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# A note on the transition probability of ASEP

By

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**Abstract**

We discuss a few properties of the transition probabilities of the ASEP. Starting from the formulation of the process and its connection to the XXZ spin chain, we explain some formulas for the transition probability and how they are useful for studying current fluctuations.

## § 1. Introduction

The one-dimensional asymmetric simple exclusion process(ASEP) is a stochastic model of many particles in which each particle tries to perform the asymmetric random walk but with an exclusion interaction among them so that if the target site is occupied the jump does not happen. In spite of its simplicity the ASEP shows nontrivial behaviors due to its nonlinearity and noise. The ASEP was introduced as a model for biopolymerization [7]. Later the system has been studied from several contexts of mathematics and physics such as stochatic interacting particle systems in probability theory [9,10] and nonequilibrium statistical physics [24]. It also has a lot of applications to traffic flow, surface growth, etc.

It has turned out that the ASEP is "exactly solvable" in the sense that some physical quantities for several situations (initial and boundary conditions) can be obtained explicitly. For instance the stationary measure can be written in the form of matrix products [3, 15]. In the last decade, time dependent current fluctuations have been studied extensively. First Johansson showed that it is related to the distributions from

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random matrix theory [8]. A lot of generalizations have been obtained since then. At the same time the understanding of the underlying mathematical structure has deepened considerably. A key property is that there is a determinantal structure which is not apparent in the original formulation of ASEP. In the case of [8], it appeared in the combinatorics of Young tableaux. In [14] it was found that one can extend the model to a system of non-colliding walkers. It is similar to the Dyson's Brownian motion in random matrix theory and can be studied by standard methods. More recently, for ASEP with general parameter value, the Fredholm determinant appears after taking the thermodynamics limit [25, 26].

In these development, the transition probability of the process has played an important role. This is because this quantity has a simple physical meaning and at the same time shows a structure which makes the process tractable. It is the probability that the particles are at certain positions at time $t$ given that they started from certain positions initially.

In this paper, we explain some properties of the transition probability of the ASEP. We start from a few formulations of the process and explain the connection to the XXZ spin chain. Then we introduce the transition probability and give a few formulas for it. For the case of TASEP in which particles hop only in one direction it is written as a single determinant. For general ASEP it is written as a summation over a permutation of the order of the particle number. We discuss the same construction works for the XXZ chain and mention an alternative way to derive it.

## § 2. Formulation of the model

There are several ways of formulating the ASEP. One is to focus on the time evolution of occupation numbers. For simplicity let us first consider the ASEP on a finite lattice $[L] = \{1, 2, \dots, L\}, L \in \mathbb{Z}_+ := \{1, 2, \dots\}$ with reflective boundaries. Let us denote by $\eta_x$ the occupation number at site (or vertex) $x$, i.e., if the site $x$ is occupied by a particle $\eta_x = 1$ and if it is empty $\eta_x = 0$. Then the particle configuration of the whole system is specified by $\eta = \{\eta_1, \cdots, \eta_L\}$. In ASEP, $\eta$ is a stochastic process. Let us denote the probability that the particle configuration is $\eta$ at time $t$ by $P(\eta, t) = P(\eta_1, \cdots, \eta_L; t)$. The dynamics of ASEP is defined by giving the evolution equation for this probability.

(2.1)
$$\frac{d}{dt}P(\eta_1, \dots, \eta_L; t) = -\sum_{j=1}^{L-1} \sum_{\tau_j=0,1} \sum_{\tau_{j+1}=0,1} h_{\eta_j, \eta_{j+1}; \tau_j, \tau_{j+1}} P(\eta_1, \dots, \tau_j, \tau_{j+1}, \cdots, \eta_L; t).$$

This is called the Kolmogorov's forward equation in stochastic analysis and the master equation in physics literature. The off-diagonal (diagonal) elements of the matrix $h$

represent the transition rates into (out of) configuration $\eta$. For the ASEP, the non-zero elements of the matrices are

$$h_{0,1;1,0} = -p, \quad h_{1,0;1,0} = p, \tag{2.2}$$

$$h_{1,0;0,1} = -q, \quad h_{0,1;0,1} = q. \tag{2.3}$$

The parameter $p$ (resp. $q$) represents the hopping rate to right (resp. left). When $p = 0$ or $q = 0$, the process is called the TASEP (totally ASEP). One sees that the sums of each column of the matrices are zero which represents the conservation of probability.

The equation (2.1) can be written in a compact form. We employ the formulation using the quantum spin chain language, in which the particle occupation in ASEP is identified as spin-up and down state in spin-1/2 representation, i.e., the empty site and the occupied one correspond to up-spin and down-spin respectively. See e.g. [24]. Let us define the Pauli matrices by

$$\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.4}$$

We also set

$$\begin{aligned} s^+ &= \frac{1}{2}(\sigma^x + i\sigma^y) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad s^- = \frac{1}{2}(\sigma^x - i\sigma^y) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\ s^z &= \frac{1}{2}\sigma^z = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad n = \frac{1}{2} - s^z = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{2.5}$$

Let us introduce a vector $|0\rangle$ which corresponds to the empty system. One can construct the state with occupation number $\eta = \{\eta_1, \dots, \eta_L\}$ by

$$|\eta\rangle = |\eta_1, \dots, \eta_L\rangle = (s_1^-)^{\eta_1} \cdots (s_L^-)^{\eta_L} |0\rangle. \tag{2.6}$$

Here $s_x^-, x \in [L]$ means it acts nontrivially only on the space of site $x$ as a $2 \times 2$ matrix $s^-$ in (2.5).

Let us construct the vector which represents the probability distribution of the system at time $t$ by

$$|P(t)\rangle = \sum_\eta P(\eta, t)|\eta\rangle \tag{2.7}$$

where $\sum_\eta$ means the summation over all particle configuration. The time evolution (2.1) is compactly written as

$$\frac{d}{dt}|P\rangle = -H|P\rangle. \tag{2.8}$$

The transition rate matrix is given by

$$H = \sum_{j=1}^{L} h_{j,j+1} \tag{2.9}$$

where $h_{j,j+1}$ acts nontrivially only on the space of sites $j, j+1$ as a $4 \times 4$ matrix

$$h = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & q & -p & 0 \\ 0 & -q & p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.10}$$

Notice that the master equation (2.8) formally looks the same as the (imaginary-time) Schrödinger equation. The "Hamiltonian" is now not in general hermitian so that one always has to keep in mind that many facts for hermitian matrices may not hold for stochastic processes. Nevertheless the formal analogy of the master equation with the Schrödinger equation has been useful for studying stochastic processes.

Because the number of particles is conserved, one can also specify the configuration of the system by giving positions of the particles. One can construct an $n$ particle state with positions at $x_1, \dots, x_n$ by

$$|x_1, \dots, x_n\rangle = s^-_{x_1} \cdots s^-_{x_n} |0\rangle. \tag{2.11}$$

Formally one can consider the ASEP on $\mathbb{Z}$ by taking $L \to \infty$ limit in the above. For instance the transition rate matrix would be given by

$$H = \sum_{j \in \mathbb{Z}} h_{j,j+1} \tag{2.12}$$

with (2.10). But there is a difficulty in considering the dynamics of infinitely many particles at the same time. For ASEP on $\mathbb{Z}$, the time evolution of ASEP is better described using the backward Kolmogorov's equation. Instead of considering the evolution of the probability of a configuration, one looks at the evolution of a function $f(\eta)$ which only depends on the finite number of $\eta_x$'s. The generator $L$ for the time evolution of ASEP is given by

$$Lf(\eta) = \sum_{x \in \mathbb{Z}} \left( p\eta_x(1-\eta_{x+1}) + q(1-\eta_x)\eta_{x+1} \right) \left( f(\eta^{x,x+1}) - f(\eta) \right). \tag{2.13}$$

Here

$$\eta^{xy}(u) = \begin{cases} \eta(y), & \text{if } u = x, \\ \eta(x), & \text{if } u = y, \\ \eta(u), & \text{if } u \neq x, y. \end{cases} \tag{2.14}$$

For more details see [9].

## § 3. Transition probability

A basic quantity in the analysis of stochastic process is the transition probability. For the case of ASEP it is the probability that $n$ particles starting from $y_1, \dots, y_n$ at time 0 are on sites $x_1, \dots, x_n$ at time $t$. Using the formalism in the last section, it is written as

$$G(x_1, \dots, x_n; t | y_1, \dots, y_n; 0) = \langle x_1, \dots, x_n | e^{-tH} | y_1, \dots, y_n \rangle \tag{3.1}$$

with $\langle x_1, \dots, x_n| = \langle 0| s^+_{x_1} \cdots s^+_{x_n}$. We sometimes abbreviate this as $G(\{x\}_n; t|\{y\}_n; 0)$ with the understanding $\{x\}_n = (x_1, \dots, x_n)$. We often omit the depencence on $y$ as well. One could also consider the transition probability for particle occupation $\eta$, but it is often more useful to focus on the dynamics of the particles. This is true in particular for case with reflective boundaries or the infinite case because the number of particles is conserved. For open boundary with particle input and output at the boundaries, this is less useful because then one has to consider when each particle enters (or exit) the system.

For stochastic interacting particle systems, it is in general difficult to obtain an expression for the transition probability. But for the ASEP one can find useful expressions. As we will see shortly, for the TASEP on $\mathbb{Z}$, the transition probability can be written as a single determinant. For ASEP with general $p$ and $q$, the formula using the Bethe ansatz contains a summation over permulations and is more involved than for TASEP. But we can still utilize it to study current fluctuations. For the ASEP on a finite lattice, one can still apply the Bethe ansatz. But this time one has to solve the Bethe ansatz equation and the study of scaling behaviors have not been achieved.

Let us write down the master equation for the ASEP on $\mathbb{Z}$. We start from a simple case of one particle ($N = 1$) for which it reads

$$\frac{d}{dt} G(x; t) = pG(x-1; t) + qG(x+1; t) - (p+q)G(x; t). \tag{3.2}$$

This is the master equation for a single asymmetric random walk. It can be solved using Fourier series and the solution is given by

$$G(x; t | y; 0) = e^{-(p+q)t} \left(\frac{q}{p}\right)^{-\frac{x-y}{2}} I_{x-y}\left(2\sqrt{\frac{q}{p}}\right), \tag{3.3}$$

where $I_n(x)$ is the modified Bessel function. Next for $N = 2$, we have to consider two

cases separately. When $x_2 - x_1 \geq 2$, the master equation reads

$$\begin{aligned}\frac{d}{dt}G(x_1,x_2;t) &= pG(x_1-1,x_2;t)+qG(x_1+1,x_2;t)+pG(x_1,x_2-1;t)\\ &\quad +qG(x_1,x_2+1;t)-2(p+q)G(x_1,x_2;t).\end{aligned} \tag{3.4}$$

When $x_2 = x_1 + 1$, due to the exclusion rule, the equation is

$$\frac{d}{dt}G(x_1,x_1+1;t) = pG(x_1-1,x_1+1;t)+qG(x_1,x_1+2;t)-(p+q)G(x_1,x_2;t). \tag{3.5}$$

The initial condition for the transition probability is

$$G(x_1,x_2;t|y_1,y_2;0) = \delta_{x_1y_1}\delta_{x_2y_2}. \tag{3.6}$$

The transition probability is determined as the solution to (3.4),(3.5),(3.6). It is a little cumbersome that one has to deal with the two equations (3.4),(3.5) separately. But the second one can be replaced by a boundary condition for $G(x_1,x_2;t)$. Setting $x_2 = x_1+1$ in (3.4) one gets

$$\begin{aligned}\frac{d}{dt}G(x_1,x_1+1;t) &= pG(x_1-1,x_1+1;t)+qG(x_1+1,x_1+1;t)+pG(x_1,x_1;t)\\ &\quad +qG(x_1,x_1+2;t)-2(p+q)G(x_1,x_1+1;t).\end{aligned} \tag{3.7}$$

Comparing (3.7) with (3.5), we have

$$pG(x_1,x_1,t)+qG(x_1+1,x_1+1;t) = (p+q)G(x_1,x_1+1;t). \tag{3.8}$$

This means that instead of considering (3.4),(3.5) for $x_1 < x_2$, one can consider (3.4) with the boundary condition (3.8) for $x_1 \leq x_2$ and focus on the case $x_1 < x_2$.

For general $N$ the situation is similar to the $N = 2$ case. The main master equation reads

$$\begin{aligned}\frac{d}{dt}G(x_1,\ldots,x_N;t) &= \sum_{i=1}^{N}\Big(pG(\ldots,x_i-1,\ldots;t)+qG(\ldots,x_i+1,\ldots,t)\\ &\quad -(p+q)G(\ldots,x_i,\ldots;t)\Big).\end{aligned} \tag{3.9}$$

One has to solve this with the boundary condition

$$pG(\ldots,x_i,x_i,\ldots;t)+qG(\ldots,x_i+1,x_i+1,\ldots;t) = (p+q)G(\ldots,x_i,x_{i+1},\ldots;t) \tag{3.10}$$

and the initial condition,

$$G(x_1,\ldots,x_N;t=0) = \prod_{i=1}^{N}\delta_{x_iy_i}. \tag{3.11}$$

## § 4. ASEP and XXZ

In this section we explain the relation between the ASEP and XXZ spin chain. For the moment let us consider the ASEP on $[L]$ with reflective boundary condition. To see the connection it is useful to rewrite the transition rate matrix (2.9) in terms of the spin matrices (2.5). It reads

$$H_{\mathrm{ASEP}} = -\sum_{j=1}^{L-1} \left[ p s_j^+ s_{j+1}^- + q s_j^- s_{j+1}^+ - p n_j (1-n_{j+1}) - q(1-n_j) n_{j+1} \right]. \tag{4.1}$$

Now let us set, with $\tau = p/q$,

$$U = \tau^{-\frac{1}{2}\sum_{j=1}^{L-1}(j-1)n_j}. \tag{4.2}$$

Then one finds

$$-U H_{\mathrm{ASEP}} U^{-1} = H_{\mathrm{XXZ}} \sqrt{pq} \tag{4.3}$$

where

(4.4)

$$H_{\mathrm{XXZ}} = \frac{1}{2}\sum_{j=1}^{L-1} \left[ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \frac{\tau^{1/2}+\tau^{-1/2}}{2}(\sigma_j^z \sigma_{j+1}^z - 1) + \frac{\tau^{1/2}-\tau^{-1/2}}{2}(\sigma_j^z - \sigma_{j+1}^z) \right]$$

is the Hamiltonian of the XXZ spin chain. This means that the ASEP and the XXZ chain are related by the similarity transformation.

One has to pay attention to the boundary conditions. In the above, the ASEP is the one with reflective boundary condition and the XXZ chain is the one with a special value of boundary magnetic fields. It is known that this case corresponds to the existence of the $U_q(sl_2)$ symmetry [13]. For the ASEP one can show there is a duality in ASEP [23], which turned out to be useful for studying the current fluctuations of ASEP [6]. We remark that the ASEP with periodic boundary condition is not equivalent to the XXZ spin chain with periodic boundary condition and that the ASEP with open boundary condition is related to the XXZ chain with off-diagonal boundary terms [5, 22].

By applying the Jordan-Wigner transformation, the Hamiltonian can be written in terms of fermion operators. For the XXZ chain, the Hamiltonian contains quartic term in the fermion operators and in this sense it is not a free fermion.

## § 5. TASEP Case

For TASEP, Schütz found the following formula for the transition probability [23].

$$G(x_1, \ldots, x_N; t | y_1, \ldots, y_N; 0) = \det\left(F_{l-j}(x_l - y_j; t)\right)_{1 \le j,l \le N} \tag{5.1}$$

Here the function $F_n(x,t)$ appearing as a matrix element of the determinant can be written as a contour integral,

$$F_n(x,t) = \frac{1}{2\pi i}\int_{0,1} dz \frac{1}{z^{x+1}} \left(1-1/z\right)^n e^{-(1-z)t} \tag{5.2}$$

where the contour in the integral means the contour enclosing the poles of the integrand at $z = 0, 1$ anticlockwise.

It is not difficult to check the following properties of $F_n(x,t)$:

$$F_{n+1}(x;t) = \sum_{y=x}^{\infty} F_n(y,t), \tag{5.3}$$

$$\frac{d}{dt}F_n(x,t) = F_n(x-1;t) - F_n(x;t). \tag{5.4}$$

Then using these one can show that the determinant in (5.1) satisfies (3.4),(3.8), (3.6), i.e. it gives the transition probability for TASEP.

Notice that the determinant has a "nice" structure with respect to the indices $j, l$. If this were a Slater determinant of the form $\det(a_{l-j})$, it directly shows a free fermionic nature of TASEP and the analysis would have been easy. But the determinant in (5.1) is not really a Slater determinant, so one needs to find a reformulation to further proceed.

In fact the TASEP has some free fermion properties. First it became manifest in [8] where the distribution of the integrated current for the step initial condition can be written as a random matrix integral. Then the above determinantal transition probability was utilized [2,11] to reproduce and generalize the results in [8]. Later it was realized that the above determinant formula has a "hidden" structure of non-colliding walkers, which allowed one to study the current fluctuations for other initial conditions such as the alternating initial conditions [1, 16].

## § 6. ASEP Case

For ASEP, the transition probability is given by [25]

$$G(\{x\}_k, t|\{y\}_k, 0) = \sum_{\sigma\in S_k} \int_{C_r} \cdots \int_{C_r} d\xi_1 \cdots d\xi_k A_\sigma \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\sum_i \epsilon(\xi_i)t} \tag{6.1}$$

where $S_k$ is a set of all permutations of order $k$, $\epsilon(\xi) = p/\xi + q\xi - (p+q)$ and

$$A_\sigma = \mathrm{sgn}\sigma \frac{\prod_{i<j}(p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - (p+q)\xi_{\sigma(i)})}{\prod_{i<j}(p + q\xi_i\xi_j - (p+q)\xi_i)}. \tag{6.2}$$

$C_r$ is a contour enclosing the origin anticlockwise with a radius small enough that all the poles in $A_\sigma$ are not included in $C_r$.

The proof is given in [25]. Here to illustrate how it works, we consider the $N = 2$ case. For $N = 2$, the LHS of (6.1) reads

(6.3)
$$G(x_1, x_2; t|y_1, y_2; 0) = \int_{C_r}\int_{C_r} d\xi_1 d\xi_2 \xi_1^{-y_1-1}\xi_2^{-y_2-1}(\xi_1^{x_1}\xi_2^{x_2} + A_{21}\xi_2^{x_1}\xi_1^{x_2})e^{(\epsilon(\xi_1)+\epsilon(\xi_2))t}$$

with

$$A_{21} = -\frac{p + q\xi_1\xi_2 - (p+q)\xi_2}{p + q\xi_1\xi_2 - (p+q)\xi_1}. \tag{6.4}$$

We check this satisfies the master equation (3.4), the boundary condition (3.8) and the initial condition (3.6). The master equation is easily checked using the expression of $\epsilon(\xi)$. Next we check (3.8). We see

$$\begin{aligned} &pG(x_1, x_1; t) + qG(x_1 + 1, x_1 + 1; t) - (p+q)G(x_1, x_1 + 1; t) \\ &= \int_{C_r}\int_{C_r} d\xi_1 d\xi_2 \xi_1^{x_1-y_1-1}\xi_2^{x_2-y_2-1} e^{(\epsilon(\xi_1)+\epsilon(\xi_2))t} \\ &\quad \times \{p(1 + A_{21}) + q\xi_1\xi_2(1 + A_{21}) - (p+q)(\xi_2 + A_{21}\xi_1)\}. \end{aligned} \tag{6.5}$$

Here the factor in the parenthesis is seen to be zero due to (6.4). Finally we check the initial condition (3.6). Setting $t = 0$ in (6.3), we get

$$G(x_1, x_2, 0) = \int_{C_r}\int_{C_r} d\xi_1 d\xi_2 \xi_1^{-y_1-1}\xi_2^{-y_2-1}\left(\xi_1^{x_1}\xi_2^{x_2} + A_{21}\xi_2^{x_1}\xi_1^{x_2}\right). \tag{6.6}$$

One can easily see that the first term gives $\delta_{x_1y_1}\delta_{x_2y_2}$. So we want to show that the second term is zero, i.e.,

$$\int_{C_r}\int_{C_r} d\xi_1 d\xi_2 \frac{p + q\xi_1\xi_2 - (p+q)\xi_2}{p + q\xi_1\xi_2 - (p+q)\xi_1}\xi_2^{x_1-y_2-1}\xi_1^{x_2-y_1-1} = 0. \tag{6.7}$$

We change the integration variable from $\xi_2$ to $\eta$ by

$$\eta = \xi_1\xi_2. \tag{6.8}$$

The integration is over a circle of radius $r^2$. Then the LHS of (6.7) is

$$\begin{aligned} &\int_{C_r} d\xi_1 \int_{C_{r^2}} \xi_1 d\eta \frac{p + q\eta - (p+q)\eta/\xi_1}{p + q\eta - (p+q)\xi_1}\left(\frac{\eta}{\xi_1}\right)^{x_1-y_2-1}\xi_1^{x_2-y_1-1} \\ &= \int_{C_r} d\xi_1 \int_{C_{r^2}} d\eta \frac{p + q\eta - (p+q)\eta/\xi_1}{p + q\eta - (p+q)\xi_1}\left(\frac{\eta}{\xi_1}\right)^{x_1-y_2-1}\eta^{x_1-y_2-1}\xi_1^{x_2-x_1+y_2-y_1-1} \end{aligned} \tag{6.9}$$

Let us consider the $\xi_1$ integration first. Since $x_2 - x_1 + y_2 - y_1 - 1 \geq 1$, there is no pole at $\xi_1 = 0$. In addition, for small enough $r$, the denominator is bounded away from zero since $|\xi_1| \leq r, |\eta| = r^2$.

It is interesting that the full transition probability (without any approximation) is obtained by choosing the contours appropriately. More surprisingly, though the formula (6.1) looks rather involved, it can be used to study the current fluctuations of the ASEP [25, 26].

## § 7. XXZ Case

Next we consider a similar quantity for the XXZ spin chain, i.e., we define the Green's function of the XXZ chain by (3.1) with $-H$ replaced by $H_{\mathrm{XXZ}}$ (4.4). One can obtain the Green's function for XXZ chain from that for ASEP by using the relation (4.3). Here we repeat the previous arguments for ASEP again for XXZ chain. In terms of (2.5), the Hamilotonian of the XXZ chain is writen as

$$H_{\mathrm{XXZ}} = \sum_{j=1}^{L-1} \left( s_j^+ s_{j+1}^- + s_j^- s_{j+1}^+ - \frac{\tau^{1/2}+\tau^{-1/2}}{2}(n_j + n_{j+1} - 2n_j n_{j+1}) \right.$$
$$\left. + \frac{\tau^{1/2}-\tau^{-1/2}}{2}(n_j - n_{j+1}) \right). \tag{7.1}$$

One sees that the (imaginary time) Schrödinger equation for the XXZ is

(7.2)
$$\frac{d}{dt}G(x_1,\dots,x_N;t) = \sum_{i=1}^{N} (G(\dots,x_i-1,\dots;t) + G(\dots,x_i+1,\dots,t) - 2\Delta G(\dots,x_i,\dots;t))$$

where $\Delta = (\tau^{1/2}+\tau^{-1/2})/2$ and

$$G(\dots,x_i,x_i,\dots;t) + G(\dots,x_i+1,x_i+1,\dots;t) = 2\Delta G(\dots,x_i,x_{i+1},\dots;t). \tag{7.3}$$

The initial condition is

$$G(x_1,\dots,x_N;t=0) = \prod_{i=1}^{N} \delta_{x_i y_i}. \tag{7.4}$$

The Green's function is given by

$$G(\{x\}_k,t|\{y\}_k,0) = \sum_{\sigma\in S_k} \int_{C_r} \cdots \int_{C_r} d\xi_1 \cdots d\xi_k A_\sigma \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\sum_i \varepsilon(\xi_i)t} \tag{7.5}$$

where $S_k$ is a set of all permutations of order $k$, $\varepsilon(\xi) = \xi + \xi^{-1} - 2\Delta$ and

$$A_\sigma = \mathrm{sgn}\sigma \frac{\prod_{i<j}(1+\xi_{\sigma(i)}\xi_{\sigma(j)} - 2\Delta\xi_{\sigma(i)})}{\prod_{i<j}(1+\xi_i\xi_j - 2\Delta\xi_i)}. \tag{7.6}$$

$C_r$ is a contour enclosing the origin anticlockwise with a radius small enough that all the poles in $A_\sigma$ are not included in $C_r$. Just as for the ASEP one can readily check that

(7.5) is the Green's function for the XXZ chain. (But as far as the authors know this fact has not been written.)

Here we constructed the Green's function by directly confirming the Schrödinger equation. But it should be possible to obtain the Green's function by summing up all the excited states of the Hamiltonian. From the point of view of (7.5) this corresponds to changing the radius of the contours and taking the contributions from poles in the $S$-matrix into account appropriately. Here we consider the $N = 2$ case as an illustration. When $N = 2$, (7.5) is

$$\begin{aligned} G(x_1, x_2; t|y_1, y_2; 0) = &\int_{C_r} \frac{dz_1}{2\pi i} \int_{C_r} \frac{dz_2}{2\pi i} \frac{1}{z_1^{y_1+1} z_2^{y_2+1}} \\ &\times \left( z_1^{x_1} z_2^{x_2} - \frac{1 + z_1 z_2 - 2\Delta z_2}{1 + z_1 z_2 - 2\Delta z_1} z_1^{x_2} z_2^{x_1} \right) e^{(\varepsilon(z_1)+\varepsilon(z_2))t}. \end{aligned} \tag{7.7}$$

By considering the pole at $z_1 = 1/(2\Delta - z_2)$, what we get is

$$\begin{aligned} &G(x_1, x_2; t|y_1, y_2; 0) \\ &= \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} e^{-i(k_1 y_1 + k_2 y_2)} \\ &\quad \times \left( e^{ik_1 x_1 + ik_2 x_2} - \frac{1 + e^{i(k_1+k_2)} - 2\Delta e^{ik_1}}{1 + e^{i(k_1+k_2)} - 2\Delta e^{ik_2}} e^{i(k_2 x_1 + k_1 x_2)} \right) e^{(\varepsilon(z_1)+\varepsilon(z_2))t} \\ &\quad + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} du (e^{2v} - 1) e^{iu(x_1+x_2-y_1-y_2)} e^{-v(x_2-x_1+y_1-y_2)} e^{-\varepsilon_u t} \end{aligned} \tag{7.8}$$

where

$$e^v = \frac{\Delta}{\cos u}, \quad \varepsilon_u = 2\left( \Delta - \frac{\cos^2 u}{\Delta} \right). \tag{7.9}$$

It is possible to reproduce this by summing the contributions of the eigenstates of the XXZ chain. Moreover the same construction would work for general $N$ as well.

## § 8. Concluding remarks

In this article we have explained some properties of the transition probabilities of the ASEP. As already mentioned in the main text, the transition probability is now a very useful tool to study time dependent fluctuation properties of the ASEP. For the TASEP, the first derivation of the current fluctuation was through a series of mappings from TASEP to a combinatorics of semi-standard Young tableaux [8, 17]. Later the determinantal transition probability by Schütz has turned out to be very useful for various generalizations. For the ASEP with general parameter values, the current fluctuations were computed based on the transition probability. Recenlty it has

also led to the solution of the KPZ equation, a well-known equation describing surface growth phenomena [4, 18–21]. For the moment it seems rather difficult to generalize to various initial and boundary conditions. It would be useful to understand further the structure behind the solvability of ASEP. For instance for a similar but different directed polymer problem, the relation of the system to quantum Toda lattice has been clarified [12].

Another point in the paper was the relation between the ASEP and the XXZ spin chain. The connection itself has been known for some time already, but the real progress based on this observation seems to be just on the way.

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