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Spectral correlation functions for chaotic systems

By

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Abstract

The n-level spectral correlation functions for chaotic quantum systems are calculated by using a recently proposed scheme called the extended diagonal approximation (EDA). The EDA is a natural extension of the diagonal approximation, which was invented by Berry in order to semiclassically evaluate the 2-level correlation function. When the time reversal invariance of the chaotic systems is broken, the EDA yields $n \times n$ determinant expressions for the n-level correlation functions, which exactly agree with the predictions of the random matrix theory. On the other hand, when the system is time reversal invariant, only the leading terms of the random matrix predictions are reproduced.

§ 1. The diagonal approximation for the spectral correlation function

Let us consider a two-dimensional bounded quantum system which is chaotic in the classical limit. The time-reversal invariance is supposed to be broken. We are interested in the distribution of the energy levels E_j, which are the eigenvalues of the system Hamiltonian H. The semiclassical theory describes the limit $\hbar \to 0$ and shows that the energy level density

$$\rho(E) = \sum_j \delta(E - E_j) \tag{1.1}$$

can be written in the form

$$\rho(E) = -\frac{1}{\pi} \mathrm{Im}\; g(E^+), \tag{1.2}$$

where $E^{\pm} = E \pm i\kappa$ with an infinitesimal positive number κ and[1]

$$g(E) = -i\pi\bar{\rho}(E) - \frac{i}{\hbar} \sum_\gamma A_\gamma \mathrm{e}^{iS_\gamma(E)/\hbar} \tag{1.3}$$

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with a sum over the classical periodic orbits γ. The classical action and stability amplitude are denoted by S_γ and A_γ, respectively. The smoothed energy level density $\bar{\rho}(E)$ is

$$\bar{\rho}(E) = \frac{\Omega(E)}{(2\pi\hbar)^2}, \tag{1.4}$$

where $\Omega(E)$ is the phase space volume of the energy shell.

The physical quantity we aim to calculate is the n-level spectral correlation function

$$R_n(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) = \frac{1}{\bar{\rho}^n} \left\langle \prod_{j=1}^{n} \rho(E + \epsilon_j) \right\rangle. \tag{1.5}$$

The brackets $\langle \cdot \rangle$ stand for an average over the windows of the center energy E and energy differences ϵ_j. The correlation functions describe the energy level fluctuation around the smoothed density $\bar{\rho}$. In the semiclassical limit $\hbar \to 0$ with fixed $\omega_j = \bar{\rho}\epsilon_j$, the random matrix theory (RMT) predicts a universal determinant form[2, 3, 4, 5]

$$R_n^{(\mathrm{GUE})}(\omega_1, \omega_2, \cdots, \omega_n) = \det \left[\frac{\sin\{\pi(\omega_j - \omega_k)\}}{\pi(\omega_j - \omega_k)} \right]_{j,l=1,2,\cdots,n}, \tag{1.6}$$

which was derived from the Gaussian Unitary Ensemble (GUE) of random matrices[6]. For example, the 2-level correlation function is

$$R_2^{(\mathrm{GUE})}(\omega_1, \omega_2) = 1 - \left[\frac{\sin\{\pi(\omega_1 - \omega_2)\}}{\pi(\omega_1 - \omega_2)} \right]^2. \tag{1.7}$$

In this article we explain attempts to reproduce the RMT predictions in the semiclassical framework.

Let us denote a complex conjugate by an asterisk. As we can readily see that

$$\begin{aligned} \rho(E) &= \frac{i}{2\pi} \left\{ g(E^+) - g(E^+)^* \right\} \\ &= \bar{\rho}(E) + \frac{1}{2\pi\hbar} \sum_{\gamma} \left\{ A_\gamma e^{iS_\gamma(E^+)/\hbar} + A_\gamma^* e^{-iS_\gamma(E^-)/\hbar} \right\}, \end{aligned} \tag{1.8}$$

the 2-level correlation function can be written as

$$R_2(\epsilon_1, \epsilon_2) = 1 + \frac{1}{(2\pi\hbar\bar{\rho})^2} \left\langle \sum_{\gamma} \sum_{\gamma'} A_\gamma A_{\gamma'}^* e^{i\left\{S_\gamma(E+\epsilon_1) - S_{\gamma'}(E+\epsilon_2)\right\}/\hbar} + \mathrm{c.c.} \right\rangle, \tag{1.9}$$

where highly oscillatory terms in the limit $\hbar \to 0$, which vanish after averaging, are omitted.

In order to evaluate the above 2-level correlation function, Berry introduced a useful scheme called the diagonal approximation[7], in which one only calculates the diagonal terms with $\gamma = \gamma'$. Then we find

$$R_2(\epsilon_1, \epsilon_2) = 1 + \frac{1}{(2\pi\hbar\bar{\rho})^2} \left\langle \sum_\gamma |A_\gamma|^2 \mathrm{e}^{i\{S_\gamma(E+\epsilon_1) - S_\gamma(E+\epsilon_2)\}/\hbar} + \text{c.c.} \right\rangle . \tag{1.10}$$

In terms of a relative coordinate

$$\omega = \omega_1 - \omega_2 = \bar{\rho}(\epsilon_1 - \epsilon_2), \tag{1.11}$$

we obtain

$$\{S_\gamma(E+\epsilon_1) - S_\gamma(E+\epsilon_2)\}/\hbar = T_\gamma(\epsilon_1 - \epsilon_2)/\hbar. = T_\gamma\omega/(\hbar\bar{\rho}) \tag{1.12}$$

in the limit $\hbar \to 0$. Here $T_\gamma = \mathrm{d}S_\gamma/\mathrm{d}E$ is the period of the periodic orbit γ. It follows that

$$R_2(\omega) = 1 + \frac{1}{(2\pi\hbar\bar{\rho})^2} \left\langle \sum_\gamma |A_\gamma|^2 \left\{ \mathrm{e}^{iT_\gamma\omega/(\hbar\bar{\rho})} + \mathrm{e}^{-iT_\gamma\omega/(\hbar\bar{\rho})} \right\} \right\rangle . \tag{1.13}$$

Let us now calculate the form factor (the Fourier transform of the 2-level correlation function) defined as

$$K(\tau) = \int_{-\infty}^{\infty} \{R_2(\omega) - 1\} \, \mathrm{e}^{-2\pi i\omega\tau} \mathrm{d}\omega. \tag{1.14}$$

Putting the semiclassical formula (1.13), we can readily derive

$$K(\tau) = \frac{1}{T_H} \sum_\gamma |A_\gamma|^2 \left\{ \delta(T_\gamma - \tau T_H) + \delta(T_\gamma + \tau T_H) \right\}, \tag{1.15}$$

where $T_H = 2\pi\hbar\bar{\rho}$ is the Heisenberg time.

It is known that the ergodicity of chaotic dynamics implies a sum rule (the Hannay-Ozorio de Almeida sum rule)[8]

$$\sum_\gamma |A_\gamma|^2 \delta(T_\gamma - T) = T, \tag{1.16}$$

so that we arrive at

$$K(\tau) = |\tau|, \tag{1.17}$$

which should be in agreement with the RMT prediction.

As one can show that

$$\int_{-\infty}^{\infty} K_{\rm RM}(\tau) e^{2\pi i\omega\tau} d\tau = \left[1 - \left\{\frac{\sin(\pi\omega)}{\pi\omega}\right\}^2\right] - 1, \quad \omega \neq 0 \tag{1.18}$$

with

$$K_{\rm RM}(\tau) = \begin{cases} |\tau|, |\tau| \leq 1, \\ 1, \quad |\tau| > 1, \end{cases} \tag{1.19}$$

$K_{\rm RM}(\tau)$ is the RMT prediction of the form factor. Therefore the diagonal approximation gives a form factor in agreement with the RMT prediction only in the region $|\tau| \leq 1$. In order to describe the region of large $|\tau|$, we need a more sophisticated scheme, in which off-diagonal contributions are incorporated.

Let us write the stability amplitude as $A_\gamma = T_\gamma F_\gamma$. It is known that the dependence of F_γ on large periods T_γ is

$$F_\gamma \sim e^{-\lambda T_\gamma/2} e^{-\pi i n_\gamma/2}, \quad \lambda > 0 \tag{1.20}$$

in chaotic systems, where n_γ is an integer. Hence the Hannay-Ozorio de Almeida sum rule leads to

$$T = \sum_\gamma T_\gamma^2 |F_\gamma|^2 \delta(T_\gamma - T) \sim \int dT' \rho_{\rm PO}(T')(T')^2 e^{-\lambda T'} \delta(T' - T), \tag{1.21}$$

where $\rho_{\rm PO}(T)$ is the density function of the periodic orbits. This relation implies an estimate

$$\rho_{\rm PO}(T) \sim \frac{e^{\lambda T}}{T} \tag{1.22}$$

for large T. Then we find

$$\begin{aligned} \sum_\gamma \left| A_\gamma e^{iS_\gamma(E)/\hbar} \right| &= \sum_\gamma T_\gamma |F_\gamma| \\ &\sim \int dT' \rho_{\rm PO}(T') T' e^{-\lambda T'/2} \\ &\sim \int dT' e^{\lambda T'/2}, \end{aligned} \tag{1.23}$$

so that the second term of the semiclassical formula (1.3) is not absolutely convergent. The Riemann-Siegel lookalike formula, which converts the infinite sum into a finite sum, was introduced by Berry and Keating in order to overcome this difficulty[9, 10, 11]. One can use the resulting finite sum to semiclassically calculate the energy levels[12]. We shall see in the following sections that such a resummation formula is also useful in the analysis of the spectral correlation functions[13, 14, 15, 16].

§ 2. The Riemann-Siegel lookalike formula

Let us now introduce a quantity $\Delta(E)$ as

$$\Delta(E) = \exp\left(\int^{E} g(E')\mathrm{d}E'\right), \tag{2.1}$$

which satisfies

$$g(E) = -\left.\frac{\partial}{\partial\epsilon}\frac{\Delta(E)}{\Delta(E+\epsilon)}\right|_{\epsilon=0}. \tag{2.2}$$

We moreover define the generating function Z_n as

$$Z_n = \left(\frac{1}{2\pi\bar{\rho}i}\right)^n \left\langle \prod_{j=1}^{n} \left\{ \frac{\Delta(E+\eta_j)}{\Delta(E^+ + \epsilon_j)} - \frac{\Delta(E+\eta_j)^*}{\Delta(E^+ + \epsilon_j)^*} \right\} \right\rangle. \tag{2.3}$$

Then the n-level correlation functions are expressed as

$$\begin{aligned} R_n(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) &= \left(\frac{i}{2\pi\bar{\rho}}\right)^n \left\langle \prod_{j=1}^{n} \left\{ g(E^+ + \epsilon_j) - g(E^+ + \epsilon_j)^* \right\} \right\rangle \\ &= \left.\frac{\partial^n}{\partial\epsilon_1\partial\epsilon_2\cdots\partial\epsilon_n} Z_n\right|_{\boldsymbol{\eta}=\boldsymbol{\epsilon}}, \end{aligned} \tag{2.4}$$

where

$$\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n), \quad \boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_n). \tag{2.5}$$

Using the semiclassical formula (1.3) for $g(E)$, we can derive

$$\int^{E} g(E')\mathrm{d}E' = -i\pi\bar{N}(E) - \frac{i}{\hbar}\sum_{\gamma}\int^{E} \mathrm{d}E' A_\gamma \mathrm{e}^{iS_\gamma(E')/\hbar}, \tag{2.6}$$

where

$$\bar{N}(E) = \int^{E} \bar{\rho}(E')\mathrm{d}E' \tag{2.7}$$

is the cumulative energy level density. It can easily be seen that

$$\frac{\partial}{\partial E}\left(F_\gamma \mathrm{e}^{iS_\gamma(E)/\hbar}\right) = \frac{i}{\hbar} A_\gamma \mathrm{e}^{iS_\gamma(E)/\hbar} \tag{2.8}$$

in the limit $\hbar \to 0$. Therefore

$$\int^{E} g(E')\mathrm{d}E' = -i\pi\bar{N}(E) - \sum_{\gamma} F_\gamma \mathrm{e}^{iS_\gamma(E)/\hbar} \tag{2.9}$$

holds. Now $\Delta(E^+)$ can be evaluated as

(2.10)
$$\begin{aligned}\Delta(E^+) &= \exp\left\{-i\pi\bar{N}(E^+) - \sum_\gamma F_\gamma \mathrm{e}^{S_\gamma(E^+)/\hbar}\right\} \\ &= \mathrm{e}^{-i\pi\bar{N}(E^+)}\left\{1 - \sum_\gamma F_\gamma \mathrm{e}^{iS_\gamma(E^+)/\hbar} + \frac{1}{2}\sum_\gamma\sum_{\gamma'} F_\gamma F_{\gamma'} \mathrm{e}^{i\{S_\gamma(E^+)+S_{\gamma'}(E^+)\}/\hbar}\right. \\ &\quad \left. - \frac{1}{6}\sum_\gamma\sum_{\gamma'}\sum_{\gamma''} F_\gamma F_{\gamma'} F_{\gamma''} \mathrm{e}^{i\{S_\gamma(E^+)+S_{\gamma'}(E^+)+S_{\gamma''}(E^+)\}/\hbar} + \cdots \right\}.\end{aligned}$$

Let us define a pseudo-orbit (a set of periodic orbits)

$$A = \{\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_{n_A}\} \tag{2.11}$$

and the corresponding quantities

$$\begin{aligned}F_A &= F_{\gamma_1} F_{\gamma_2} F_{\gamma_3} \cdots F_{\gamma_{n_A}}, \\ S_A &= S_{\gamma_1} + S_{\gamma_2} + S_{\gamma_3} + \cdots + S_{\gamma_{n_A}}.\end{aligned} \tag{2.12}$$

Then we find

$$\Delta(E^+) = \mathrm{e}^{-i\pi\bar{N}(E^+)} \sum_A F_A (-1)^{n_A}\ \mathrm{e}^{iS_A(E^+)/\hbar}. \tag{2.13}$$

We can similarly expand the inverse of $\Delta(E)$ as

$$\begin{aligned}\Delta(E^+)^{-1} &= \exp\left\{i\pi\bar{N}(E^+) + \sum_\gamma F_\gamma \mathrm{e}^{S_\gamma(E^+)/\hbar}\right\} \\ &= \mathrm{e}^{i\pi\bar{N}(E^+)} \sum_A F_A\ \mathrm{e}^{iS_A(E^+)/\hbar}.\end{aligned} \tag{2.14}$$

We are now in a position to introduce the Riemann-Siegel lookalike formula, which converts the sum over A into a finite sum. It is an analogue of the Riemann-Siegel formula for the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{2.15}$$

According to the Riemann hypothesis, nontrivial zeros of $\zeta(z)$ are located on the critical line $\mathrm{Re}z = 1/2$. It is conjectured that the correlation functions among the nontrivial zeros are asymptotically identical to the correlation functions of the GUE of random matrices[17]. Assuming that the Riemann hypothesis is true, one can write the zero density $\rho_{\mathrm{RZ}}(E)$ ($z = 1/2 - iE$, E real) in the form

$$\rho_{\mathrm{RZ}}(E) = \bar{\rho}_{\mathrm{RZ}}(E) - \frac{1}{\pi}\mathrm{Re}\sum_{p}\sum_{r=1}^{\infty} \log p \ \mathrm{e}^{-\frac{r}{2}\log p}\mathrm{e}^{irE\log p} \tag{2.16}$$

with

$$\bar{\rho}_{\mathrm{RZ}}(E) = \frac{1}{2\pi}\log\left(\frac{E}{2\pi}\right), \tag{2.17}$$

where p is a prime number. On the other hand, using the formulas (1.8) and (1.20), we obtain an estimate of the energy level density

$$\rho(E) \sim \bar{\rho}(E) + \frac{1}{\pi\hbar}\mathrm{Re}\sum_{p}\sum_{r=1}^{\infty} T_p \mathrm{e}^{-\lambda T_\gamma/2}\mathrm{e}^{iS_\gamma/\hbar}\mathrm{e}^{-\pi i n_\gamma/2}. \tag{2.18}$$

A periodic orbit γ is supposed to be an r times repetition of a primitive periodic orbit p with a period T_p , so that $T_\gamma = rT_p$. We used the fact that the contributions from the orbits with $r > 1$ is negligible, because of the exponential factor $\mathrm{e}^{-\lambda T_\gamma/2}$. Except the difference of the sign, the second terms of (2.16) and (2.18) are identified, if we put

$$\hbar = 1, \quad \lambda = 1, \quad T_\gamma = r\log p, \quad S_\gamma = rE\log p, \quad n_\gamma = 0. \tag{2.19}$$

Therefore one can expect that the energy level density of chaotic systems can similarly be treated as the zero density of the Riemann zeta function. The relations (2.19) leads to a guess that the quantity

$$\Delta_{\mathrm{RZ}}(E) = -\mathrm{e}^{-i\pi\bar{N}_{\mathrm{RZ}}(E)}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\mathrm{e}^{iE\log n} \tag{2.20}$$

with

$$\bar{N}_{\mathrm{RZ}}(E) = \int_0^E \bar{\rho}_{\mathrm{RZ}}(E')\mathrm{d}E' \tag{2.21}$$

is the counterpart of $\Delta(E)$ in (2.13).

In the theory of the Riemann zeta function, the quantity $\Delta_{\mathrm{RZ}}(E)$ is known to satisfy a resummation formula

$$\Delta_{\mathrm{RZ}}(E) \sim -\sum_{n=1}^{n^\sharp}\frac{1}{\sqrt{n}}\mathrm{e}^{-i\pi\bar{N}_{\mathrm{RZ}}(E)+iE\log n} - c.c. \tag{2.22}$$

in the limit $E \to \infty$. This resummation formula is called the Riemann-Siegel formula. The upper limit $n^\sharp$ of the finite sum is the stationary point of the exponent

$$f_{\mathrm{RZ}}(E) = -i\pi\bar{N}_{\mathrm{RZ}}(E) + iE\log n. \tag{2.23}$$

Hence the equation

$$\left.\frac{\mathrm{d}f_{\mathrm{RZ}}(E)}{\mathrm{d}E}\right|_{n=n^\sharp} = -i\pi\bar{\rho}_{\mathrm{RZ}}(E) + i\log n^\sharp = 0 \tag{2.24}$$

determines $n^\sharp$ as the integer part of $\sqrt{E/(2\pi)}$.

One can now infer an analogous relation for the semiclassical $\Delta(E)$ as

$$\Delta(E) = \mathrm{e}^{-i\pi\bar{N}(E)} \sum_A^\sharp F_A(-1)^{n_A} \mathrm{e}^{iS_A(E)/\hbar} + c.c. \tag{2.25}$$

for a real E. This resummation formula is called the Riemann-Siegel lookalike formula after (2.22)[9, 10, 11]. A sharp ($\sharp$) on the summation symbol stands for an upper limit of the finite sum over pseudo-orbits, which is the stationary point of the exponent

$$f(E) = -i\pi\bar{N}(E) + iS_A(E)/\hbar. \tag{2.26}$$

Let us now define T_A as

$$T_A = T_{\gamma_1} + T_{\gamma_2} + T_{\gamma_3} + \cdots + T_{\gamma_{n_A}} \tag{2.27}$$

and the upper limit of T_A as $T_A^\sharp$. Then the stationary point equation

$$\left.\frac{\mathrm{d}f(E)}{\mathrm{d}E}\right|_{T_A=T_A^\sharp} = -i\pi\bar{\rho}(E) + i\frac{T_A^\sharp}{\hbar} = 0 \tag{2.28}$$

determines the upper limit $T_A^\sharp$ as a half of the Heisenberg time

$$T_A^\sharp = \pi\hbar\bar{\rho} = T_H/2. \tag{2.29}$$

§ 3. Chaotic systems without time reversal invariance

Let us come back to the generating function (2.3). Putting (2.14) and (2.25) into (2.3), we find an expression

(3.1)

$$Z_n = \left(\frac{1}{2\pi\bar{\rho}i}\right)^n \sum_{\substack{\sigma_1=\pm1\\ \tau_1=\pm1}} \sum_{\substack{\sigma_2=\pm1\\ \tau_2=\pm1}} \cdots \sum_{\substack{\sigma_n=\pm1\\ \tau_n=\pm1}} \left\langle \exp\left[i\pi\sum_{j=1}^n \{\sigma_j\bar{N}(E+\epsilon_j) - \tau_j\bar{N}(E+\eta_j)\}\right]\right.$$

$$\left. \times \prod_{j=1}^n \left\{ \sigma_j \sum_{A_j} F_{A_j}^{(\sigma_j)} \mathrm{e}^{i\sigma_j S_{A_j}(E+\epsilon_j)/\hbar} \sum_{\substack{B_j\\ (T_{B_j}<T_H/2)}} F_{B_j}^{(\tau_j)} (-1)^{n_{B_j}} \mathrm{e}^{i\tau_j S_{B_j}(E+\eta_j)/\hbar} \right\} \right\rangle,$$

where $F_A^{(1)} = F_A$ and $F_A^{(-1)} = F_A^*$.

As there are highly oscillatory phase factors due to large $\bar{N}$ in the semiclassical limit $\hbar \to 0$, most of the terms in the sum over σ_j and τ_j vanish after averaging. In order to find the surviving terms, we expand the exponent of the first phase factor in (3.1) as

$$\begin{aligned} \exp\left[i\pi\sum_{j=1}^{n}\{\sigma_j\bar{N}(E+\epsilon_j)-\tau_j\bar{N}(E+\eta_j)\}\right] \\ \sim \exp\left[i\pi\bar{N}(E)\sum_{j=1}^{n}(\sigma_j-\tau_j)+i\pi\bar{\rho}(E)\sum_{j=1}^{n}(\sigma_j\epsilon_j-\tau_j\eta_j)\right]. \end{aligned} \tag{3.2}$$

Hereafter we only treat the terms satisfying

$$\sum_{j=1}^{n}(\sigma_j-\tau_j)=0, \tag{3.3}$$

because they survive due to the absence of highly oscillatory phase factors.

Moreover, using four subsets of $\{1,2,3,\cdots,n\}$

$$I=\{j|\sigma_j=1\},\quad J=\{j|\sigma_j=-1\},\quad K=\{j|\tau_j=1\},\quad L=\{j|\tau_j=-1\}, \tag{3.4}$$

we can rewrite the second phase factor in (3.1) as

$$\begin{aligned} &\exp\left[i\sum_{j=1}^{n}\left\{\sigma_j S_{A_j}(E+\epsilon_j)+\tau_j S_{B_j}(E+\eta_j)\right\}/\hbar\right] \\ &=\exp\left[i\left\{\sum_{j\in I}S_{A_j}(E+\epsilon_j)-\sum_{k\in J}S_{A_k}(E+\epsilon_k)\right.\right. \\ &\qquad\left.\left.+\sum_{j\in K}S_{B_j}(E+\eta_j)-\sum_{k\in L}S_{B_k}(E+\eta_k)\right\}/\hbar\right]. \end{aligned} \tag{3.5}$$

For most choices of pseudo-orbits, this phase factor rapidly oscillates in the semiclassical limit and the corresponding terms vanish after averaging. Only the terms with almost vanishing exponents survive. In order to choose such terms, let us assume that the component orbits in the pseudo-orbits A_j ($j \in I$) and B_j ($j \in K$) are the same as those in A_k ($k \in J$) and B_k ($k \in L$), neglecting repetitions. This assumption gives an approximation scheme called the extended diagonal approximation (EDA)[15]. Now the stability amplitude of each periodic orbit is multiplied with the complex conjugate to form the absolute square, and it is possible to omit the upper limit $T_H/2$ of the sum over pseudo-orbits.

The EDA implies the decompositions of the pseudo-orbits

$$
\begin{aligned}
A_j &= \{\cup_{k\in J}(A_j\cap A_k)\}\cup\{\cup_{k\in L}(A_j\cap B_k)\}, \quad j\in I,\\
A_j &= \{\cup_{k\in I}(A_j\cap A_k)\}\cup\{\cup_{k\in K}(A_j\cap B_k)\}, \quad j\in J,\\
B_j &= \{\cup_{k\in J}(B_j\cap A_k)\}\cup\{\cup_{k\in L}(B_j\cap B_k)\}, \quad j\in K,\\
B_j &= \{\cup_{k\in I}(B_j\cap A_k)\}\cup\{\cup_{k\in K}(B_j\cap B_k)\}, \quad j\in L,
\end{aligned}
\tag{3.6}
$$

so that the sums over the pseudo-orbits are rewritten in the form

$$
\begin{aligned}
\sum_{A_j} F_{A_j}\mathrm{e}^{iS_{A_j}(E+\epsilon_j)/\hbar} &= \prod_{k\in J}\left(\sum_{A_j\cap A_k} F_{A_j\cap A_k}\mathrm{e}^{iS_{A_j\cap A_k}(E+\epsilon_j)/\hbar}\right)\\
&\quad\times\prod_{k\in L}\left(\sum_{A_j\cap B_k} F_{A_j\cap B_k}\mathrm{e}^{iS_{A_j\cap B_k}(E+\epsilon_j)/\hbar}\right), \quad j\in I,
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
\sum_{A_j} F^*_{A_j}\mathrm{e}^{-iS_{A_j}(E+\epsilon_j)/\hbar} &= \prod_{k\in I}\left(\sum_{A_j\cap A_k} F^*_{A_j\cap A_k}\mathrm{e}^{-iS_{A_j\cap A_k}(E+\epsilon_j)/\hbar}\right)\\
&\quad\times\prod_{k\in K}\left(\sum_{A_j\cap B_k} F^*_{A_j\cap B_k}\mathrm{e}^{-iS_{A_j\cap B_k}(E+\epsilon_j)/\hbar}\right), \quad j\in J,
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
&\sum_{B_j} F_{B_j}(-1)^{n_{B_j}}\mathrm{e}^{iS_{B_j}(E+\eta_j)/\hbar}\\
&= \prod_{k\in J}\left(\sum_{B_j\cap A_k} F_{B_j\cap A_k}(-1)^{n_{B_j\cap A_k}}\mathrm{e}^{iS_{B_j\cap A_k}(E+\eta_j)/\hbar}\right)\\
&\times\prod_{k\in L}\left(\sum_{B_j\cap B_k} F_{B_j\cap B_k}(-1)^{n_{B_j\cap B_k}}\mathrm{e}^{iS_{B_j\cap B_k}(E+\eta_j)/\hbar}\right), \quad j\in K
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
&\sum_{B_j} F^*_{B_j}(-1)^{n_{B_j}}\mathrm{e}^{-iS_{B_j}(E+\eta_j)/\hbar}\\
&= \prod_{k\in I}\left(\sum_{B_j\cap A_k} F^*_{B_j\cap A_k}(-1)^{n_{B_j\cap A_k}}\mathrm{e}^{-iS_{B_j\cap A_k}(E+\eta_j)/\hbar}\right)\\
&\times\prod_{k\in K}\left(\sum_{B_j\cap B_k} F^*_{B_j\cap B_k}(-1)^{n_{B_j\cap B_k}}\mathrm{e}^{-iS_{B_j\cap B_k}(E+\eta_j)/\hbar}\right), \quad j\in L.
\end{aligned}
\tag{3.10}
$$

When we put these formulas into (3.1), the exponents of the phase factors almost vanish. Introducing the scaled variables

$$\omega_j = \bar{\rho}\epsilon_j, \quad \xi_j = \bar{\rho}\eta_j, \tag{3.11}$$

and using the asymptotic relation

$$\{S_A(E+\epsilon_j) - S_A(E+\epsilon_k)\}/\hbar = T_A(\epsilon_1 - \epsilon_2)/\hbar = T_A(\omega_j - \omega_k)/(\hbar\bar{\rho}) \tag{3.12}$$

in the limit $\hbar \to 0$, we obtain

$$\begin{aligned} Z_n = \left(\frac{1}{2\pi\bar{\rho}i}\right)^n & \sum_{\substack{\sigma_j,\tau_j \\ (\sum_{j=1}^n \sigma_j = \sum_{j=1}^n \tau_j)}} \prod_{j=1}^n \sigma_j \mathrm{e}^{i\pi(\sigma_j\omega_j - \tau_j\xi_j)} \\ & \times \left\langle \frac{\prod_{j\in I,k\in L} \zeta_D(-i(\omega_j - \xi_k)/(\hbar\bar{\rho})) \prod_{j\in K,k\in J} \zeta_D(-i(\xi_j - \omega_k)/(\hbar\bar{\rho}))}{\prod_{j\in I,k\in J} \zeta_D(-i(\omega_j - \omega_k)/(\hbar\bar{\rho})) \prod_{j\in K,k\in L} \zeta_D(-i(\xi_j - \xi_k)/(\hbar\bar{\rho}))} \right\rangle, \end{aligned} \tag{3.13}$$

where $\zeta_D(s)$ is the dynamical zeta function

$$\zeta_D(s) = \sum_A |F_A|^2 (-1)^{n_A} \mathrm{e}^{-sT_A} = \exp\left(-\sum_\gamma |F_\gamma|^2 \mathrm{e}^{-sT_\gamma}\right). \tag{3.14}$$

It follows from the asymptotic estimates (1.20) and (1.22) that

$$\begin{aligned} \log\zeta_D(s) &= -\sum_\gamma |F_\gamma|^2 \mathrm{e}^{-sT_\gamma} \\ &\sim -\int_{T_0}^\infty \mathrm{d}T \rho_{\mathrm{PO}}(T) \mathrm{e}^{-(\lambda+s)T} \\ &\sim -\int_{T_0}^\infty \mathrm{d}T \frac{\mathrm{e}^{-sT}}{T} \\ &= -\int_{sT_0}^\infty \mathrm{d}u \frac{\mathrm{e}^{-u}}{u} \\ &\sim \log(sT_0), \quad s \to 0, \end{aligned} \tag{3.15}$$

where T_0 is the shortest period. Therefore we find

$$\zeta_D(s) \propto s, \quad s \to 0. \tag{3.16}$$

Then it is straightforward to see that

(3.17)
$$R_n(\omega_1,\omega_2,\cdots,\omega_n) = \bar{\rho}^n \frac{\partial^n}{\partial\omega_1\partial\omega_2\cdots\partial\omega_n} Z_n\Big|_{\boldsymbol{\omega}=\boldsymbol{\xi}}$$
$$= \left(\frac{1}{2\pi i}\right)^n \frac{\partial^n}{\partial\omega_1\partial\omega_2\cdots\partial\omega_n} \sum_{\substack{I+J=K+L=\{1,2,\cdots,n\} \\ (|J|=|L|)}} (-1)^{|J|}$$
$$\times\, \mathrm{e}^{i\pi\left(\sum_{j\in I}\omega_j - \sum_{j\in J}\omega_j - \sum_{j\in K}\xi_j + \sum_{j\in L}\xi_j\right)} \frac{\prod_{\substack{j\in I \\ k\in L}}(\omega_j-\xi_k)\prod_{\substack{j\in K \\ k\in J}}(\xi_j-\omega_k)}{\prod_{\substack{j\in I \\ k\in J}}(\omega_j-\omega_k)\prod_{\substack{j\in K \\ k\in L}}(\xi_j-\xi_k)}\Bigg|_{\boldsymbol{\omega}=\boldsymbol{\xi}}$$

in the limit $\hbar\to 0$, where

$$\boldsymbol{\omega} = (\omega_1,\omega_2,\cdots,\omega_n), \quad \boldsymbol{\xi} = (\xi_1,\xi_2,\cdots,\xi_n) \tag{3.18}$$

and $|J|$ and $|L|$ are the numbers of the elements of J and L, respectively.

Let us for example calculate the 2-level correlation function. The above equation can be decomposed into the form

$$R_2(\omega_1,\omega_2) = \sum_{\substack{I+J=K+L=\{1,2\} \\ (|J|=|L|)}} q(I,K), \tag{3.19}$$

where

$$q(\emptyset,\emptyset) = q(\{1,2\},\{1,2\}) = \frac{1}{4}, \tag{3.20}$$

$$q(\{1\},\{1\}) = q(\{2\},\{2\}) = \frac{1}{4}\left[1 - \frac{1}{\{\pi(\omega_1-\omega_2)\}^2}\right], \tag{3.21}$$

$$q(\{1\},\{2\}) = \frac{\mathrm{e}^{2\pi i(\omega_1-\omega_2)}}{4\{\pi(\omega_1-\omega_2)\}^2} \tag{3.22}$$

and

$$q(\{2\},\{1\}) = \frac{\mathrm{e}^{-2\pi i(\omega_1-\omega_2)}}{4\{\pi(\omega_1-\omega_2)\}^2}. \tag{3.23}$$

Therefore we obtain

$$R_2(\omega_1,\omega_2) = 1 - \frac{2 - \mathrm{e}^{2\pi i(\omega_1-\omega_2)} - \mathrm{e}^{-2\pi i(\omega_1-\omega_2)}}{4\{\pi(\omega_1-\omega_2)\}^2} = 1 - \left[\frac{\sin\{\pi(\omega_1-\omega_2)\}}{\pi(\omega_1-\omega_2)}\right]^2, \tag{3.24}$$

which is in agreement with the RMT formula (1.7). Moreover it can be proved that the n-level correlation functions (3.17) are in general identical to the RMT determinant expressions (1.6)[15]. Thus it can be concluded that the EDA fully reproduces the RMT predictions, when the time reversal invariance is broken.

§ 4. Chaotic systems with time reversal invariance

So far we have only discussed chaotic systems without time reversal invariance. When the system is time reversal invariant, the cancellations occur not only between the same actions, but also between S_γ and $S_{\gamma^{(-)}}$, where $\gamma^{(-)}$ is the time reverse of the periodic orbit γ. Let us define the time reverse $A^{(-)}$ of a pseudo-orbit A as a set of the time reversed components of A.

Then the EDA implies the decompositions

$$
\begin{aligned}
A_j &= \{\cup_{k\in J}\cup_{r=\pm 1}(A_j\cap A_k^{(r)})\}\cup\{\cup_{k\in L}\cup_{r=\pm 1}(A_j\cap B_k^{(r)})\}, \quad j\in I,\\
A_j &= \{\cup_{k\in I}\cup_{r=\pm 1}(A_j\cap A_k^{(r)})\}\cup\{\cup_{k\in K}\cup_{r=\pm 1}(A_j\cap B_k^{(r)})\}, \quad j\in J,\\
B_j &= \{\cup_{k\in J}\cup_{r=\pm 1}(B_j\cap A_k^{(r)})\}\cup\{\cup_{k\in L}\cup_{r=\pm 1}(B_j\cap B_k^{(r)})\}, \quad j\in K,\\
B_j &= \{\cup_{k\in I}\cup_{r=\pm 1}(B_j\cap A_k^{(r)})\}\cup\{\cup_{k\in K}\cup_{r=\pm 1}(B_j\cap B_k^{(r)})\}, \quad j\in L,
\end{aligned}
\tag{4.1}
$$

where $A_j^{(1)}=A_j$, $A_j^{(-1)}=A_j^{(-)}$, $B_j^{(1)}=B_j$, $B_j^{(-1)}=B_j^{(-)}$. Hence we can rewrite the sum over the pseudo-orbits as

$$
\begin{aligned}
\sum_{A_j}F_{A_j}\mathrm{e}^{iS_{A_j}(E+\epsilon_j)/\hbar} = &\prod_{k\in J}\prod_{r=\pm 1}\left(\sum_{A_j\cap A_k^{(r)}}F_{A_j\cap A_k^{(r)}}\mathrm{e}^{iS_{A_j\cap A_k^{(r)}}(E+\epsilon_j)/\hbar}\right)\\
&\times\prod_{k\in L}\prod_{r=\pm 1}\left(\sum_{A_j\cap B_k^{(r)}}F_{A_j\cap B_k^{(r)}}\mathrm{e}^{iS_{A_j\cap B_k^{(r)}}(E+\epsilon_j)/\hbar}\right), \quad j\in I,
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
\sum_{A_j}F^*_{A_j}\mathrm{e}^{-iS_{A_j}(E+\epsilon_j)/\hbar} = &\prod_{k\in I}\prod_{r=\pm 1}\left(\sum_{A_j\cap A_k^{(r)}}F^*_{A_j\cap A_k^{(r)}}\mathrm{e}^{-iS_{A_j\cap A_k^{(r)}}(E+\epsilon_j)/\hbar}\right)\\
&\times\prod_{k\in K}\prod_{r=\pm 1}\left(\sum_{A_j\cap B_k^{(r)}}F^*_{A_j\cap B_k^{(r)}}\mathrm{e}^{-iS_{A_j\cap B_k^{(r)}}(E+\epsilon_j)/\hbar}\right), \quad j\in J,
\end{aligned}
\tag{4.3}
$$

$$(4.4)\qquad \sum_{B_j} F_{B_j}(-1)^{n_{B_j}} \mathrm{e}^{iS_{B_j}(E+\eta_j)/\hbar}$$
$$= \prod_{k\in J}\prod_{r=\pm 1}\left(\sum_{B_j\cap A_k^{(r)}} F_{B_j\cap A_k^{(r)}}(-1)^{n_{B_j\cap A_k^{(r)}}} \mathrm{e}^{iS_{B_j\cap A_k^{(r)}}(E+\eta_j)/\hbar}\right)$$
$$\times \prod_{k\in L}\prod_{r=\pm 1}\left(\sum_{B_j\cap B_k^{(r)}} F_{B_j\cap B_k^{(r)}}(-1)^{n_{B_j\cap B_k^{(r)}}} \mathrm{e}^{iS_{B_j\cap B_k^{(r)}}(E+\eta_j)/\hbar}\right), \quad j\in K$$

and

$$(4.5)\qquad \sum_{B_j} F^*_{B_j}(-1)^{n_{B_j}} \mathrm{e}^{-iS_{B_j}(E+\eta_j)/\hbar}$$
$$= \prod_{k\in I}\prod_{r=\pm 1}\left(\sum_{B_j\cap A_k^{(r)}} F^*_{B_j\cap A_k^{(r)}}(-1)^{n_{B_j\cap A_k^{(r)}}} \mathrm{e}^{-iS_{B_j\cap A_k^{(r)}}(E+\eta_j)/\hbar}\right)$$
$$\times \prod_{k\in K}\prod_{r=\pm 1}\left(\sum_{B_j\cap B_k^{(r)}} F^*_{B_j\cap B_k^{(r)}}(-1)^{n_{B_j\cap B_k^{(r)}}} \mathrm{e}^{-iS_{B_j\cap B_k^{(r)}}(E+\eta_j)/\hbar}\right), \quad j\in L.$$

Putting these formulas into (3.1) and using the asymptotic estimate (3.16), we arrive at

(4.6)

$$R_n(\omega_1,\omega_2,\cdots,\omega_n) = \bar{\rho}^n \frac{\partial^n}{\partial\omega_1\partial\omega_2\cdots\partial\omega_n} Z_n\Big|_{\boldsymbol{\omega}=\boldsymbol{\xi}}$$
$$= \left(\frac{1}{2\pi i}\right)^n \frac{\partial^n}{\partial\omega_1\partial\omega_2\cdots\partial\omega_n} \sum_{\substack{I+J=K+L=\{1,2,\cdots,n\}\\ (|J|=|L|)}} (-1)^{|J|}$$
$$\times\ \mathrm{e}^{i\pi\left(\sum_{j\in I}\omega_j-\sum_{j\in J}\omega_j-\sum_{j\in K}\xi_j+\sum_{j\in L}\xi_j\right)} \frac{\displaystyle\prod_{\substack{j\in I\\ k\in L}}(\omega_j-\xi_k)^2\prod_{\substack{j\in K\\ k\in J}}(\xi_j-\omega_k)^2}{\displaystyle\prod_{\substack{j\in I\\ k\in J}}(\omega_j-\omega_k)^2\prod_{\substack{j\in K\\ k\in L}}(\xi_j-\xi_k)^2}\Bigg|_{\boldsymbol{\omega}=\boldsymbol{\xi}}$$

in the limit $\hbar\to 0$. The existence of the factors $(\omega_j-\xi_k)^2$ and $(\xi_j-\omega_k)^2$ implies that only the terms with $I=K$ and $J=L$ survive. Therefore one can simplify this expression as

$$R_n(\omega_1, \omega_2, \cdots, \omega_n) \\ = \left(\frac{1}{2\pi i}\right)^n \frac{\partial^n}{\partial\omega_1 \partial\omega_2 \cdots \partial\omega_n} \sum_{I+J=\{1,2,\cdots,n\}} (-1)^{|J|} \\ \times \, \mathrm{e}^{i\pi\{\sum_{j\in I}(\omega_j - \xi_j) - \sum_{k \in J}(\omega_k - \xi_k)\}} \prod_{\substack{j\in I \\ k \in J}} \frac{(\omega_j - \xi_k)^2(\xi_j - \omega_k)^2}{(\omega_j - \omega_k)^2(\xi_j - \xi_k)^2}\Bigg|_{\boldsymbol{\omega}=\boldsymbol{\xi}} . \tag{4.7}$$

The 2-level and 3-level correlation functions are for example evaluated as

$$R_2(\omega_1, \omega_2) = 1 - \frac{1}{\{\pi(\omega_1 - \omega_2)\}^2} \tag{4.8}$$

and

$$R_3(\omega_1, \omega_2, \omega_3) = 1 - \frac{1}{\{\pi(\omega_1 - \omega_2)\}^2} - \frac{1}{\{\pi(\omega_2 - \omega_3)\}^2} - \frac{1}{\{\pi(\omega_3 - \omega_1)\}^2}. \tag{4.9}$$

When the chaotic system is time reversal invariant, the quantum Hamiltonian can be represented as a real symmetric matrix, so that the corresponding random matrix model is the Gaussian Orthogonal Ensemble (GOE). It is known that the RMT prediction of the n-level correlation function is[18]

$$R_n^{(\mathrm{GOE})}(\omega_1, \omega_2, \cdots, \omega_n) = \mathrm{Pf} \begin{bmatrix} D(\omega_j, \omega_l) & S(\omega_l, \omega_j) \\ -S(\omega_j, \omega_l) & -I(\omega_j, \omega_l) \end{bmatrix}_{j,l=1,2,\cdots,n} . \tag{4.10}$$

Here Pf stands for a Pfaffian and

$$S(\omega, \xi) = \frac{\sin\{\pi(\omega - \xi)\}}{\pi(\omega - \xi)}, \tag{4.11}$$

$$D(\omega, \xi) = \int_0^1 \mathrm{d}u \; u \;\sin\{\pi u(\omega - \xi)\} \tag{4.12}$$

and

$$I(\omega, \xi) = \int_1^\infty \mathrm{d}v \; \frac{1}{v} \;\sin\{\pi v(\omega - \xi)\}. \tag{4.13}$$

Let us suppose that the energy levels are mutually far apart ($\omega_j - \omega_l \sim X$ and X is large). Then we can expand the RMT prediction as

$$R_2^{(\mathrm{GOE})}(\omega_1, \omega_2) = 1 - \frac{1}{\{\pi(\omega_1 - \omega_2)\}^2} + O(X^{-3}) \tag{4.14}$$

and

(4.15)

$$R_3^{(\mathrm{GOE})}(\omega_1,\omega_2,\omega_3) = 1 - \frac{1}{\{\pi(\omega_1-\omega_2)\}^2} - \frac{1}{\{\pi(\omega_2-\omega_3)\}^2} - \frac{1}{\{\pi(\omega_3-\omega_1)\}^2} + O(X^{-3}).$$

It is observed that the EDA reproduces only the leading terms of the RMT predictions, when the chaotic system is time reversal invariant. The remaining terms of the 2-level correlation function are known to be recovered by using a diagrammatic method[13, 16]. An extension of such a method to treat the general n-level correlation functions is an interesting problem.

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