathcal{O}(n^{-1})\ , n\to\infty \tag{4.14}$$

for each $\phi\in C_0(\mathbf{R})$.

It should be noted that the speed of L^2-convergence cannot be faster than estimated in (4.14). This is because at the final stages of the proofs of (c) and (d), we used the well known equality

$$\lim_{n\to\infty}\Big(1-\frac{c}{n}\Big)^n = e^{-c}\ .$$

But concerning this equality, we also know

$$\Big(1-\frac{c}{n}\Big)^n - e^{-c} \sim \frac{e^{-c}c^2}{2n}\ . \tag{4.15}$$

Anyway, we can conclude from (4.14) that

$$\sum_{n=1}^{\infty}\mathbf{E}\Big[\Big\{\{\mathcal{L}_\omega^{(n^2)}(\phi)-\mathcal{L}(\phi)\Big\}^2\Big] \le \text{const.}\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty \tag{4.16}$$

for any $\phi\in C_0(\mathbf{R})$. Hence by the standard argument using a countable dense subclass of $C_0(\mathbf{R})$, we obtain the following

Proposition 4.2. *For $\mathbf{Q}$-almost every $\omega \in \Omega$, one has*

$$\lim_{n\to\infty} \mathcal{L}_\omega^{(n^2)}(\phi) = \mathcal{L}_0(\phi) \tag{4.17}$$

for any $\phi \in C_0(\mathbf{R})$, so that the law of $\Xi_{(\omega,\cdot)}^{(n^2)}$ under the uniform distribution μ on $(0,1)$ converges to the law $\mathbf{P}$ of the Poisson point process with intensity measure dx.

To complete the proof of Theorem 4.1, let us fix an $\omega \in \Omega$ for which the assertion of the above proposition holds, and pick any $\phi \in C_0^+(\mathbf{R})$. For any k satisfying $n^2 < k \le (n+1)^2$, we can write

$$\begin{aligned}\Xi_{(\omega,t)}^{(k)}(\phi) &= \sum_{j=1}^{k} \phi(k(X_j(\omega) - t)) \\ &= \sum_{j=1}^{n^2} \phi\Big(\frac{k}{n^2}\cdot n^2(X_j(\omega)-t)\Big) + \sum_{j=n^2+1}^{k} \phi\Big(k(X_j(\omega)-t)\Big)\ .\end{aligned} \tag{4.18}$$

If we let $\psi_{n,k}(x) = \phi(\frac{k}{n^2}x) - \phi(x)$, then $\psi_{n,k} \in C_0(\mathbf{R})$ and it holds that

$$\max_{n^2<k\le(n+1)^2} \sup_{x\in\mathbf{R}} |\psi_{n.k}(x)| \longrightarrow 0 \quad (n\to\infty)\ . \tag{4.19}$$

Moreover, there exists a compact interval $[a,b]$ such that the supports of $\psi_{n,k}$ and ϕ are all contained in $[a,b]$. Then we have

$$\begin{aligned}\Xi_{(\omega,t)}^{(k)}(\phi) &= \Xi_{(\omega,t)}^{(n^2)}(\phi) + \sum_{j=1}^{n^2} \psi_{n,k}(n^2(X_j(\omega)-t)) + \sum_{j=n^2+1}^{k} \phi(k(X_j(\omega)-t)) \\ &=: \Xi_{(\omega,t)}^{(n^2)}(\phi) + I_{n,k}(t) + J_{n,k}(t)\ ,\end{aligned} \tag{4.20}$$

and hence

$$\begin{aligned}\Big|\mathcal{L}_\omega^{(k)}(\phi) - \mathcal{L}_\omega^{(n^2)}(\phi)\Big| &= \Big|\int_0^1 \exp\Big\{-\Xi_{(\omega,t)}^{(k)}(\phi)\Big\}dt - \int_0^1 \exp\Big\{-\Xi_{(\omega,t)}^{(n^2)}(\phi)\Big\}dt\Big| \\ &= \Big|\int_0^1 \exp\Big\{-\Xi_{(\omega,t)}^{(n^2)}(\phi)\Big\}\Big[\exp\{-I_{n,k}(t)\}\exp\{-J_{n,k}(t)\} - 1\Big]dt\Big| \\ &\le \int_0^1 |\exp\{-I_{n,k}(t)\} - 1|dt + \int_0^1 |\exp\{-J_{n,k}(t)\} - 1|dt \\ &= L_{n,k}^{(1)} + L_{n,k}^{(2)}\ .\end{aligned}$$

Clearly we have

$$\begin{aligned}\int_0^1 |J_{n,k}(t)|dt &\le \|\phi\|_\infty \int_0^1 \sum_{j=n^2+1}^{k} 1_{[a,b]}(k(X_j(\omega)-t))dt \\ &\le \|\phi\|_\infty (b-a)\frac{2n+1}{n^2} \longrightarrow 0\ ,\end{aligned} \tag{4.21}$$

so that

$$\max_{n^2<k\le(n+1)^2} L_{n,k}^{(2)} \le \|\phi\|_\infty (b-a)\frac{2n+1}{n^2} \longrightarrow 0\ . \tag{4.22}$$

To estimate $L_{n,k}^{(1)}$, we note that for any $\varepsilon > 0$, there exist an $N \in \mathbf{N}$ and an $\eta \in C_0^+(\mathbf{R})$ with $0 \le \eta(x) < \varepsilon$ such that for all $n > N$, one has

$$\max_{n^2<k\le(n+1)^2} |\psi_{n,k}(x)| \le \eta(x)\ . \tag{4.23}$$

Then noting the inequality $|e^{-s}-1| \le e^{|s|}-1$ and Proposition 4.2, we see

$$\int_0^1 |\exp\{-I_{n,k}(t)-1|dt \le \int_0^1 \exp\Big(\Xi_{(\omega,t)}^{(n^2)}(\eta)\Big)dt - 1 \longrightarrow \mathcal{L}_0(-\eta) - 1\ . \tag{4.24}$$

But since we can assume supp $\eta \subset [a-1, b+1]$, the right hand side tends to 0 as $\varepsilon \searrow 0$. Hence we get $\max_{n^2<k\le(n+1)^2} L_{n,k}^{(1)} \to 0$ $(n \to 0)$, as desired. □

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