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Fully noncommutative discrete Liouville equation

By

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Abstract

A fully noncommutative version of the discrete Liouville equation is suggested, based on a class of representations of mapping class groups of punctured surfaces arising from certain set-theoretical solutions of the Pentagon equation.

§ 1. Introduction

The discrete Liouville equation [2] has the form

$$\chi_{m,n-1}\chi_{m,n+1} = (1+\chi_{m-1,n})(1+\chi_{m+1,n}), \tag{1.1}$$

where the "discrete space-time" is represented by the integer lattice $\mathbb{Z}^2$ and the dynamical field $\chi_{m,n}$ is a strictly positive real function on this lattice. To see that this is a discretized version of the Liouville equation, we take a small positive parameter ϵ as the lattice spacing of the discretized space-time, and consider the combination

$$\phi_\epsilon(x,t) = -\log(\epsilon^2\chi_{m,n})$$

in the limit, where $\epsilon \to 0$, $m, n \to \infty$ in such a way that the products $x = m\epsilon$, and $t = n\epsilon$ are kept fixed. If a solution $\chi_{m,n}$ of the discrete Liouville equation is such that such a limit exists, then the limiting value $\phi_0(x,t)$ solves the Liouville equation

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = -2e^\phi. \tag{1.2}$$

The analytically continued version of it with imaginary time variable $t \to it$ is the equation

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$$\frac{\partial^2\phi}{\partial z\partial\bar{z}} = \frac{1}{2}e^{\phi}, \quad z = x + it, \tag{1.3}$$

which describes surfaces of constant negative curvature. Indeed, if $p\colon \mathbb{H} \to \Sigma$ is a universal covering map for a hyperbolic surface Σ, where $\mathbb{H}$ is the upper half plane with the standard Poincaré metric ds^2, and $\sigma\colon U \to \mathbb{H}$, $U \subset \Sigma$, a local section of p, then, the pull-back metric $\sigma^* ds^2$ in conformal form $e^{\phi}|dz|^2$, z being a local complex coordinate on U, gives a solution ϕ of the Liouville equation (1.3) on U.

In this paper, using the connection of the discrete Liouville equation with the mapping class dynamics in Teichmüller space [1, 5], we describe a fully noncommutative version of the discrete Liouville equation. The construction is based on the combinatorial settings of the quantum Teichmüller theory [3, 4] and the quantum theory of the discrete Liouville equation [2] (see also [5] for a review).

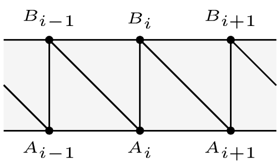
§ 2. Discrete Liouville equation and Teichmüller space

The key instrument in our construction will be the realization of the discrete Liouville equation as a mapping class group dynamics in Teichmüller space. Following the paper [5], we describe this result in the case of an infinite strip.

Consider a strip with marked points on its boundary as pair of topological spaces $S = (\mathbb{R} \times I, \mathbb{Z} \times \partial I)$, where $I = [0, 1]$ is the unit interval in $\mathbb{R}$. Elements of the subset $\mathbb{Z} \times \partial I$ are marked points, and in the sequel they will be denoted as

$$A_k = (k, 0), \quad B_k = (k, 1), \quad k \in \mathbb{Z}.$$

Additionally, we choose the triangulation of S shown in this picture



and we associate real positive variables $\{f_k\}_{k\in\mathbb{Z}}$ with its internal edges such that f_{2i-1} is associated with the edge A_iB_i and f_{2i} with the edge $A_{i+1}B_i$. These variables will be identified with the shear coordinates in the corresponding Teichmüller space of hyperbolic structures in S as follows.

Orientation preserving realizations of the strip S as an ideal geodesic strip in the hyperbolic plane $\mathbb{H}$ bijectively correspond to orientation preserving embeddings of the marked points into the boundary of the hyperbolic plane, $g\colon \mathbb{Z} \times \partial I \to \partial\mathbb{H}$, considered up to overall $PSL_2\mathbb{R}$ transformations. In the upper half-space model of $\mathbb{H}$, we can assume that

$$g(\mathbb{Z} \times \partial I) \subset \mathbb{R} \subset \partial\mathbb{H}, \quad g(A_i) < g(A_{i+1}) < g(B_{j+1}) < g(B_j), \ \forall i, j \in \mathbb{Z},$$

and define

$$f_{2i-1} = [g(B_i), g(A_{i+1}), g(A_i), g(B_{i-1})],$$
$$f_{2i} = [g(B_i), g(B_{i+1}), g(A_{i+1}), g(A_i)], \quad i \in \mathbb{Z},$$

where

$$[z_1, z_2, z_3, z_4] \equiv -(z_1 - z_2)(z_2 - z_3)^{-1}(z_3 - z_4)(z_4 - z_1)^{-1}$$

is a cross-ratio of four numbers.

The mapping class group of S is given by all orientation preserving selfhomeomorphisms preserving the set of marked points, not necessarily point-wise. We are interested in the mapping class $[f]$ which fixes the bottom marked points $\mathbb{Z} \times \{0\}$ point-wise and cyclically permutes the top marked points $\mathbb{Z} \times \{1\}$:

$$A_i \mapsto A_i, \quad B_i \mapsto B_{i+1}, \quad i \in \mathbb{Z}.$$

It is represented by the explicit linear map

$$f\colon \mathbb{R} \times I \to \mathbb{R} \times I, \quad f(x,t) = (x+t, t).$$

The non-quantum version of the result of [1], adapted to the case of our infinite strip, can be stated as the following theorem.

Theorem 2.1 ([1, 5]). *The discrete dynamical system on the Teichmüller space of the strip S, corresponding to the mapping class $[f]$, is described by the discrete Liouville equation (1.1) on the sublattice $m + n = 1 \pmod 2$ with the evolution step being identified with the translation along a "light-cone":*

$$\chi_{m,n} \mapsto \chi'_{m,n} = \chi_{m-1,n+1}.$$

Indeed, under a flip, the shear coordinates transform according to the formulae [3]:

$$a' = a/(1 + 1/e),\ d' = d/(1 + 1/e),\ b' = b(1 + e),\ c' = c(1 + e),\ e' = 1/e, \tag{2.1}$$

where the variables are shown in Figure 1, and all other variables staying unchanged. We remark that this transformation law still applies even if some of the sides of the quadrilateral are a part of the boundary. The only modification is that there is no coordinate associated to a boundary edge, and thus there is nothing to be transformed on this edge.

From Figure 2 and the transformation law (2.1) it follows that the mapping class $[f]$ acts in the Teichmüller space according to the following formulae

$$f_{2j} \mapsto f'_{2j} = 1/f_{2j-1}, \quad f_{2j+1} \mapsto f'_{2j+1} = f_{2j}(1 + f_{2j-1})(1 + f_{2j+1}). \tag{2.2}$$

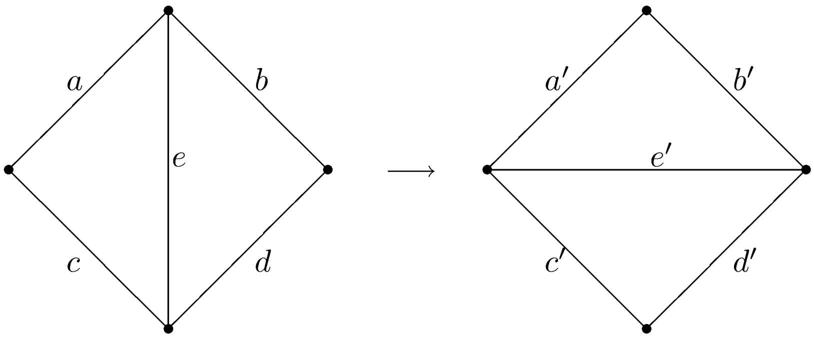


Figure 1. A flip transformation corresponding to equations (2.1).

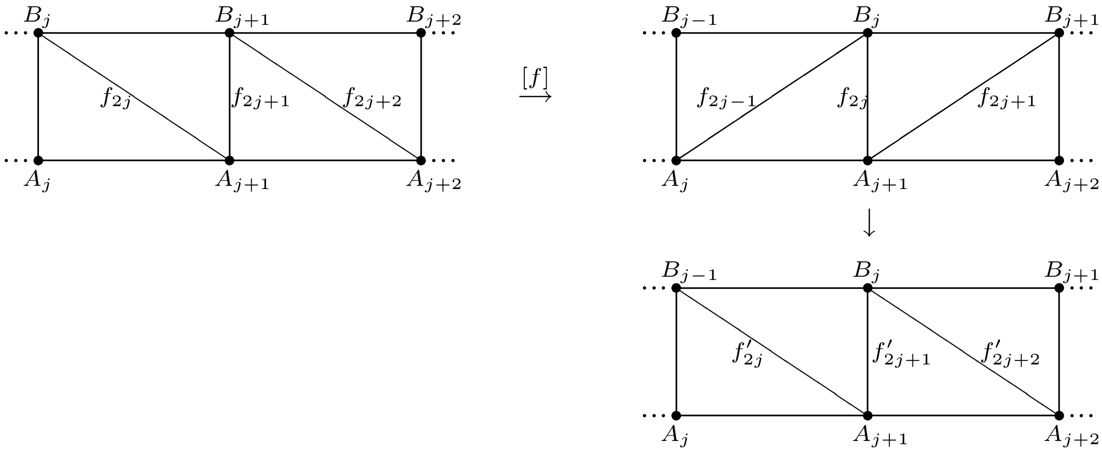


Figure 2. The action of the mapping class $[f]$ on the triangulated strip: it is identical on the bottom boundary and a shift to the right by one spacing on the top boundary.

If we identify the variables $\{f_k\}_{k\in\mathbb{Z}}$ with the initial data for the discrete Liouville equation (1.1) on the sublattice $m+n=1 \pmod 2$ along the zig-zag line $n\in\{-1,0\}$ according to the formulae

$$f_m = \begin{cases} \chi_{m,0} & \text{if } m=1 \pmod 2; \\ 1/\chi_{m,-1} & \text{otherwise,} \end{cases}$$

then, the transformation formulae (2.2) exactly correspond to the light-cone evolution:

$$\chi_{m,n} \mapsto \chi'_{m,n} = \chi_{m-1,n+1}$$

for the time instants $n\in\{-1,0\}$.

§ 3. Mapping class group representations

Let Σ be an oriented surface with a set of punctures P. We assume that it admits ideal triangulations. Fix an index set J of cardinality equal to that of the set of triangles in ideal triangulations of Σ. In particular, for a surface of finite type $\Sigma = \Sigma_{g,s}$ of genus g and s punctures, we have $|J| = 2(2g - 2 + s)$. We will denote by $J!$ the set of all bijections of the set J to itself.

Definition 3.1. A *decorated ideal triangulation* of Σ is an ideal triangulation τ, where all triangles are provided with a marked corner, and a bijective map

$$\bar{\tau}\colon J \ni j \mapsto \bar{\tau}_j \in T(\tau)$$

is fixed. Here $T(\tau)$ is the set of all triangles of τ.

Graphically, the marked corner of a triangle $\bar{\tau}_j$ is indicated by an asterisk and the index j is put inside the triangle. The set of all decorated ideal triangulations of Σ is denoted by Δ_Σ.

§ 3.1. Groupoid of decorated ideal triangulations

Recall that if a group G freely acts on a set X then there is an associated groupoid defined as follows. The objects are the G-orbits in X, while morphisms are G-orbits in $X \times X$ with respect to the diagonal action. Denote by $[x]$ the object represented by the element $x \in X$ and $[x, y]$ the morphism represented by the pair of elements $(x, y) \in X \times X$. Two morphisms $[x, y]$ and $[u, v]$, are composable if and only if $[y] = [u]$ and their composition is $[x, y][u, v] = [x, gv]$, where $g \in G$ is the unique element sending u to y. The inverse and the identity morphisms are given respectively by $[x, y]^{-1} = [y, x]$ and $\mathrm{id}_{[x]} = [x, x]$. In what follows, products of the form $[x_1, x_2][x_2, x_3] \cdots [x_{n-1}, x_n]$ will be written as $[x_1, x_2, x_3, \ldots, x_{n-1}, x_n]$.

Remarking that the mapping class group $\mathcal{M}_\Sigma$ of Σ freely acts on Δ_Σ, denote by $\mathcal{G}_\Sigma$ the corresponding groupoid, called the *groupoid of decorated ideal triangulations.* It admits a presentation with three types of generators and four types of nontrivial relations.

The generators are of the form $[\tau, \tau^\sigma]$, $[\tau, \rho_i\tau]$, and $[\tau, \omega_{ij}\tau]$, where τ^σ is obtained from τ by replacing the ordering map $\bar{\tau}$ by the map $\bar{\tau} \circ \sigma$, where $\sigma \in J!$ is a permutation of the set J, $\rho_i\tau$ is obtained from τ by changing the marked corner of the triangle $\bar{\tau}_i$ as in Figure 3, and $\omega_{ij}\tau$ is obtained from τ by applying the flip transformation in the quadrilateral composed of the triangles $\bar{\tau}_i$ and $\bar{\tau}_j$ as in Figure 4.

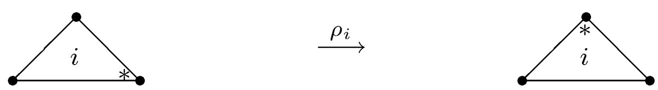


Figure 3. Transformation ρ_i.

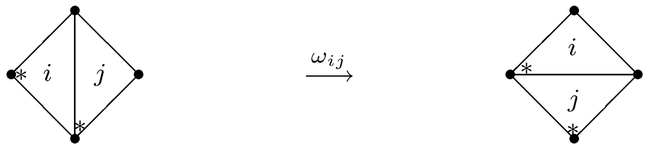


Figure 4. Transformation ω_{ij}.

There are two sets of relations satisfied by these generators. The first set is as follows:

$$[\tau, \tau^\alpha, (\tau^\alpha)^\beta] = [\tau, \tau^{\alpha\beta}], \quad \alpha, \beta \in J!, \tag{3.1}$$

$$[\tau, \rho_i\tau, \rho_i\rho_i\tau, \rho_i\rho_i\rho_i\tau] = \mathrm{id}_{[\tau]}, \tag{3.2}$$

$$[\tau, \omega_{ij}\tau, \omega_{ik}\omega_{ij}\tau, \omega_{jk}\omega_{ik}\omega_{ij}\tau] = [\tau, \omega_{jk}\tau, \omega_{ij}\omega_{jk}\tau], \tag{3.3}$$

$$[\tau, \omega_{ij}\tau, \rho_i\omega_{ij}\tau, \omega_{ji}\rho_i\omega_{ij}\tau] = [\tau, \tau^{(ij)}, \rho_j\tau^{(ij)}, \rho_i\rho_j\tau^{(ij)}]. \tag{3.4}$$

The first two relations are evident, while the other two are shown graphically in Figures 5, 6.

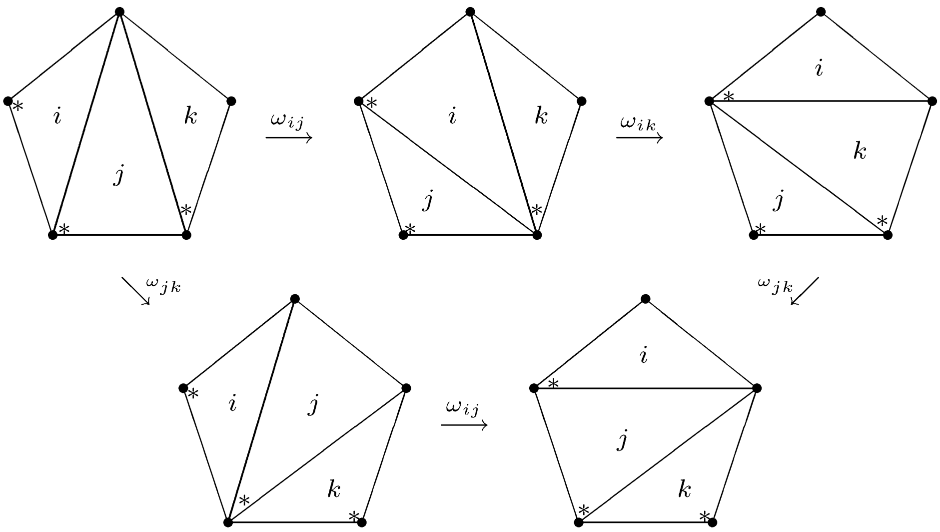


Figure 5. Pentagon relation (3.3).

The following commutation relations fulfill the remaining second set of relations:

$$[\tau, \rho_i\tau, (\rho_i\tau)^\sigma] = [\tau, \tau^\sigma, \rho_{\sigma^{-1}(i)}\tau^\sigma], \tag{3.5}$$

$$[\tau, \omega_{ij}\tau, (\omega_{ij}\tau)^\sigma] = [\tau, \tau^\sigma, \omega_{\sigma^{-1}(i)\sigma^{-1}(i)}\tau^\sigma], \tag{3.6}$$

$$[\tau, \rho_j\tau, \rho_i\rho_j\tau] = [\tau, \rho_i\tau, \rho_j\rho_i\tau], \tag{3.7}$$

$$[\tau, \rho_i\tau, \omega_{jk}\rho_i\tau] = [\tau, \omega_{jk}\tau, \rho_i\omega_{jk}\tau], \; i \notin \{j, k\}, \tag{3.8}$$

$$[\tau, \omega_{ij}\tau, \omega_{kl}\omega_{ij}\tau] = [\tau, \omega_{kl}\tau, \omega_{ij}\omega_{kl}\tau], \; \{i, j\} \cap \{k, l\} = \emptyset. \tag{3.9}$$

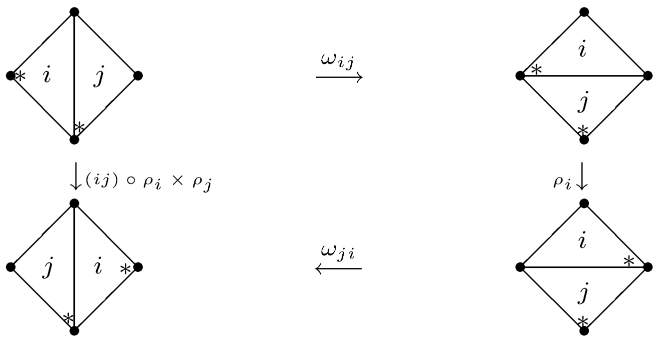


Figure 6. Inversion relation (3.4).

§ 3.2. Semisymmetric T-matrices

Let $\mathcal{C} = (\mathcal{C}, \otimes, s)$ be a symmetric (strict) monoidal category. A *T-matrix* in $\mathcal{C}$ is a pair (V, T), where V is an object of $\mathcal{C}$ and $T \in \operatorname{End}(V \otimes V)$ satisfies the the following *Pentagon* identity in $\operatorname{End}(V^{\otimes 3})$:

$$T_{12}T_{13}T_{23} = T_{23}T_{12}.$$

A *semisymmetric T-matrix* in $\mathcal{C}$, is a triple (V, T, A), where (V, T) is a T-matrix in $\mathcal{C}$ and $A \in \operatorname{End}(V)$ is such that

$$A^3 = \mathrm{id}_V, \quad T(A \otimes \mathrm{id}_V)s_{V,V}T = A \otimes A.$$

In what follows, we will call the element A of a semisymmetric T-matrix the *rotation* operator. The importance of semisymmetric T-matrices comes from the following theorem.

Theorem 3.2. *Let (V, T, A) be a semisymmetric T-matrix. Then there exists a unique homomorphism of groupoids from the groupoid of decorated ideal triangulations $\mathcal{G}_\Sigma$ into the automorphism group $\operatorname{Aut}(V^{\otimes J})$ such that*

$$[\tau, \tau^\sigma] \mapsto P_\sigma, \quad [\tau, \rho_i\tau] \mapsto A_i, \quad [\tau, \omega_{ij}\tau] \mapsto T_{ij}.$$

§ 3.3. Set-theoretical semisymmetric T-matrices

A semisymmetric T-matrix is called *set-theoretical* if the underlying category is the category of sets with the monoidal structure given by the cartesian product. In this case, the map $T\colon V^2 \to V^2$ corresponds to two binary operations $V^2 \to V$

$$(x, y)T = (xy, x * y)$$

satisfying the equations

$$(xy)z = x(yz), \quad x * (yz) = (x * y)((xy) * z), \quad (x * y) * ((xy) * z) = y * z.$$

Here we use the unusual convention that the maps act from right to left. Let us denote also

$$(x)A = \hat{x}, \quad (x)A^2 = \check{x}.$$

A group G is called *group with addition* if it is provided with an associative and commutative binary operation called *addition* with respect to which the group multiplication is distributive.

One can show that no finite group can be a group with addition. The set of positive real numbers $\mathbb{R}_{>0}$ is naturally a group with addition as well as its subgroup of positive rationals $\mathbb{Q}_{>0}$. The group of integers $\mathbb{Z}$ is also a group with addition where the addition is the maximum operation $\max(m, n)$. An example of a non Abelian group with addition is given by the group of upper-triangular real two-by-two matrices with positive reals on the diagonal. The addition here is given by the usual matrix addition.

Proposition 3.3. *Let G be a group with addition and $c \in G$ a central element (for example, the identity element 1). Then there exists a set-theoretical semisymmetric T matrix with the underlying set G^2 and the following structural operations*

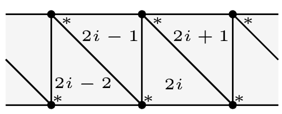
$$xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2),$$
$$x * y = \left((1 + y_2x_2^{-1}x_1)^{-1}y_1, (1 + y_2x_2^{-1}x_1)^{-1}y_2x_2^{-1}\right),$$

$$\hat{x} = \widehat{(x_1, x_2)} = (cx_1^{-1}x_2, x_1^{-1}).$$

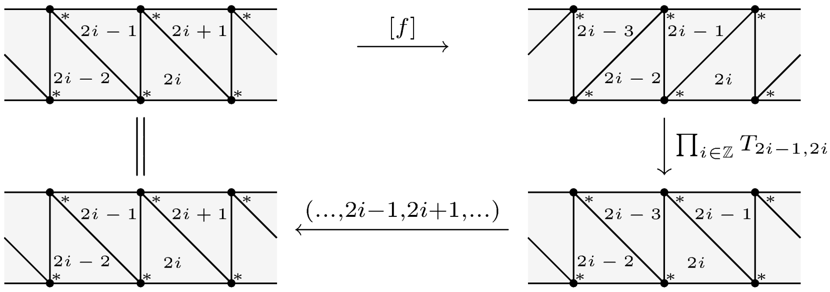
The *ratio coordinates* in the Teichmüller space introduced in [4] correspond to the positive real numbers $\mathbb{R}_{>0}$ considered as a group with addition.

§ 4. Fully noncommutative discrete Liouville equation

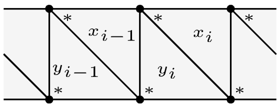
Following Section 2, we apply Theorem 3.2 to realize the discrete dynamical system corresponding to the mapping class $[f]$ of the infinite strip $S = (\mathbb{R} \times I, \mathbb{Z} \times \partial I)$. We choose the following decorated ideal triangulation:



where we use the index set $J = \mathbb{Z}$. Realization of the mapping class $[f]$ through a T-matrix (V, T) is obtained from the following commutative diagram:



Notice that in this case the rotation operator is not used and the dynamical system can be defined for any T-matrix. In particular, for a set-theoretical T-matrix (X, T), we can color our triangulation with values an element g of the the set X^J:



where $x_i = g(2i + 1)$ and $y_i = g(2i)$. Then the associated mapping class dynamics is described by the equations

$$x_{i,t+1} = x_{i-1,t}y_{i,t}, \quad y_{i,t+1} = x_{i-1,t} * y_{i,t}, \quad i, t \in \mathbb{Z}.$$

§ 4.1. Liouville dynamics in groups with addition

Let G be a group with addition. Associated to Proposition 3.3 evolution equation of Liouville type is given by four G-valued fields:

$$x_{i,m,n}, \; y_{i,m,n}, \quad i \in \{1, 2\}, \; m, n \in \mathbb{Z}$$

satisfying four equations

$$x_{1,m,n+1} = x_{1,m-1,n}y_{1,m,n}, \quad y_{2,m,n+1}x_{2,m,n+1} = y_{2,m,n},$$
$$w_{m,n} \equiv y_{2,m,n+1}y^{-1}_{2,m-1,n}y_{2,m-1,n-1}y^{-1}_{2,m,n} = x^{-1}_{1,m-1,n+1}x_{1,m,n+2}x^{-1}_{1,m,n+1}x_{1,m-1,n},$$
$$w_{m,n} + y_{2,m,n+1}x_{1,m-1,n} = 1$$

Suppose our group with addition G is embedded into a ring R. Then, defining a new field

$$\eta_{m,n} = y_{2,m,n}y^{-1}_{2,m-1,n-1},$$

we rewrite our evolution system in the form

$$\chi_{m+1,n+1} = (1 + \chi_{m+1,n})\bar{\chi}^{-1}_{m,n-1}(1 + \bar{\chi}_{m,n})$$

where we use the notation

$$\chi_{m,n} \equiv (\eta_{m,n+1}^{-1}\eta_{m,n} - 1)^{-1}, \quad \bar{\chi}_{m,n} \equiv (\eta_{m,n}\eta_{m,n+1}^{-1} - 1)^{-1}.$$

In the the case of a commutative ring, we obviously have $\bar{\chi}_{m,n} = \chi_{m,n}$ and we recover the discrete Liouville equation (1.1) in a slightly differently parameterized space-time lattice.

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