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On free regular infinitely divisible distributions

By

Noriyoshi Sakuma *

Abstract

Free regular infinitely divisible distributions are the images of classical infinitely divisible distributions on $\mathbb{R}_+$ under the Bercovici-Pata bijection. We show the time independent properties of the distributions of a free Lévy process whose distribution at time 1 is free regular infinitely divisible, that is, the supports of free regular infinitely divisible distributions are concentrated on positive real line. In contrast, the distributions of a free Lévy process based on the standard positive Wigner distribution w_+ are not always concentrated on $\mathbb{R}_+$. We also show that $\mathrm{w}_+ \boxtimes \mathrm{w}_+$ and $\mathrm{w}_+ \boxtimes \mathrm{w}$ are not free infinitely divisible where w is the standard Wigner law.

§ 1. Introduction and Preliminaries

In [9], which treats the free infinite divisibility of the free mixture of the Wigner law, the free counterpart of the classical, positively supported, infinitely divisible distributions called "free regular infinitely divisible distributions" play a key role. In this paper, we discuss certain properties of them and give some counterexamples.

First, we start by setting notation and recalling well-known facts. $\mathbb{E}_X$ means the expectation with respect to random variable X. Let $\mathcal{P}$ denote the set of all Borel probability measures on $\mathbb{R}$ and let $\mathcal{P}_+$ and $\mathcal{P}_s$ be the sets of all Borel probability measures with support in $\mathbb{R}_+ = [0, \infty)$ and of all symmetric Borel probability measures (*i.e.* $\mu(B) = \mu(-B)$ for all Borel set B in $\mathbb{R}$), respectively. Denote the set of all infinitely divisible distributions on $\mathbb{R}$ by I^*. Basic facts of I^* are found in [10].

In the following, we explain the main tools of free probability. For details, see [5]. Let $\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $\mathbb{C}^- := \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$. For any probability

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Supported by Grant-in-Aid for JSPS Fellows
*Department of Mathematics, Keio University, Yokohama, Japan
e-mail: noriyosi@math.keio.ac.jp

measure μ, the Cauchy transform $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ is defined by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\mu(\mathrm{d}x), \quad z \in \mathbb{C}^+.$$

and the reciprocal Cauchy transform of $\mu \in \mathcal{P}$ is defined by

$$F_\mu(z) = \frac{1}{G_\mu(z)}, \quad z \in \mathbb{C}^+.$$

$F_\mu(z)$ has a right inverse $F_\mu^{-1}(z)$ on the region $\Gamma_{\eta,M}$ w.r.t. composition operation, where

$$\Gamma_{\eta,M} := \{z \in \mathbb{C} : |\mathrm{Re}(z)| < \eta \mathrm{Im}(z), \mathrm{Im}(z) > M\},$$

for some $\eta > 0$ and some $M > 0$.

Let $\boxplus$ denote the free additive convolution. For the free additive convolution $\boxplus$, the Voiculescu transform and the free cumulant transform

$$\varphi_\mu(z) := F_\mu^{-1}(z) - z, \quad \mathcal{C}_\mu^{\boxplus}(z) := zF_\mu^{-1}(1/z) - 1, \quad z \in \Gamma_{\eta,M},$$

of $\mu \in \mathcal{P}$ play the role of the cumulant transform $\mathcal{C}_\mu^*(z) := \log\left(\int_{\mathbb{R}} e^{itx}\mu(dx)\right)$ in classical probability theory because $\varphi_{\mu\boxplus\nu}(z) = \varphi_\mu(z) + \varphi_\nu(z)$ and $\mathcal{C}_{\mu\boxplus\nu}^{\boxplus}(z) = \mathcal{C}_\mu^{\boxplus}(z) + \mathcal{C}_\nu^{\boxplus}(z)$. We have a characterization, which is called "Lévy–Khintchine representation", for the Fourier transform of $\mu \in I^*$: $\mu \in I^*$ iff there exists a unique set of $a_\mu \geq 0$, $b_\mu \in \mathbb{R}$ and a measure ν_μ on $\mathbb{R}$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \min(1, |x|^2)\nu_\mu(dx) < \infty$) such that

$$\mathcal{C}_\mu^*(t) = -\frac{a_\mu}{2}t^2 + ib_\mu t + \int_{\mathbb{R}} \left(e^{itx} - 1 - itx1_{|x|\leq 1}\right)\nu_\mu(dx).$$

ν_μ is called the Lévy measure. When we consider infinitely divisible distributions with respect to the free additive convolution $\boxplus$, we have "free" Lévy–Khintchine representation for their free cumulant transform as follows: μ is a free infinitely divisible distribution iff there exist a unique set of $a_\mu \geq 0$, $b_\mu \in \mathbb{R}$ and a Lévy measure ν_μ on $\mathbb{R}$ such that

$$\mathcal{C}_\mu^{\boxplus}(z) = b_\mu z + a_\mu z^2 + \int_{\mathbb{R}} \left(\frac{1}{1-zx} - 1 - zx1_{[-1,1]}(x)\right)\nu_\mu(\mathrm{d}x), \quad z \in \mathbb{C}^- . \tag{1.1}$$

We denote the set of all free infinitely divisible distributions on $\mathbb{R}$ by $I^{\boxplus}$. It is said that a probability measure σ on $[0,\infty)$ is a free regular infinitely divisible distribution iff

$$\mathcal{C}_\sigma^{\boxplus}(z) = b_\sigma z + \int_{\mathbb{R}_+} \left(\frac{1}{1-zx} - 1\right)\nu_\sigma(\mathrm{d}x), \quad z \in \mathbb{C}^- , \tag{1.2}$$

with $b_\sigma \geq 0$ and a measure concentrated on $\mathbb{R}_+$ and satisfying $\int_0^\infty \min(1,x)\nu_\sigma(\mathrm{d}x) < \infty$. Not all nonnegative free infinitely divisible distributions are regular. We denote the set of all free regular distributions by $I^{\boxplus}_{r+}$. This class is a counterpart of classical infinitely divisible distributions on $\mathbb{R}_+$. $\sigma \in I^* \cap \mathcal{P}_+$ iff there exist $b_\sigma \geq 0$, $\nu_\sigma((-\infty, 0]) = 0$ and $\int_0^\infty \min(1,x)\nu_\sigma(\mathrm{d}x) < \infty$

$$\mathcal{C}^*_\sigma(t) = ib_\sigma t + \int_{\mathbb{R}_+} \left(e^{itx} - 1\right) \nu_\sigma(dx).$$

We write the class $I^* \cap \mathcal{P}_+$ by I^*_+.

Let Λ be the Bercovici-Pata bijection between I^* and $I^{\boxplus}$. It is such that if $\mu \in I^*$ has a characteristic triplet (a_μ, ν_μ, b_μ), then $\Lambda(\mu)$ is in $I^{\boxplus}$ with triplet (a_μ, ν_μ, b_μ). For details and examples, see [9].

We can find some examples which are free infinitely divisible distribution concentrated on $\mathbb{R}_+$ but are not free regular infinitely divisible. A typical example is as follow: a probability measure $\mathrm{w}_{b,a}$ is a Wigner distribution with the mean $b \in \mathbb{R}$ and variance $a > 0$ if it has the density

$$f_{\mathrm{w}_{b,a}}(x) = \frac{1}{2\pi a}\sqrt{4a - (x-b)^2}\,1_{[b-\sqrt{4a}, b+\sqrt{4a}]}(x)$$

If the left edge of the density of Wigner distribution is positive, that is $b - \sqrt{4a} \geq 0$, the density is concentrated on $\mathbb{R}_+$. Especially, if $a = 1$ and $b = 2$, we call it the standard positive Wigner distribution. Let w and w^+ denote the standard Wigner law (i.e. $a = 1$ and $b = 0$) and the standard positive Wigner distribution, respectively.

To study the distribution of products of free independent random variables, there is another useful analytic tool called the "S-transform", which is defined as follows. For $\mu \in \mathcal{P}_+$, define

$$\Psi_\mu(z) = \int_{\mathbb{R}} \frac{zx}{1 - zx}\mu(\mathrm{d}x) = z^{-1}G_\mu(z^{-1}) - 1, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{1.3}$$

It was proved in [5] that for a probability measure μ with support on $\mathbb{R}_+$ and $\mu(\{0\}) < 1$, the function $\Psi_\mu(z)$ has a unique inverse $\chi_\mu(z)$ in the left-half plane $i\mathbb{C}^+$ and $\Psi_\mu(i\mathbb{C}^+)$ is a region contained in the circle with diameter $(\mu(\{0\}) - 1, 0)$. In this case the S-transform of μ is defined as $S_\mu(z) = \chi_\mu(z)\frac{1+z}{z}$. It satisfies

$$z = \mathcal{C}^{\boxplus}_\mu\left(zS_\mu(z)\right) \tag{1.4}$$

for sufficiently small $z \in \Psi_\mu(i\mathbb{C}^+)$.

Following [5], the "free multiplicative convolution" of probability measures μ_1 and μ_2 supported on $\mathbb{R}_+$ is defined as the probability measure $\mu_1 \boxtimes \mu_2$ concentrated on $\mathbb{R}_+$ such that

$$S_{\mu_1 \boxtimes \mu_2}(z) = S_{\mu_1}(z)S_{\mu_2}(z) \tag{1.5}$$

for $z \in \Psi_{\mu_1}(i\mathbb{C}^+) \cap \Psi_{\mu_2}(i\mathbb{C}^+)$.

In [1], the free multiplicative convolutions are defined on $\mathcal{P}_+ \times \mathcal{P}_s$. In [9], free multiplicative mixtures of the Wigner law $\rho \boxtimes \mathrm{w}$ with $\rho \in \mathcal{P}_+$ were introduced and it showed that $\rho \boxtimes \mathrm{w}$ is free infinitely divisible iff $\rho \boxtimes \rho \in I_{r+}^{\boxplus}$.

In this paper, we give some properties of the class of free regular infinitely divisible distributions in section 2. In Section 3, we show that $\mathrm{w}^+ \boxtimes \mathrm{w}^+$ and $\mathrm{w}^+ \boxtimes \mathrm{w}$ is not free infinitely divisible.

§ 2. An observation from the Bercovici-Pata bijection with random matrix model

The following observation tells us the time independent property of the free regular infinitely divisible distributions.

In [4] and [6], random matrix models for free infinitely divisible distributions were constructed by the idea based on the Bercovici-Pata bijection. We write the set of $d \times d$ Hermitian matrices and the closed cone of $d \times d$ non-negative definite Hermitian matrices by $\mathbb{H}_d$ and $\mathbb{H}_d^+$, respectively. We define an inner product in $\mathbb{H}_d$ as $\langle A, B\rangle_{\mathbb{H}_d} := \mathrm{Tr}(AB^*)$, where $A, B \in \mathbb{H}_d$ and Tr means the trace of matrix. First we take a $*$-infinitely divisible distribution μ. Let M_d be a random matrix with its Fourier transform

$$\mathbb{E}_{M_d}[\exp(i\mathrm{Tr}(A_d M_d))] = \exp(d\mathbb{E}_u[\mathcal{C}_\mu^*(\langle u, A_d u\rangle_{\mathbb{C}^d})]), A_d \in \mathbb{H}_d,$$

where u is a random vector uniformly distributed on the unit sphere of $\mathbb{C}^d$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$ be the random eigenvalues of M_d. Then the random spectral distribution $\mu_d = \frac{1}{d}\sum_{k=1}^d \delta_{\lambda_k}$ of M_d converges to a deterministic probability distribution ρ in probability as $d \to \infty$ and $\rho = \Lambda(\mu)$. If we consider the random matrix model for the free regular infinitely divisible distribution μ, the random matrix model is concentrated on the closed cone of non-negative definite matrix. Then the sums and limits in distribution of them are also non-negative definite. Therefore, the free regular infinitely divisible distributions must be concentrated on $\mathbb{R}_+$. On the other hand, if μ is in $I_{r+}^{\boxplus}$, then $\mu^{\boxplus t} 1_d$ is also in $I_{r+}^{\boxplus}$ for any $t \geq 0$. Consequencely, we have the following theorem.

Theorem 2.1. *(1) Let μ be in $I_{r+}^{\boxplus}$. Then $\mu^{\boxplus t} \in \mathcal{P}_+$ for all $t \geq 0$.*
(2) $\mu \boxtimes \mu$ is also free infinitely divisible.

Proof. (1) For every $\mu \in I_+^*$, μ^{*t} is also in I_+^*. From the definition of $I_{r+}^{\boxplus}$, for any $\rho \in I_{r+}^{\boxplus}$, there exists $\rho \in I_+^*$ such that $\rho = \Lambda(\mu)$. Therefore, it is enough to show that M_d^μ is concentrated on $\mathbb{H}_d^+$ if $\mu \in I_+^*$. Since $\mu \in I_+^*$, there exist $b_\mu \geq 0$ and a measure

ν_μ satisfying $\nu_\sigma((-\infty,0])=0$ and $\int_0^\infty \min(1,x)\nu_\mu(\mathrm{d}x)<\infty$ such that

$$\mathcal{C}^*_\mu(t)=ib_\mu t+\int_{\mathbb{R}_+}\left(e^{itx}-1\right)\nu_\mu(dx).$$

We have, for any $A\in\mathbb{H}_d$,

$$\mathbb{E}_{M_d}[\exp(i\mathrm{Tr}(A_dM_d^\mu))]=\exp\left\{d\mathbb{E}_u\left[ib_\mu\langle u,A_du\rangle_{\mathbb{C}^d}+\int_0^\infty\left(e^{i\langle u,A_du\rangle_{\mathbb{C}^d}x}-1\right)\nu_\mu(\mathrm{d}x)\right]\right\}.$$

Then, by writing $V_d=uu^*$ and by using the polar decomposition we have

$$\begin{aligned}\mathbb{E}[\exp(i\mathrm{Tr}(A_dM_d^\mu))]&=\exp\left\{d\mathbb{E}_u\left[ib_\mu\mathrm{Tr}(uu^*A_d)+\int_0^\infty\left(e^{i\mathrm{Tr}(uu^*A)x}-1\right)\nu_\mu(\mathrm{d}x)\right]\right\}\\&=\exp\left\{d\mathbb{E}_{V_d}\left[ib_\mu\mathrm{Tr}(A_dV_d)+\int_0^\infty\left(e^{i\mathrm{Tr}(V_dA_d)x}-1\right)\nu_\mu(\mathrm{d}x)\right]\right\}\\&=\exp\left\{i\langle A_d,b_\mu 1_d\rangle_{\mathbb{H}_d}+\int_{\mathbb{H}_d^+}\left(e^{i\langle A_d,X_d\rangle_{\mathbb{H}_d}}-1\right)\widetilde{\nu}_\mu(\mathrm{d}X_d)\right\},\end{aligned}$$

where 1_d is identity matrix, $X_d=xV_d$ and $\widetilde{\nu}_\mu=d\rho_{V_d}\nu_\mu$. Here ρ_{V_d} is the distribution of V_d on $\mathbb{H}_d^+$. Since V_d is in $\mathbb{H}_d^+$, $\widetilde{\nu}_\mu(\mathbb{H}_d\backslash\mathbb{H}_d^+)=0$ and $\int_{\mathbb{H}_d^+}\min(1,||X_d||)\widetilde{\nu}_\mu(\mathrm{d}X_d)<\infty$. $b_\mu 1_d$ is also in $\mathbb{H}_d^+$. Thus M_d^μ is concentrated on $\mathbb{H}_d^+$ by Proposition 3.1. in [2].

(2) In Belinschi and Nica [3, eq (3.9)], they showed the following remarkable property related to the free multiplicative and additive convolutions. For any probability measures μ_1 and μ_2 on $\mathbb{R}_+$,

$$D_t(\mu_1\boxtimes\mu_2)^{\boxplus t}=(\mu_1^{\boxplus t})\boxtimes(\mu_2^{\boxplus t}),\quad t\geq 1, \tag{2.1}$$

where $D_t\mu$ is the dilation of a measure μ by t, that is $D_t\mu(B)=\mu\left(\frac{1}{t}B\right)$ for any Borel set B. Hence, we have (2). □

Remark. For the general polar decomposition of characteristic function of this model and concrete examples, see [7].

§ 3. Positive Wigner distributions

When we consider the $\boxplus$-convolution semi-group of the positive Wigner distributions, they are not always concentrated on $\mathbb{R}_+$;

$$(\mathrm{w}^+)^{\boxplus t}=\frac{1}{2\pi t}\sqrt{4t-(x-2t)^2}1_{[2t-2\sqrt{t},2t+2\sqrt{t}]}(x).$$

Proposition 3.1. *If $t\geq 1$, $(\mathrm{w}^+)^{\boxplus t}\in\mathcal{P}_+$. Otherwise, $(\mathrm{w}^+)^{\boxplus t}\notin\mathcal{P}_+$.*

A similar example can be found in Hasebe [8, Chapter 10]. From Theorem 2.1 (1), these facts also show that these examples are not free regular infinitely divisible. We find some different properties which cannot be found if we condsider free regular infinitely divisible distributions.

We state the main theorem and give the proof.

Theorem 3.2. *Let* $\mu = \mathrm{w}^+ \boxtimes \mathrm{w}^+$*. Then* μ *is not free infinitely divisible.*

When we check the free infinite divisibility of a Borel probability measure, the following result by Bercovici and Voiculescu [5] is useful.

Proposition 3.3. *A Borel probability measure* μ *on* $\mathbb{R}$ *is in* $I^{\boxplus}$ *iff the Voiculescu transform of* μ *extends to an analytic function* $\varphi_\mu(z) : \mathbb{C}^+ \to \mathbb{C}^-$*.*

Proof of Theorem 3.2. The free cumulant transform of w^+ is

$$\mathcal{C}^{\boxplus}_{\mathrm{w}^+}(z) = z^2 + 2z.$$

From the equation (1.4), $\mathcal{C}^{\boxplus}_{\mathrm{w}^+}(zS_{\mathrm{w}^+}(z)) = z$, we have

$$S_{\mathrm{w}^+}(z) = \frac{\sqrt{1+z}-1}{z}.$$

Therefore,

$$S_\mu(z) = \frac{2+z-2\sqrt{1+z}}{z^2}.$$

and use (1.4) again, we get the free cumulant transform and the Voiculescu transform of μ as follows:

$$\varphi_\mu(z) = \frac{4z^2}{(1-z)^2}.$$

If we set $z = 1/3 + i/5$,

$$\mathrm{Im}(\varphi_\mu(1/3+i/5)) = \frac{14760}{11881}.$$

So the range of φ_μ is not in $\mathbb{C}^-$. It does not satisfy Proposition 3.3. Hence we conclude that μ is not free infinitely divisible. $\square$

From this theorem, we obtain the followings.

Corollary 3.4. *The free multiplicative convolution of free infinitely divisible distributions may not be free infinitely divisible.*

The following example is related to the free type W distribution in [9].

Corollary 3.5. *The distribution* $\mu = \mathrm{w}^+ \boxtimes \mathrm{w}$ *is not free infinitely divisible.*

Proof. It can be shown by Theorem 2.1 and [9], Theorem 22. □

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