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A note on a few processes related to Dyson's Brownian motion

By

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Abstract

Some connections between processes related to Dyson's Brownian motion are explained. Starting from the Brownian motion of hermitian matrices, we consider noncolliding Brownian motions, the interlacing structure of DBM with different number of particles, a system of ordered reflection, the case with a boundary and their relations.

§ 1. Introduction

Let us consider a time dependent random matrix $H = H(t)$ of size n of the form,

(1.1)
$$H(t) = \begin{bmatrix} B_{11}(t) & \frac{1}{\sqrt{2}}(B_{12}^{(R)}(t) + iB_{12}^{(I)}(t)) & \cdots & \frac{1}{\sqrt{2}}(B_{1n}^{(R)}(t) + iB_{1n}^{(I)}(t)) \\ \frac{1}{\sqrt{2}}(B_{21}^{(R)}(t) - iB_{21}^{(I)}(t)) & B_{22}(t) & \cdots & \frac{1}{\sqrt{2}}(B_{2n}^{(R)}(t) + iB_{2n}^{(I)}(t)) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2}}(B_{n1}^{(R)}(t) - iB_{n1}^{(I)}(t)) & \frac{1}{\sqrt{2}}(B_{n2}^{(R)}(t) - iB_{n2}^{(I)}(t)) & \cdots & B_{nn}(t) \end{bmatrix},$$

where $B_{jj}, 1 \le j \le n, B_{jk}^{(R)} = B_{kj}^{(R)}, B_{jk}^{(I)} = B_{kj}^{(I)}, 1 \le j < k \le n$ are independent Brownian motions. The stochastic dynamics of the n eigenvalues of H denoted by $X_1 \le X_2 \le \ldots \le X_n$ is described by the stochastic differential equation(SDE),

$$dX_i = dB_i + \sum_{\substack{1 \le j \le m \\ j \ne i}} \frac{dt}{X_i - X_j}, \quad 1 \le i \le m, \tag{1.2}$$

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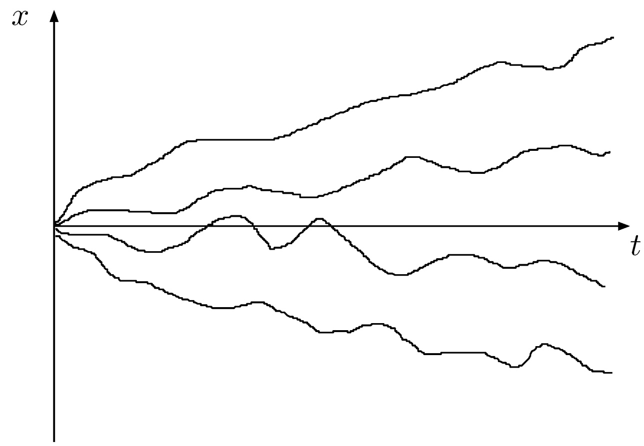


Figure 1. Dyson's BM

where $B_i, 1 \le i \le m$ are independent one dimensional Brownian motions[2]. This is known as Dyson's Brownian motion (DBM). The process satisfies $X_1(t) < X_2(t) < \cdots < X_m(t)$ for all $t > 0$. The process X can be started from the origin, i.e., one can take $X_i(0) = 0, 1 \le i \le m$ [12]. Pictorially this looks like Fig. 1.

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. ¿From the SDE (1.2), one sees that the transition density $p_t^+(x, x')$ of this process from x to x' during time interval t satisfies

$$\frac{\partial}{\partial t} p_t^+ = \frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} p_t^+ + \sum_{i=1}^n \sum_{j \ne i} \frac{1}{x_i - x_j} \cdot \frac{\partial p_t^+}{\partial x_i}. \tag{1.3}$$

Due to the nonclliding properties of the process, it also satisfies

$$p_t^+|_{x_i = x_{i+1}} = 0, \quad 1 \le i \le n-1. \tag{1.4}$$

If we set

$$h_n^{(A)}(x) = \prod_{1 \le i < j \le n} (x_j - x_i), \tag{1.5}$$

(1.3) is rewritten as

$$\frac{\partial}{\partial t} p_t^+ = \frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} p_t^+ + \sum_{i=1}^n \frac{\partial}{\partial x_i} \log h_n^{(A)}(x) \cdot \frac{\partial p_t^+}{\partial x_i}. \tag{1.6}$$

Dyson's BM can be constructed from noncolliding Brownian motion through Doob's h-transformation using the function (1.5) [4]. By the Karlin-McGregor formula[6, 5], the transition density of the noncolliding Brownian motion with n particles is given by

$$p_t(x, x') = \det(\phi_t(x_i, x_j'))_{1 \le i,j \le n} \tag{1.7}$$

with

$$\phi_t(x,x') = \frac{1}{\sqrt{2\pi t}} e^{-(x-x')^2/(2t)}. \tag{1.8}$$

This satisfies

$$\frac{\partial}{\partial t} p_t = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} p_t \tag{1.9}$$

and

$$p_t|_{x_i=x_{i+1}} = 0. \tag{1.10}$$

One notices that $h_n^{(A)}$ in (1.5) is harmonic for (1.7), that is,

$$\int_{-\infty}^{\infty} dx' p_t(x,x') h_n^{(A)}(x') = h_n^{(A)}(x). \tag{1.11}$$

Proof. This is easily seen from the fact that $h_n^{(A)}$ is a Vandermonde determinant, the Heine identity,

$$\int dx_1 \cdots dx_n \det(f_j(x_k))_{1\le j,k\le n} \det(g_j(x_k))_{1\le j,k\le n} = n! \det\left(\int dx f_j(x) g_k(x)\right)_{1\le j,k\le n} \tag{1.12}$$

for nice functions f_j, g_j and

$$\int_{-\infty}^{\infty} \phi_t(x,x')(x')^n dx' = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^{n-k} \tag{1.13}$$

where

$$\gamma_k = \int_{-\infty}^{\infty} \frac{e^{-x^2/(2t)}}{\sqrt{2\pi t}} x^k dx. \tag{1.14}$$

□

We define the h-transform of $p_t(x,x')$ by

$$p_t^+(x,x') := \frac{h_n^{(A)}(x')}{h_n^{(A)}(x)} p_t(x,x'). \tag{1.15}$$

Then one has

Proposition 1.1. *RHS of (1.15) satisfies (1.6).*

Proof. We abbreviate the superscript (A) in $h_n^{(A)}$. From the definition (1.15) and (1.9) we have

$$\frac{\partial}{\partial t} p_t^+(x,x') = \frac{1}{2}\frac{h_n(x')}{h_n(x)}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} p_t. \tag{1.16}$$

By a straightforward computation one sees

(1.17)
$$\frac{\partial}{\partial x_i} p_t^+ = -\frac{\partial}{\partial x_i}\log h_n(x)\cdot p_t^+ + \frac{h_n(x')}{h_n(x)}\frac{\partial}{\partial x_i} p_t,$$

(1.18)
$$\frac{\partial^2}{\partial x_i^2} p_t^+ = \frac{h_n(x')}{h_n(x)}\frac{\partial^2}{\partial x_i^2} p_t d - 2\frac{\partial}{\partial x_i}\log h_n(x)\frac{\partial}{\partial x_i} p_t^+ + \left\{-\frac{\partial^2}{\partial x_i^2}\log h_n(x) - \left(\frac{\partial}{\partial x_i}\log h_n(x)\right)^2\right\} p_t^+.$$

Here one computes

$$-\frac{\partial^2}{\partial x_i^2}\log h_n(x) - \left(\frac{\partial}{\partial x_i}\log h_n(x)\right)^2 = -2\sum_{j,k(\neq i)}\frac{1}{(x_i-x_j)(x_i-x_k)}. \tag{1.19}$$

Noticing

$$\sum_i \sum_{j,k(\neq i)} \frac{1}{(x_i-x_j)(x_i-x_k)} = 0, \tag{1.20}$$

we see that the RHS of (1.15) satisfies (1.6). □

The determinantal structure is the key to the tractability of the process.

At a fixed time t, the random matrix $H(t)$ is nothing but the Gaussian unitary ensemble [10, 3]. For instance the distribution of the position of the top particle $X_n(t)$ is given by an n fold integral as

$$\Pr[X_n(t)\le x_0] = \frac{1}{Z_N(t)}\int_{-\infty}^{x_0} dx_1 \cdots \int_{-\infty}^{x_0} dx_N \prod_{1\le i<j\le N}(x_i-x_j)^2 \prod_{i=1}^N e^{-\frac{x_i^2}{2t}} \tag{1.21}$$

where $Z_N(t)$ is the normalization constant.

In the following we introduce a few processes related to DBM and discuss connections between them. The main aim of this article is to give a short summary and explanation of them with figures. The argument of seeing interlacing properties in propositions 2.1, 4.3, 4.4 has not appeared and should be useful for further studies. There are many references on related topics. For recent developments, see for instance [8, 9].

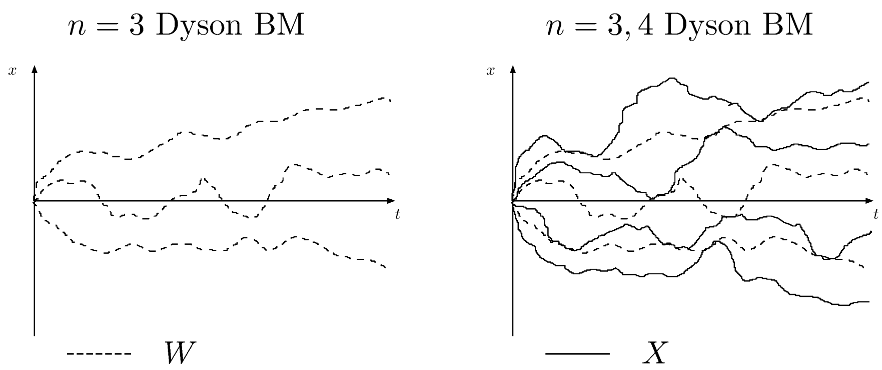


Figure 2. Interlacing processes.

§ 2. Interlacing

In the last section we kept the number of particles fixed. In [16] Warren observed there is an interesting interlacing in DBMs for different number of particles. Let $W_j, 1 \le j \le n$ be the Dyson's BM with n particles starting from the origin. Then let $X_j, 1 \le j \le n+1$ be s.t. each X_j performs a BM with the conditions that they interlace W_j, i.e. $X_1 \le W_1 \le X_2 \le \ldots \le W_n \le X_{n+1}$. The interlacing is maintained by prescribing that X_j is reflected from W_{j-1} and W_j, where the reflection means the Skorokhod construction. Now suppose we forget about the original W particles and focus on X particles. One can show X is the Dyson's BM with $n+1$ particles. See Fig. 2 for an example of $n = 3$. In the left figure, we see trajectories of the three particle DBM starting from the origin. Next, in the right figure, we add four new particles. The lowest one X_1 starts from the origin, performs a BM and is reflected by W_1. The next two (X_2, X_3) are also BMs starting from the origin and are reflected by the X particles below and above them (by W_1, W_2 for X_2 and W_2, W_3 for X_3). The top one X_4 starts from the origin, performs a BM and is reflected by W_3. The above statement says that the dynamics of the X particles are distributed as a DBM with four particles.

Let us consider a system of $2n+1$ particles in which the $W_j, 1 \le j \le n$ are replaced by noncolliding BM with n particles. The transition density $q_t((x,w),(x',w'))$ of this process with $2n+1$ particles satisfies

$$\frac{\partial}{\partial t} q_t = \frac{1}{2}\left(\sum_{i=1}^{n} \frac{\partial^2}{\partial w_i^2} + \sum_{i=1}^{n+1} \frac{\partial^2}{\partial x_i^2}\right) q_t. \tag{2.1}$$

Due to the noncolliding properties of W particles and the interlacing condition using the Skorokhod, it also satisfies the boundary conditions,

$$q_t|_{w_i = w_{i+1}} = 0, \quad \frac{\partial}{\partial x_i} q_t|_{x_i = w_i} = 0, \quad \frac{\partial}{\partial x_{i+1}} q_t|_{x_{i+1} = w_i} = 0. \tag{2.2}$$

The transition density for X is given by integrating over the positions of W particles and applying the h-transformation to X particles as

$$p_t^+(x,x') = \frac{h_{n+1}^{(A)}(x')}{h_{n+1}^{(A)}(x)} \int_{W^n(x)} q_t((x,w),(x',w'))dw. \tag{2.3}$$

Here the integration is over $W^n(x) = \{(w_1,\ldots,w_n) \in \mathbb{R}^n | x_1 \le w_1 \le x_2 \le \ldots \le w_n \le x_{n+1}\}$ for a given $x \in \mathbb{R}^{n+1}$. That this is the transition density for DBM with $n+1$ particles can be shown by performing the integral [16]. Here we see this in a different way.

Proposition 2.1. *RHS of (2.3) satisfies (1.6) and (1.4) for $n+1$ particles.*

Proof. First we see

$$\begin{aligned}\frac{\partial}{\partial t}p_t^+(x,x') = \frac{1}{2}\frac{h_{n+1}(x')}{h_{n+1}(x)}\Bigg\{\int_{W^n(x)} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}q_t dw + \sum_{i=1}^n \Bigg(\int_{W^{(n)}(x):w_i=x_{i+1}} \frac{\partial}{\partial w_i}q_t dw^{(n-1)} \\ - \int_{W^{(n)}(x):w_i=x_i} \frac{\partial}{\partial w_i}q_t dw^{(n-1)}\Bigg)\Bigg\}. \end{aligned}\tag{2.4}$$

Here $\{W^{(n)}(x) : w_i = x_{i+1}\}$ is $W^{(n)}(x)$ but with the condition that the ith component w_i is fixed to be x_{i+1}; the meaning of $\{W^{(n)}(x) : w_i = x_i\}$ is analogous. We also have

$$\begin{aligned}\frac{\partial}{\partial x_i}p_t^+(x,x') = -\frac{\partial}{\partial x_i}\log h_{n+1}(x)p_t^+ + \frac{h_{n+1}(x')}{h_{n+1}(x)}\Bigg\{\int_{W^n(x)} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}q_t dw + \Bigg(\int \frac{\partial}{\partial x_i}q_t^+ dw \\ + \int_{W^{(n)}(x):w_{i-1}=x_i} \frac{\partial}{\partial w_i}q_t dw^{(n-1)} - \int_{W^{(n)}(x):w_i=x_i} \frac{\partial}{\partial w_i}q_t dw^{(n-1)}\Bigg)\Bigg\}, \end{aligned}\tag{2.5}$$

$$\begin{aligned}\frac{\partial^2}{\partial x_i^2}p_t^+(x,x') = \frac{h_{n+1}(x')}{h_{n+1}(x)}\frac{\partial^2}{\partial x_i^2}q_t^+ - 2\frac{\partial}{\partial x_i}\log h_{n+1}(x)\cdot\frac{\partial}{\partial x_i}p_t^+ - \frac{\partial^2}{\partial x_i^2}\log h_{n+1}(x)\cdot p_t^+ \\ - \left(\frac{\partial}{\partial x_i}\log h_{n+1}(x)\right)^2 q_t^+ \\ + \frac{h_{n+1}(x')}{h_{n+1}(x)}\left(\int_{W^{(n)}(x):w_{i-1}=x_i} \frac{\partial}{\partial w_i}q_t dw^{(n-1)} - \int_{W^{(n)}(x):w_i=x_i} \frac{\partial}{\partial w_i}q_t dw^{(n-1)}\right). \end{aligned}\tag{2.6}$$

Combining these, we see that RHS of (2.3) satisfies (1.3) and (1.4) for $n+1$ particles. □

By repeating this interlacing procedure from $n = 1$ to n, one can construct a process of $\frac{n}{2}(n+1)$ particles. The position of particles, $x_i^k \in \mathbb{R}, 1 \le i \le k \le n$ satisfy the constraint $x_i^{k+1} \le x_i^k \le x_{i+1}^k$, which is known as the Gelfand-Tsetlin cone. See Fig. 3. Hence we now have a stochastic dynamics of particles on the GT cone. By

$$\begin{array}{ccccccc}
 & & & x_1^1 & & & \\
 & & x_1^2 & & x_2^2 & & \\
 & x_1^3 & & x_2^3 & & x_3^3 & \\
 \iddots & & & \vdots & & & \ddots \\
x_1^n \quad x_2^n & x_3^n & & \cdots & & x_{n-1}^n & x_n^n
\end{array}$$

Figure 3. Gelfand-Tsetlin cone

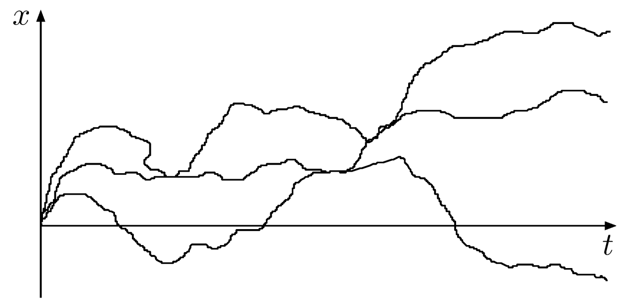


Figure 4. Z process

construction the dynamics of n particles $x^n = (x_1^n, \ldots, x_n^n)$ of the n-th row of the GT cone is the DBM of n particles.

§ 3. DNLS model

Instead of looking at a row, one can focus on the n particles $x_i^i, 1 \le i \le n$ on the upper right line of the GT cone. One sees that this is a Markov process. In this Z process, $Z_1 \le Z_2 \le \ldots \le Z_n$, Z_1 is a Brownian motion and Z_{j+1} is reflected by Z_j, $1 \le j \le n-1$. Here the reflection again means the Skorokhod construction to push Z_{j+1} up from Z_j. More precisely,

$$\begin{aligned} Z_1(t) &= B_1(t), \\ Z_j(t) &= \sup_{0 \le s \le t} (Z_{j-1}(s) + B_j(t) - B_j(s)), \quad 2 \le j \le n, \end{aligned} \tag{3.1}$$

where $B_i, 1 \le i \le n$ are independent Brownian motions, each starting from 0. For a schematic figure, see Fig. 4.

The totally asymmetric simple exclusion process (TASEP) is a stochastic process on $\mathbb{Z}$ in which each particle tries to hop to the right neighboring site with rate 1 under the exclusion interaction among particles, i.e., each site can be either occupied by a particle or is empty. In the diffusion scaling each particle tends to a BM and the exclusion interaction is replaced by a reflective wall. The Z process can be considered as a continuous version of TASEP.

For TASEP, a determinantal formula for the transition probability was found by Schütz [14] and has turned out to be useful for studying fluctuations of TASEP. The generator of ASEP is known to be equivalent (modulo a similarity transformation) to the Hamiltonian of XXZ spin chain. Similarly the generator of the Z process is a special case of the quantum version of the derivative non-linear Schrödinger(DNLS) model with imaginary coupling [13]. Let us set $\phi_t^{(k)}(y) = \frac{d^k}{dy^k}\phi_t(y)$ for $k \geq 0$ and $\phi_t^{(-k)}(y) = (-1)^k \int_y^\infty \frac{(z-y)^{k-1}}{(k-1)!}\phi_t(z)dz$ for $k \geq 1$.

Proposition 3.1. *The transition densities $q_t(x,x')$ from $x = (x_1,\dots,x_n)$ at $t=0$ to $x'=(x'_1,\dots,x'_n)$ at t of the Z process can be written as*

$$q_t(x,x') = \det\{a_{i,j}(x_i,x'_j)\}_{1\leq i,j\leq n} \tag{3.2}$$

where $a_{i,j}$ is given by

$$a_{i,j}(x,x') = \phi_t^{(j-i)}(x-x'). \tag{3.3}$$

Proof. One has to check the Kolomogorov (or master) equation, boundary conditions and initial conditions. This was done in [16] and [13]. ☐

Suppose we are interested in the distribution of Z_n. A direct way to compute would be to use the above transition densities and follow the arguments in [11]. But now looking at the GT cone, one can also use the fact that it is the same as the distribution of the nth particle in the n-particle DBM. It is given by (1.21). The same picture is also true for the TASEP. In [17] a discrete space stochastic process was introduced on the GT cone. If one focuses on the nth row, it is a process related to the Charlier ensemble; on the other hand, the dynamics on the upper right line is TASEP. This implies that the distribution of a particle in TASEP is the same as that of the top particle in the Charlier ensemble. In this way one has a clear understanding why random matrix expression appears in the studies of TASEP.

§ 4. Dyson's BM with a boundary and interlacing

One can introduce similar DBM type non-colliding system of m particles in the presence of a wall at the origin [4, 7, 15]. The dynamics of the positions of the m particles $X^{(C)} = (X_1^{(C)},\dots,X_m^{(C)})$ satisfying $0 < X_1(t) < X_2(t) < \cdots < X_m(t)$ for all $t > 0$ are described by the stochastic differential equation,

$$dX_i^{(C)} = dB_i + \frac{dt}{X_i^{(C)}} + \sum_{\substack{1\leq j\leq m \\ j\neq i}} \left(\frac{1}{X_i^{(C)} - X_j^{(C)}} + \frac{1}{X_i^{(C)} + X_j^{(C)}} \right) dt, \ 1 \leq i \leq m. \tag{4.1}$$

Compare this with (1.2). This process is referred to as Dyson's Brownian motion of type C. It can be interpreted as a system of m Brownian particles conditioned to never collide with each other or the wall. See Fig. 5.

One can also consider the case where the wall above is replaced by a reflecting wall[7]. The dynamics of the positions of the m particles $X^{(D)} = (X_1^{(D)}, \ldots, X_m^{(D)})$ satisfying $0 \leq X_1(t) < X_2(t) < \cdots < X_m(t)$ for all $t > 0$, is described by the stochastic differential equation,

(4.2)

$$dX_i^{(D)} = dB_i + \frac{1}{2}\mathbf{1}_{(i=1)}dL(t) + \sum_{\substack{1\leq j\leq m \\ j\neq i}} \left(\frac{1}{X_i^{(D)} - X_j^{(D)}} + \frac{1}{X_i^{(D)} + X_j^{(D)}} \right) dt, \ 1 \leq i \leq m,$$

where $L(t)$ denotes the local time of $X_1^{(D)}$ at the origin. This process will be referred to as Dyson's Brownian motion of type D. See Fig. 6.

Th processes can be realized as those of eigenvalues of certain random matrix ensembles. Let $B_{ij}^k(t), \tilde{B}_{ij}^k(t), 0 \leq k \leq 3, 1 \leq i \leq j \leq N$ be independent one-dimensional standard Brownian motions starting from the origin. Put

$$s_{ij}^k(t) = \begin{cases} \frac{1}{\sqrt{2}} B_{ij}^k(t), & i < j, \\ B_{ii}^k(t), & i = j, \end{cases}, \quad a_{ij}^k(t) = \begin{cases} \frac{1}{\sqrt{2}} \tilde{B}_{ij}^k(t), & i < j, \\ 0, & i = j, \end{cases} \tag{4.3}$$

with $s_{ij}^k(t) = s_{ji}^k(t)$ and $a_{ij}^k(t) = -a_{ji}^k(t)$ for $i > j$. Let us also introduce the Pauli matrices,

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.4}$$

and set

$$H^{(C)}(t) = ia^0(t) \otimes \sigma_0 + s^1(t) \otimes \sigma_1 + s^2(t) \otimes \sigma_2 + s^3(t) \otimes \sigma_3, \tag{4.5}$$

$$H^{(D)}(t) = ia^0(t) \otimes \sigma_0 + is^1(t) \otimes \sigma_1 + is^2(t) \otimes \sigma_2 + s^3(t) \otimes \sigma_3. \tag{4.6}$$

Then it is known that the processes of eigenvalues of $H^{(C)}$ and $H^{(D)}$ are the DBM of type C and D [7].

The transition density of the positions, $0 \leq X_1 \leq X_2 \leq \ldots \leq X_n$, satisfies (1.6) with $h_n^{(A)}$ replaced by $h_n^{(b)}$ where (b) is either (C) or (D) and

$$h_n^{(C)}(x) = \prod_{i=1}^n x_i \prod_{1\leq i<j\leq n} (x_j^2 - x_i^2), \tag{4.7}$$

$$h_n^{(D)}(x) = \prod_{1\leq i<j\leq n} (x_j^2 - x_i^2). \tag{4.8}$$

The transition density satisfies (1.4) with an additional condition at the origin,

$$p_t^+|_{x_1=0} = 0 \tag{4.9}$$

for type C and

$$\frac{\partial}{\partial x_1} p_t^+|_{x_1=0} = 0 \tag{4.10}$$

for type D.

The Dyson's BM of type C and D can be constructed from noncolliding BM with a boundary through Doob's h-transform using (4.7) and (4.8) as in (1.15). Let us introduce

$$\phi_t^{(C)}(x,x') = \frac{1}{\sqrt{2\pi t}}\left(e^{-(x-x')^2/(2t)} - e^{-(x+x')^2/(2t)}\right), \tag{4.11}$$

$$\phi_t^{(D)}(x,x') = \frac{1}{\sqrt{2\pi t}}\left(e^{-(x-x')^2/(2t)} + e^{-(x+x')^2/(2t)}\right). \tag{4.12}$$

The transition density of the noncolliding Brownian motion with n particles is given by

$$p_t^{(b)}(x,x') = \det(\phi_t^{(b)}(x_i,x_j'))_{1\le i,j\le n}. \tag{4.13}$$

This satisfies

$$\frac{\partial}{\partial t} p_t^{(b)} = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} p_t^{(b)} \tag{4.14}$$

and

$$p_t^{(C)}|_{x_i=x_{i+1}} = 0, \quad p_t^{(C)}|_{x_1=0} = 0 \tag{4.15}$$

for type C and

$$p_t^{(D)}|_{x_i=x_{i+1}} = 0, \quad \frac{\partial}{\partial x_1} p_t^{(D)}|_{x_1=0} = 0 \tag{4.16}$$

for type D.

One notices that $h_n^{(b)}$ in (1.5) is harmonic for (4.13), that is,

$$\int_{-\infty}^{\infty} dx' p_t^{(b)}(x,x') h_n^{(b)}(x') = h_n^{(b)}(x). \tag{4.17}$$

Proof. Notice a simple identity

$$\int_{-\infty}^{\infty} \frac{x^n}{\sqrt{2\pi t}} e^{-(x-x')^2/(2t)} dx = \int_0^{\infty} \frac{x^n}{\sqrt{2\pi t}}\left(e^{-(x-x')^2/(2t)} + (-1)^n e^{-(x+x')^2/(2t)}\right) dx. \tag{4.18}$$

(1.13) is rewritten as

$$\int_0^\infty \phi_t^{(C)}(x,x')(x')^{2m}dx' = \sum_{k=0}^{2m+1}\binom{2m+1}{2l}\gamma_{2l}x^{2m+1-2l}, \tag{4.19}$$

$$\int_0^\infty \phi_t^{(D)}(x,x')(x')^{2m}dx' = \sum_{k=0}^{2m}\binom{2m}{2l}\gamma_{2l}x^{2(m-l)}. \tag{4.20}$$

¿From this we know that $h_n^{(b)}(x)$ in (4.8), which is the Vandermonde of $\{1, x^2, \ldots, x^{2(n-1)}\}$ or $\{x, x^3, \ldots, x^{2n-1}\}$ are harmonic to $h_n^{(b)}$ in (4.13). □

We define the h-transform of $p_t^{(b)}(x,x')$ by

$$p_t^{(b)+}(x,x') := \frac{h_n^{(b)}(x')}{h_n^{(b)}(x)}p_t^{(b)}(x,x'). \tag{4.21}$$

Then one has

Proposition 4.1. *RHS of (4.21) satisfies (1.6) with $h^{(A)}$ replaced by $h^{(b)}$.*

Proof. The proof is analogous to that of proposition 1.1. The difference is h_n. For C one computes

$$\frac{\partial}{\partial x_i}\log h_n^{(C)}(x) = \sum_{j(\neq i)}\left(\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j}\right), \tag{4.22}$$

$$\frac{\partial^2}{\partial x_i^2}\log h_n^{(C)}(x) = -\sum_{j(\neq i)}\left(\frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2}\right). \tag{4.23}$$

For D one finds

$$\frac{\partial}{\partial x_i}\log h_n^{(D)}(x) = \frac{1}{x_i} + \sum_{j(\neq i)}\left(\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j}\right), \tag{4.24}$$

$$\frac{\partial^2}{\partial x_i^2}\log h_n^{(D)}(x) = -\frac{1}{x_i^2} - \sum_{j(\neq i)}\left(\frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2}\right). \tag{4.25}$$

For both C and D, one has

$$\sum_i\left(-\frac{\partial^2}{\partial x_i^2}\log h_n^{(b)}(x) - \left(\frac{\partial}{\partial x_i}\log h_n^{(b)}(x)\right)\right) = 0. \tag{4.26}$$

Hence we see that the RHS of (4.21) satisfies (1.6). □

One can again consider the interlacing of these processes. Given the DBM of type C with n particles, we can construct the DBM of type D with n particles. Consider a system of $2n$ particles in which n W particles are noncolliding BM with a reflective wall at the origin, and n X particles are interlacing with W particles s.t. $W_1 \le X_1 \le W_2 \le \ldots \le W_n \le X_n$. The transition density $q_t((x,w),(x',w'))$ for the whole system with $2n$ particles is the solution to

$$\frac{\partial}{\partial t} q_t = \frac{1}{2}\left(\sum_{i=1}^{n} \frac{\partial^2}{\partial w_i^2} + \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right) q_t \tag{4.27}$$

and

$$q_t|_{w_i=w_{i+1}} = 0, \quad \frac{\partial}{\partial w_1} q_t|_{w_1=0} = 0, \quad \frac{\partial}{\partial x_i} q_t|_{x_i=w_i} = 0, \quad \frac{\partial}{\partial x_i} q_t|_{x_i=w_{i-1}} = 0. \tag{4.28}$$

The transition density for X is given by

$$p_t^+(x,x') = \frac{h_{n+1}^{(D)}(x')}{h_{n+1}^{(D)}(x)} \int_{W^n(x)} q_t((x,w),(x',w'))dw \tag{4.29}$$

where $W^n(x) = \{(w_1,\ldots,w_n) \in \mathbb{R}^n | 0 \le w_1 \le x_1 \le w_2 \le \ldots \le w_n \le x_n\}$.

Proposition 4.2. *RHS of (4.29) satisfies (1.6) with $h^{(A)}$ replaced by $h^{(D)}$ and (4.10) for n particles.*

The proof is similar to that of proposition 2.1. Notice that one does not have to find an expression of the transition densities of $2n$ particles as in [16]. This is an advantage of verifying (1.3) to see the interlacing structure.

Similarly given the DBM of type D with n particles, we can construct the DBM of type C with $n+1$ particles. The transition density $q_t((x,w),(x',w'))$ for the whole system with $2n+1$ particles is the solution to

$$\frac{\partial}{\partial t} q_t = \frac{1}{2}\left(\sum_{i=1}^{n} \frac{\partial^2}{\partial w_i^2} + \sum_{i=1}^{n+1} \frac{\partial^2}{\partial x_i^2}\right) q_t \tag{4.30}$$

and

$$q_t|_{w_i=w_{i+1}} = 0, \quad q_t|_{w_1=0} = 0, \quad \frac{\partial}{\partial x_i} q_t|_{x_i=w_{i-1}} = 0, \quad \frac{\partial}{\partial x_i} q_t|_{x_i=w_i} = 0. \tag{4.31}$$

By construction the transition density for X is given by

$$p_t^+(x,x') = \frac{h_n^{(C)}(x')}{h_n^{(C)}(x)} \int_{W^n(x)} q_t((x,w),(x',w'))dw \tag{4.32}$$

where $W^n(x) = \{(w_1,\ldots,w_n) \in \mathbb{R}^n | 0 \le x_1 \le w_1 \le x_2 \le \ldots \le w_n \le x_{n+1}\}$.

Proposition 4.3. *RHS of (4.32) satisfies (1.6) with $h^{(a)}$ replaced by $h^{(C)}$ and (4.9) for n particles.*

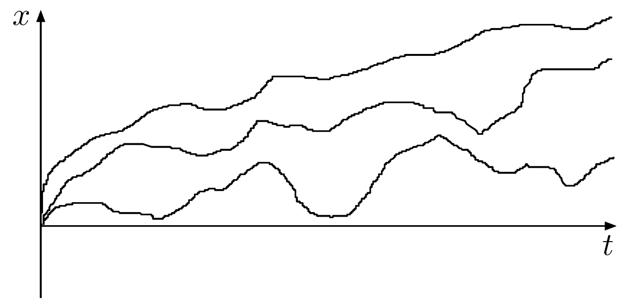


Figure 5. Dyson's BM with an absorbing boundary

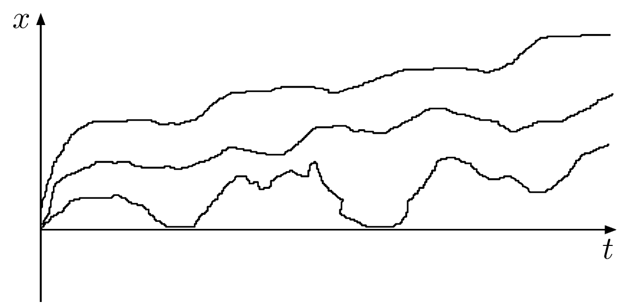


Figure 6. Dyson's BM with a reflective boundary

The proof is similar to that of proposition 2.1, 4.2.

As a special case let us consider the $n = 1$ case. Let us take a system of two particles. The first one is reflected at the origin and the second one is reflected from the first one. The transition density satisfies

$$\frac{\partial}{\partial t} q_t = \frac{1}{2}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) q_t, \tag{4.33}$$

$$\frac{\partial}{\partial z} q_t|_{z=0} = 0, \quad \frac{\partial}{\partial y} q_t|_{y=z} = 0. \tag{4.34}$$

Now integrating over the first particle, let us define

$$p_t(y,x) := \frac{x}{y}\int_0^y q_t((y,z),(x,z'))dz. \tag{4.35}$$

Then this is the transition density for the BM with absorbing boundary.

By repeating these interlacing procedures from $n = 1$ to n, one can construct a process of many particles. Let $\mathbb{K}$ denote the set with n layers $x = (x^1, x^2, \ldots, x^n)$ where $x^{2k} = (x_1^{2k}, x_2^{2k}, \ldots, x_k^{2k}) \in \mathbb{R}_+^k$, $x^{2k-1} = (x_1^{2k-1}, x_2^{2k-1}, \ldots, x_k^{2k-1}) \in \mathbb{R}_+^k$ and the intertwining relations,

$$x_1^{2k-1} \le x_1^{2k} \le x_2^{2k-1} \le x_2^{2k} \le \ldots \le x_k^{2k-1} \le x_k^{2k} \tag{4.36}$$

$$
\left|\begin{array}{cccccccccc}
x_1^1 & & & & & & & & \\
& x_1^2 & & & & & & & \\
x_1^3 & & x_2^3 & & & & & & \\
& x_1^4 & & x_2^4 & & & & & \\
x_1^5 & & x_2^5 & & x_3^5 & & & & \\
& \vdots & & \vdots & & \ddots & & & \\
x_1^n & & x_2^n & & x_3^n & \dots & x_m^n & &
\end{array}\right.
$$

Figure 7. Symplectic Gelfand-Tsetlin cone.

and

$$0 \le x_1^{2k+1} \le x_1^{2k} \le x_2^{2k+1} \le x_2^{2k} \le \ldots \le x_k^{2k} \le x_{k+1}^{2k+1} \tag{4.37}$$

This is known as the symplectic Gelfand-Tsetlin cone. See Fig. 7. Hence we now have a stochastic dynamics of particles on the symplectic GT cone.

By construction the dynamics of k particles $x^{2k} = (x_1^{2k}, \ldots, x_k^{2k})$ of the $2k$-th row of the GT cone is the DBM of type C of k particles. Those of k particles $x^{2k+1} = (x_1^{2k+1}, \ldots, x_k^{2k+1})$ of the $2k+1$-th row of the GT cone is the DBM of type D of k particles. One can focus on the k particles $x_i^i, 1 \le i \le k$ on the upper right line of the symplectic GT cone. One sees that this is a Markov process. In this Y process, $0 \le Y_1 \le Y_2 \le \ldots \le Y_n$, the interactions among Y_i's are the same as in the Z process, i.e., Y_{j+1} is reflected by Y_j, $1 \le j \le n-1$, but Y_1 is now a Brownian motion reflected at the origin (again by Skorokhod construction). See Fig. 8. Similarly to (3.1),

$$
\begin{aligned}
Y_1(t) &= B_1(t) - \inf_{0\le s\le t} B_1(s) = \sup_{0\le s\le t}(B_1(t) - B_1(s)), \\
Y_j(t) &= \sup_{0\le s\le t}(Y_{j-1}(s) + B_j(t) - B_j(s)), \quad 2 \le j \le n.
\end{aligned} \tag{4.38}
$$

The transition density of the Y process is given in [1].

§ 5. Relations

Let us take the initial conditions to be $X_i = X_i^{(C)} = X_i^{(D)} = 0$. In [1], we proved

Theorem 5.1. *The following equalities in law hold;*

$$\max_{0\le s\le t} X_{2n-1}(s) \stackrel{d}{=} X_n^{(D)}(t), \tag{5.1}$$

$$\max_{0\le s\le t} X_{2n}(s) \stackrel{d}{=} X_n^{(C)}(t), \tag{5.2}$$

$n \in \mathbb{Z}_+ = \{1, 2, \ldots\}$.

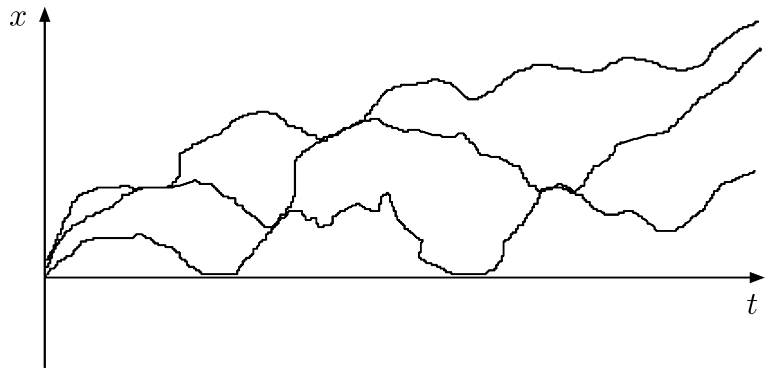


Figure 8. Y process.

The $n = 1$ case of the first equality is nothing but the well known relation between the maximum of BM and the reflective BM.

The idea of the proof of (5.1),(5.2) was the following. ¿From the arguments in section 2, we know

$$X_n(t) \stackrel{d}{=} Z_n(t) \tag{5.3}$$

and hence

$$\max_{0\le s\le t} X_n(s) \stackrel{d}{=} \max_{0\le s\le t} Z_n(s). \tag{5.4}$$

Similarly from the arguments in section 4, one has

$$Y_{2n-1}(s) \stackrel{d}{=} X_n^{(D)}(t), \tag{5.5}$$

$$Y_{2n}(s) \stackrel{d}{=} X_n^{(C)}(t), \tag{5.6}$$

$n \in \mathbb{Z}_+$. In [1] we proved (5.5), (5.6) by a generalization of Rogers-Pitman criterion. But here we would emphasize that once the dynamics on the GT cone is understood, these relations are obvious.

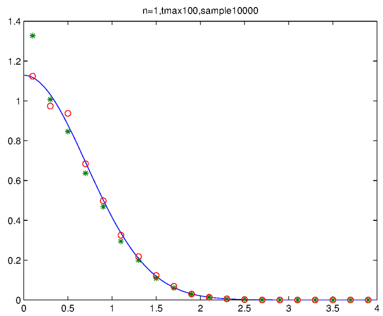
Now to prove (5.1),(5.2) it is enough to see

$$\max_{0\le s\le t} Z_n(s) \stackrel{d}{=} Y_n(t). \tag{5.7}$$

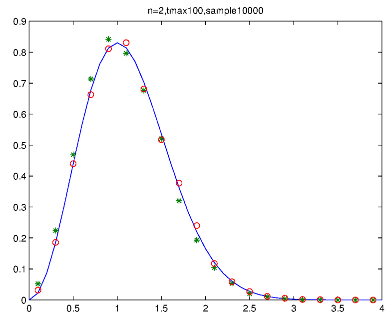
In [1] this is shown by reversing time and the order of particles. The argument does not seem to be common in random matrix theory. It would be interesting to see this in more algebraic terms.

As discussed above, both DBM an DBM with a boundary can be realized as eigenvalues of certain random matrix ensembles. We show some monte carlo results for the equalities (5.1,5.2). The LHS is represented by $*$ and the RHS by $\circ$. The densities of the RHS for $n = 1, 2$ are simple. For $n = 1$, $p(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$; for $n = 2$, $p(x) = \frac{4x^2}{\sqrt{\pi}} e^{-x^2}$. They are shown by solid curves.

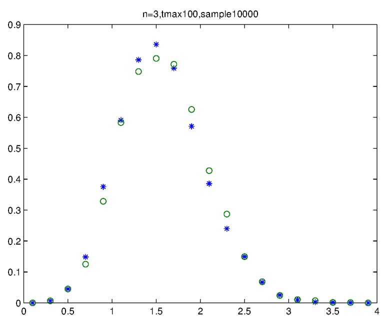
$n = 1$



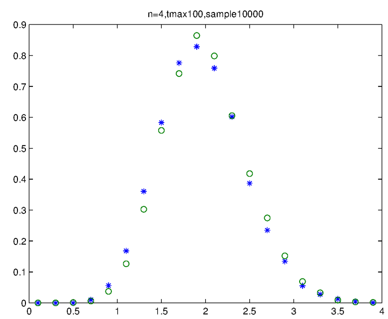
$n = 2$



$n = 3$



$n = 4$



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