RIMS Kôkyûroku Bessatsu
B28 (2011), 155–164

$X = K$ under Review

By

Cédric LECOUVEY*, Masato OKADO** and Mark SHIMOZONO***

Abstract

We review the $X = K$ conjecture and important ingredients for the proof. We also attach notes on the rank estimate for the $X = K$ theorem to hold and on the strange relation that was found to be valid without the assumption that the rank is sufficiently large. Using the latter one obtains an algorithm to calculate the image of the combinatorial R-matrix and the value of the coenergy function.

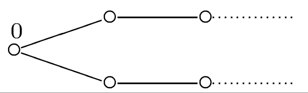
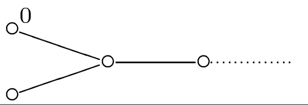
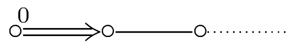
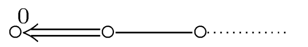
§ 1. Review on $X = K$

Let $\mathfrak{g}$ be an affine algebra of nonexceptional type and $I = \{0, 1, \ldots, n\}$ the index set of its Dynkin nodes. Let $0 \in I$ as specified in [4] and set $I_0 = I \setminus \{0\}$. For a pair (r, s) $(r \in I_0, s \in \mathbb{Z}_{>0})$ there exists a crystal $B^{r,s}$ called the Kirillov-Reshetikhin (KR) crystal [1]. It is a crystal base in the sense of Kashiwara [5] of the Kirillov-Reshetikhin module $W^{r,s}(a)$ for a suitable parameter a [11, 13] over the quantum affine algebra $U'_q(\mathfrak{g})$ without the degree operator q^d. Let B be a tensor product of KR crystals $B = B^{r_1,s_1} \otimes B^{r_2,s_2} \otimes \cdots \otimes B^{r_L,s_L}$, and for a subset J of I set $\mathrm{hw}_J(B) = \{b \in B \mid e_i b = 0 \text{ for any } i \in J\}$ where e_i is the Kashiwara operator acting on B. We call an element of $\mathrm{hw}_J(B)$ J-highest. For an I_0-weight λ we define the 1-dimensional sum $\overline{X}_{\lambda,B}(q)$ by

$$\overline{X}_{\lambda,B}(q) = \sum_{b \in \mathrm{hw}_{I_0}(B), \mathrm{wt}\, b = \lambda} q^{\overline{D}(b)}.$$

Received December 10, 2010; revised and accepted March 26, 2011.
2000 Mathematics Subject Classification(s): 17B37, 05E15, 81R10.
Key Words: *Affine crystal, Kirillov-Reshetikhin crystal, One-dimensional sum.*
Partially supported by ANR-09-JCJC-0102-01 (C.L.), Grants-in-Aid for Scientific Research No. 20540016 (M.O.), NSF DMS-0652641 and DMS-0652648 (M.S.).
*LMPT, Université François Rabelais Tours.
e-mail: cedric.lecouvey@lmpt.univ-tours.fr
**Department of Mathematical Science, Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan.
e-mail: okado@sigmath.es.osaka-u.ac.jp
***Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123 USA.
e-mail: mshimo@math.vt.edu

Here $\overline{D} : B \to \mathbb{Z}$ is the intrinsic coenergy function (see [9, §3.5]). Assume now that $n = |I_0|$ is sufficiently large. (We make an attempt to estimate n such that our main theorem holds.) Then it can be shown that $\overline{X}_{\lambda,B}(q)$ depends only on the attachment of the node 0 to the rest of the Dynkin diagram of $\mathfrak{g}$. In the table below we list all possibilities of the attachment of 0 and enumerate the corresponding nonexceptional affine algebras.

Dynkin	$\mathfrak{g}$	$\diamondsuit$
	$A_n^{(1)}$	$\varnothing$
	$B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
	$C_n^{(1)}$	$\square\square$
	$A_{2n}^{(2)}, D_{n+1}^{(2)}$	$\square$

Hence, we have four kinds of "stable" 1-dimensional sums denoted by $\overline{X}^{\diamondsuit}_{\lambda,B}(q)$ ($\diamondsuit = \varnothing, \begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square, \square$). Then the so-called $X = K$ conjecture proposed by Shimozono and Zabrocki [14, 15] is stated as follows.

Theorem 1.1 ([9]). *For $\diamondsuit \neq \varnothing$,*

$$\overline{X}^{\diamondsuit}_{\lambda,B}(q) = q^{\frac{|B|-|\lambda|}{|\diamondsuit|}} \sum_{\mu\in\mathcal{P}^{\diamondsuit}_{|B|-|\lambda|},\nu\in\mathcal{P}^{\square}_{|B|}} c^{\nu}_{\lambda\mu}\overline{X}^{\varnothing}_{\nu,B}(q^{\frac{2}{|\diamondsuit|}}).$$

Here $|B| = \sum_{j=1}^{L} r_j s_j$, $|\lambda| = \sum_i \lambda_i$ for $\lambda = (\lambda_1, \lambda_2, \ldots)$ where a non-spin weight λ is identified with a partition by the standard way, $\mathcal{P}^{\diamondsuit}_N$ =set of partitions of N tiled by $\diamondsuit$, and $c^{\nu}_{\lambda\mu}$ stands for the Littlewood-Richardson coefficient.

We sketch the proof of this theorem from [9]. Since $\overline{X}^{\diamondsuit}_{\lambda,B}(q)$ depends only on the symbol $\diamondsuit$, we choose an affine algebra $\mathfrak{g}^{\diamondsuit}$ from each kind such that $i \mapsto n - i$ ($i \in I$) gives a Dynkin diagram automorphism. Namely, we set $\mathfrak{g}^{\diamondsuit} = A_n^{(1)}, D_n^{(1)}, C_n^{(1)}, D_{n+1}^{(2)}$ for $\diamondsuit = \varnothing, \begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square, \square$. Let $\diamondsuit = \begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square$ or $\square$ from now on. Then there exists an automorphism σ on the KR crystal $B^{r,s}$ for $\mathfrak{g}^{\diamondsuit}$ satisfying

$$\sigma(e_i b) = e_{n-i}\sigma(b)$$

for any $i \in I, b \in B^{r,s}$. This automorphism σ is extended to B by $\sigma(b) = \sigma(b_1)\otimes\sigma(b_2)\otimes \cdots \otimes \sigma(b_L)$. Then the important facts for the proof are summarized as follows.

(i) σ restricts to the following bijection.

$$\left\{ \begin{matrix} I_0\text{-highest elements} \\ \text{in } B \text{ of wt}\,\lambda \end{matrix} \right\} \xrightarrow{\sigma} \left\{ \begin{matrix} I\setminus\{0,n\}\text{-highest elements} \\ \text{in } \max(B) \text{ of wt}\,\overline{\lambda} \end{matrix} \right\}$$

Here $\max(B) = \bigoplus_\gamma B(\gamma)$, where $B(\gamma)$ is the highest weight $U_q(\mathfrak{g}_{I_0}^{\diamondsuit})$-crystal of highest weight γ and γ runs over all weights with $|\gamma| = |B|$ such that $B(\gamma)$ appears in the restriction of B. Namely, $\max(B)$ is the disjoint union of classical highest weight crystals of maximal highest weights. We remark that $\mathfrak{g}_{I\setminus\{0,n\}}$ is isomorphic to A_{n-1} and set $\overline{\lambda} = (-\lambda_n, \ldots, -\lambda_1)$ if $\lambda = (\lambda_1, \ldots, \lambda_n)$.

(ii) $\overline{D}(b) = \overline{D}(\sigma(b)) + (|B| - |\mathrm{wt}\, b\,|)/|\diamondsuit|$ for $b \in \mathrm{hw}_{I_0}(B)$.

(iii) We have $[V^G(\nu)\downarrow^G_{GL_n} : V^{GL_n}(\bar{\lambda})] = \sum_{\mu\in\mathcal{P}^\diamondsuit} c^\nu_{\lambda\mu}$, where $G = SO_{2n}, Sp_{2n}, SO_{2n+1}$ for $\diamondsuit = \begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\!\square, \square$ and $V^G(\nu)$ stands for the irreducible G-module of non-spin highest weight ν.

(iv) If we represent elements of a KR crystal by Kashiwara-Nakashima tableaux [6], I_0-highest elements in $\max(B)$ contain no barred letters and can therefore be viewed as elements of type A. Under this correspondence we have $\overline{D}^{\diamondsuit}(b) = \frac{2}{|\diamondsuit|}\overline{D}^{\varnothing}(b)$.

Once these properties are established, our theorem can easily be proved as

$$\begin{aligned}
\overline{X}^{\diamondsuit}_{\lambda,B}(q) &= \sum_{b\in\mathrm{hw}_{I_0}(B),\mathrm{wt}\,b=\lambda} q^{\overline{D}(b)} \\
&\overset{(ii)}{=} q^d \sum_b q^{\overline{D}(\sigma(b))} \\
&\overset{(i)(iii)}{=} q^d \sum_{\mu\in\mathcal{P}^\diamondsuit,\nu\in\mathcal{P}} c^\nu_{\lambda\mu} \sum_{\hat{b}\in\mathrm{hw}_{I_0}(\max(B)),\mathrm{wt}\,\hat{b}=\nu} q^{\overline{D}(\hat{b})} \\
&\overset{(iv)}{=} q^d \sum_{\mu,\nu} c^\nu_{\lambda\mu} \overline{X}^{\varnothing}_{\nu,B}(q^{\frac{2}{|\diamondsuit|}})
\end{aligned}$$

where we have set $d = (|B| - |\lambda|)/|\diamondsuit|$.

Example 1.2. Consider the affine algebra $\mathfrak{g} = D_6^{(1)}$ of kind $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ and the following three elements of $B = B^{2,2} \otimes B^{3,1} \otimes B^{1,3}$. They all have weight $\lambda = (211)$. Their images by the automorphism σ are also given.

b	$\sigma(b)$
$\begin{array}{c}2\\1\end{array} \otimes \begin{array}{c}\bar{3}\\3\\1\end{array} \otimes \begin{array}{ccc}1&3&\bar{1}\end{array}$	$\begin{array}{cc}6&\bar{5}\\\bar{6}&\bar{6}\end{array} \otimes \begin{array}{c}\bar{5}\\\bar{6}\\5\end{array} \otimes \begin{array}{ccc}5&\bar{5}&\bar{4}\end{array}$
$\begin{array}{c}2\\1\end{array} \otimes \begin{array}{c}\bar{4}\\4\\3\end{array} \otimes \begin{array}{ccc}1&3&\bar{3}\end{array}$	$\begin{array}{cc}6&\bar{5}\\\bar{6}&\bar{6}\end{array} \otimes \begin{array}{c}\bar{4}\\\bar{5}\\5\end{array} \otimes \begin{array}{ccc}3&\bar{6}&\bar{3}\end{array}$
$\begin{array}{cc}2&2\\1&1\end{array} \otimes \begin{array}{c}\bar{2}\end{array} \otimes \begin{array}{ccc}1&3&\bar{1}\end{array}$	$\begin{array}{cc}\bar{5}&\bar{5}\\\bar{6}&\bar{6}\end{array} \otimes \begin{array}{c}6\\\bar{6}\\5\end{array} \otimes \begin{array}{ccc}5&\bar{5}&\bar{4}\end{array}$

By the property (i) each $\sigma(b)$ should belong to $\max(B)$. Actually, by applying the raising operators e_i $(i \in I_0)$ one finds that these three elements have the common I_0-highest element

$$\hat{b} = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 1 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline \end{array}$$

of weight $\nu = (33211)$. It is also true that they are the all I_0-highest elements in B whose images under σ belong to the same I_0-component as the above one. We can check the property (iii), since we get $c^{\nu}_{\lambda\mu} = 1$ if $\mu = (33)$, $= 2$ if $\mu = (2211)$, $= 0$ if μ are other elements in $\mathcal{P}^{\boxminus}$ and therefore

$$\sum_{\mu \in \mathcal{P}^{\boxminus}} c^{\nu}_{\lambda\mu} = 3.$$

The intrinsic coenergies $\overline{D}(b)$ are all equal and can be calculated using the property (ii) as

$$\overline{D}(b) = \overline{D}(\sigma(b)) + \frac{10-4}{2}.$$

Since $\overline{D}$ is constant on each I_0-component, we have $\overline{D}(\sigma(b)) = \overline{D}(\hat{b})$. The r.h.s. is calculated to be 4 using the knowledge of the type A crystal [14]. Therefore we obtain $\overline{D}(b) = 7$.

Remark. The so-called $X = M$ conjecture [3, 2] claims that the 1-dimensional sum $\overline{X}_{\lambda,B}(q)$ is equal to the fermionic formula $\overline{M}(\lambda, \mathbf{L}; q)$. Hence, when n is sufficiently large, one can expect that $\overline{M}(\lambda, \mathbf{L}; q)$ has a similar formula to Theorem 1.1. This is confirmed in [12]. Combining these results with [7], the $X = M$ conjecture is settled when the affine algebra is of nonexceptional type and its rank is sufficiently large.

§ 2. Rank Estimate

In this section we make an attempt to estimate n such that Theorem 1.1 holds.

Proposition 2.1. *Let ℓ be the length of λ. Then Theorem 1.1 holds if*

$$n > (2\ell + 1) + |B| - |\lambda|.$$

Proof. The obstacle for the theorem to hold lies in the fact that the property (i) is no longer valid when n is not large enough. In [9] this property is stated as Theorem 7.1. In view of the proof there one recognizes that if n is so large that $\sigma(b)$ for $b \in \mathrm{hw}_{I_0}(B)$ is contained in $\max(B)$, then everything is ok. Using row and box splittings in [9, §6] one can also reduce the proof when B is a tensor product of the simplest KR crystal

$B^{1,1}$, that is, $B = (B^{1,1})^{\otimes L}$. Hence, our task is to estimate n such that $\sigma(b)$ belongs to $\max((B^{1,1})^{\otimes L})$ for any $b \in \mathrm{hw}_{I_0}((B^{1,1})^{\otimes L})$ of weight λ.

Recall that an element of $(B^{1,1})^{\otimes L}$ can be regarded as a word of length L from the alphabet

$$\{(\phi,)1, 2, \dots, n, (0,)\overline{n}, \overline{n-1}, \dots, \overline{1}\}.$$

Here letters in parentheses are only for $\diamondsuit = \square$. Let b be a word of length L that is I_0-highest. Then the letters b lie in the set $\{(\phi,)1, 2, \dots, m, \overline{m}, \overline{m-1}, \dots, \overline{1}\}$ for some $m(\geq \ell)$. Let c_z be the number of letters z in b. Then we have $c_j - c_{\overline{j}} = \lambda_j > 0$ for $1 \leq j \leq \ell$ and $c_j = c_{\overline{j}} > 0$ for $\ell < j \leq m$. Since $\sum_{j=1}^{m}(c_j + c_{\overline{j}}) \leq L$, we have $\sum_{j=1}^{m} c_{\overline{j}} \leq \frac{L-|\lambda|}{2}$. Setting $M = \max_{\ell < j \leq m} c_{\overline{j}}$, we get

$$M + (m - \ell - 1) \leq \frac{L - |\lambda|}{2}. \tag{2.1}$$

Next recall the insertion algorithms from [8]. For a word or element of a tensor product of $B^{1,1}$ the insertion algorithm tells us the highest weight of the I_0-component the word belongs to. In our case we wish to apply this algorithm to $\sigma(b)$ to see if the shape of the resulting tableau has L nodes. This is equivalent to say that at each step of insertion of a letter to a column the resulting column remains to be admissible. This in particular means that if letter x and $\overline{x}$ coexist at position p and q in some column of height N, then we have

$$x \geq p + (N + 1 - q). \tag{2.2}$$

Let us obtain the minimal possible unbarred letter X that could appear in the course of insertion algorithms. Note that letters of $\sigma(b)$ lie in $\{n-m+1, n-m+2, \dots, n, (0,)\overline{n}, \overline{n-1}, \dots, \overline{n-m+1}\}$. Since plactic relations of [8] contain $x\overline{x}y \equiv (\overline{x-1})(x-1)y$, a pair $(n-m+1, \overline{n-m+1})$ could create $(n-m+1-M, \overline{n-m+1-M})$. Hence we can set $X = n-m+1-M$. The worst situation that could break (2.2) is that there exist pairs $(X+j-1, \overline{X+j-1})$ for any $1 \leq j \leq M+m$ in the first column during the insertion procedure. The condition for such a column to be admissible is given by

$$n \geq 2(M + m). \tag{2.3}$$

In view of (2.1) we obtain the desired result. □

§ 3. Strange Relation

In this section we show the following proposition and apply it to give an algorithm to obtain the image of the combinatorial R-matrices and the value of the coenergy function $\overline{H}$. As we see in the proof, we do not assume the rank is sufficiently large. So the algorithm can be used for any n. However, we need to restrict our affine algebras to $\mathfrak{g}^{\diamondsuit}$ ($\diamondsuit = \begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square, \square$), since we use the automorphism σ. From the same reason we exclude the KR crystals $B^{n-1,s}$ and $B^{n,s}$ for $\mathfrak{g}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = D_n^{(1)}$ and n is odd.

Proposition 3.1. *Let B be a tensor product of KR crystals. Suppose $b \in \mathrm{hw}_{I_0}(B)$. Then we have*

$$\overline{D}(b) - \overline{D}(\sigma(b)) = \frac{|B| - |\lambda(b)|}{|\diamondsuit|}.$$

Here $\lambda(b)$ stands for the partition corresponding to the weight of b.

This proposition is essentially the same as Theorem 8.1 of [9] except that we do not assume n is sufficiently large. We prepare a lemma. Let us extend the definition of $\lambda(b)$ to an arbitrary element b by $\lambda(b) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_i = (\mathrm{wt}\, b, \epsilon_i)$ and $\{\epsilon_i\}_{1 \le i \le n}$ stands for the standard basis vectors of the weight lattice. We note that $\lambda(b)$ is not necessarily a partition. Some λ_i's may be negative. Hence $|\lambda| = \sum_i \lambda_i$ may also become negative.

Lemma 3.2. *Let B_1, B_2 be single KR crystals. Let $b_1 \otimes b_2$ be an element of $B_1 \otimes B_2$ and suppose it is mapped to $b_2' \otimes b_1'$ by the affine crystal isomorphism. Then we have*

$$\overline{H}(b_1 \otimes b_2) - \overline{H}(\sigma(b_1) \otimes \sigma(b_2)) = \frac{|\lambda(b_2')| - |\lambda(b_2)|}{|\diamondsuit|}. \tag{3.1}$$

Proof. Since $B_1 \otimes B_2$ is connected, it is sufficient to show

(i) if $b_1 = u(B_1), b_2 = u(B_2)$ (see [9, §3.4] for the definition of $u(B_i)$), (3.1) holds, and

(ii) (3.1) with $b_1 \otimes b_2$ replaced by $e_i(b_1 \otimes b_2)$ holds, provided that (3.1) holds and $e_i(b_1 \otimes b_2) \neq 0$.

For (i) recall $b_1' = b_1, b_2' = b_2$ if $b_1 = u(B_1), b_2 = u(B_2)$. Since $u(B_1) \otimes u(B_2)$ can be reached from $\sigma(u(B_1)) \otimes \sigma(u(B_2))$ by applying e_i ($i \neq 0$), we have $\overline{H}(u(B_1) \otimes u(B_2)) = \overline{H}(\sigma(u(B_1)) \otimes \sigma(u(B_2))) = 0$. Hence (i) is verified.

For (ii) recall $|\lambda(e_i b)| - |\lambda(b)| = -|\diamondsuit|\,(i = 0), = |\diamondsuit|\,(i = n), = 0\,(\text{otherwise})$. If $i \neq 0, n$, both sides do not change when we replace $b_1 \otimes b_2$ with $e_i(b_1 \otimes b_2)$. If $i = 0$, the first term of the l.h.s decreases by one in case LL, increases by one in case RR, and does not change in case LR or RL. (For the meaning of LL, etc, see [9, Prop. 3.7].) The second term does not change, while the r.h.s varies in the same way as the first term of the l.h.s. The $i = n$ case is similar. $\square$

Proof of Proposition 3.1. Let $B = B^{r_1,s_1} \otimes \cdots \otimes B^{r_p,s_p}$. We prove by induction on p. When $p = 1$, the proof is the same as in [9, Th. 8.1].

Let $B = B' \otimes B^{r_p,s_p}$ and $b_1 \otimes b_2 \in B' \otimes B^{r_p,s_p}$ is mapped to $b_2' \otimes b_1' \in B^{r_p,s_p} \otimes B'$ by the affine crystal isomorphism. Then $\sigma(b_1) \otimes \sigma(b_2)$ should be mapped to $\sigma(b_2') \otimes \sigma(b_1')$. Using (3.52) of [9] we have

$$\overline{D}(b) = \overline{D}(b_1) + \overline{D}(b_2') + \overline{H}(b_1 \otimes b_2),$$
$$\overline{D}(\sigma(b)) = \overline{D}(\sigma(b_1)) + \overline{D}(\sigma(b_2')) + \overline{H}(\sigma(b_1) \otimes \sigma(b_2)).$$

On the other hand, by the previous lemma and [10, Lemma 5.2] we have

$$\overline{H}(b_1 \otimes b_2) - \overline{H}(\sigma(b_1) \otimes \sigma(b_2)) = \frac{|\lambda(b_2')| - |\lambda(b_2)|}{|\diamondsuit|}.$$

Using the induction hypothesis we obtain

$$\begin{aligned} \overline{D}(b) - \overline{D}(\sigma(b)) &= \frac{|B'| - |\lambda(b_1)|}{|\diamondsuit|} + \frac{|B^{r_p,s_p}| - |\lambda(b_2')|}{|\diamondsuit|} + \frac{|\lambda(b_2')| - |\lambda(b_2)|}{|\diamondsuit|} \\ &= \frac{|B| - |\lambda(b)|}{|\diamondsuit|} \end{aligned}$$

as desired. □

Using Proposition 3.1 we can give an algorithm to obtain the image of the combinatorial R-matrix and the value of the coenergy function $\overline{H}$. This algorithm turns out effective when it is calculated using computer. For the calculation of σ see [9, Appendix B.2]. Let $B_i = B^{r_i,s_i}$ $(i = 1, 2)$ be KR crystals. The affine crystal isomorphism

$$R : B_1 \otimes B_2 \longrightarrow B_2 \otimes B_1,$$

which is known to exist uniquely, is called the combinatorial R-matrix. For an element $b_1 \otimes b_2 \in B_1 \otimes B_2$ we wish to calculate the image $R(b_1 \otimes b_2)$. Since the application of Kashiwara operators e_i, f_i for $i \neq 0$ is not difficult, one can reduce its calculation to I_0-highest elements of $B_1 \otimes B_2$. For an element b in an I_0-component let $High(b)$ stand for the I_0-highest element and set $\Phi = High \circ \sigma$. From Proposition 3.1 and the invariance of $\overline{D}$ by classical Kashiwara operators, one has

$$\overline{D}(\Phi(b_1 \otimes b_2)) = \overline{D}(b_1 \otimes b_2) - \frac{|B_1 \otimes B_2| - |\lambda(b_1 \otimes b_2)|}{|\diamondsuit|}. \tag{3.2}$$

Note that the second term of the above relation vanishes, if and only if $b_1 \otimes b_2 \in \max(B_1 \otimes B_2)$. Since the application of Φ decreases $\overline{D}$ and $\overline{D}$ takes a finite number of values, there exists a positive integer m such that $\Phi^m(b_1 \otimes b_2) \in \max(B_1 \otimes B_2)$. Namely, there exist sequences $\mathbf{a}_1, \ldots, \mathbf{a}_m$ from I_0 and an element $\hat{b}_1 \otimes \hat{b}_2 \in \max(B_1 \otimes B_2)$ such that

$$\hat{b}_1 \otimes \hat{b}_2 = (e_{\mathbf{a}_m} \circ \sigma \circ \cdots \circ e_{\mathbf{a}_1} \circ \sigma)(b_1 \otimes b_2),$$

or equivalently,

$$b_1 \otimes b_2 = (\sigma \circ f_{\mathrm{Rev}(\mathbf{a}_1)} \circ \cdots \circ \sigma \circ f_{\mathrm{Rev}(\mathbf{a}_m)})(\hat{b}_1 \otimes \hat{b}_2).$$

Here for $\mathbf{a} = (i_1, \ldots, i_l)$ $e_{\mathbf{a}}$ stands for $e_{i_1} \cdots e_{i_l}$ ($f_{\mathbf{a}}$ is similar) and $\mathrm{Rev}(\mathbf{a}) = (i_l, \ldots, i_1)$. Since R commutes with $e_{\mathbf{a}}, f_{\mathbf{a}}$ and σ, we have

$$R(b_1 \otimes b_2) = (\sigma \circ f_{\mathrm{Rev}(\mathbf{a}_1)} \circ \cdots \circ \sigma \circ f_{\mathrm{Rev}(\mathbf{a}_m)})R(\hat{b}_1 \otimes \hat{b}_2).$$

On the other hand, for an I_0-highest element in $\max(B_1 \otimes B_2)$ the image of R is easily calculated (see [9, §9.1]). Hence, one can calculate $R(b_1 \otimes b_2)$.

We proceed to the calculation of $\overline{H}(b_1 \otimes b_2)$. Firstly, one has the relation

$$\overline{D}(b_1 \otimes b_2) = \overline{D}(b_1) + \overline{D}(b'_2) + \overline{H}(b_1 \otimes b_2), \tag{3.3}$$

where $R(b_1 \otimes b_2) = b'_2 \otimes b'_1$. The l.h.s is has been obtained in the course of the previous process and the known result of the value of $\overline{D}$ for an element in $\max(B_1 \otimes B_2)$. For I_0-highest elements b_1, b'_2 of a single KR crystal the value of $\overline{D}$ is calculated as

$$\overline{D}(b) = \frac{rs - |\lambda(b)|}{|\diamondsuit|} \quad \text{for } b \in B^{r,s}.$$

Therefore, one obtains $\overline{H}(b_1 \otimes b_2)$.

Example 3.3. Consider the affine algebra $\mathfrak{g} = D_6^{(1)}$ of kind $\boxminus$ and the following element of $B^{4,3} \otimes B^{3,3}$.

$$b_1 \otimes b_2 = \begin{array}{|c|c} \hline 4 & \\ \cline{1-1} 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \otimes \begin{array}{|c|cc} \hline \overline{2} & & \\ \cline{1-1} \overline{3} & & \\ \hline 3 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{\overline{4}} \\ \hline \end{array}$$

Then $\Phi(b_1 \otimes b_2)$ and $\Phi^2(b_1 \otimes b_2)$ are given as follows.

$$\Phi(b_1 \otimes b_2) = \begin{array}{|c|c|c|} \hline 4 & 4 & 4 \\ \hline 3 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 3 & 5 & \overline{6} \\ \hline 2 & 2 & 6 \\ \hline 1 & 1 & 5 \\ \hline \end{array}, \qquad \Phi^2(b_1 \otimes b_2) = \begin{array}{|c|c|c|} \hline 4 & 4 & 4 \\ \hline 3 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 3 & 5 & 6 \\ \hline 2 & 2 & 5 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

with

$\mathbf{a}_1 = (643546432154326432154326435643215432643546432154326435466432154326435 46),$

$\mathbf{a}_2 = (66456435464325436643215432646432154326435643215432643546432154326435466$
$43215432643546643215432643546).$

Since one knows the image of R of $\Phi^2(b_1 \otimes b_2)$ is given by

$$\begin{array}{|c|c|c|}\hline 3&3&3\\ \hline 2&2&2\\ \hline 1&1&1\\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 4&5&6\\ \hline 3&4&5\\ \hline 2&2&4\\ \hline 1&1&1\\ \hline \end{array},$$

one obtains

$$R(b_1 \otimes b_2) = b_2' \otimes b_1' = \begin{array}{|c|c|c}\cline{1-2} 3&3&\\ \cline{1-2} 2&2&\\ \hline 1&1&\multicolumn{1}{c|}{1}\\ \hline \end{array} \otimes \begin{array}{|c|c}\cline{1-1} \overline{2}&\\ \cline{1-1} \overline{3}&\\ \hline \overline{4}&\multicolumn{1}{c|}{\overline{1}}\\ \hline 4&\multicolumn{1}{c|}{4}\\ \hline \end{array}.$$

We proceed to the calculation of $\overline{H}(b_1 \otimes b_2)$. By using (3.2) twice and $\overline{D}(\Phi^2(b_1 \otimes b_2)) = 3$ one gets $\overline{D}(b_1 \otimes b_2) = 12$. Since $\overline{D}(b_1) = 3, \overline{D}(b_2') = 1$, we obtain $\overline{H}(b_1 \otimes b_2) = 8$ from (3.3).

References

[1] G. Fourier, M. Okado and A. Schilling, *Kirillov-Reshetikhin crystals for nonexceptional types*, Adv. in Math. **222** (2009) 1080–1116.

[2] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Z. Tsuboi, *Paths, crystals and fermionic formulae*, MathPhys Odyssey 2001, 205–272, Prog. Math. Phys. **23**, Birkhäuser Boston, Boston, MA, 2002.

[3] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, *Remarks on fermionic formula*, Contemporary Math. **248** (1999) 243–291.

[4] V. G. Kac, *"Infinite Dimensional Lie Algebras,"* 3rd ed., Cambridge Univ. Press, Cambridge, UK, 1990.

[5] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991) 465–516.

[6] M. Kashiwara and T. Nakashima, *Crystal graphs for representations of the q-analogue of classical Lie algebras*, J. Algebra **165** (1994), no. 2, 295–345.

[7] A. N. Kirillov, A. Schilling and M. Shimozono, *A bijection between Littlewood-Richardson tableaux and rigged configurations*, Selecta Math. (N.S.) **8** (2002) 67–135.

[8] C. Lecouvey, *Schensted-type correspondence, plactic monoid and jeu de taquin for type C_n*, J. Algebra **247** (2002) 295–331; *Schensted-type correspondences and plactic monoids for types B_n and D_n*, J. Algebraic Combin. **18** (2003) 99–133.

[9] C. Lecouvey, M. Okado and M. Shimozono, *Affine crystals, one-dimensional sums and parabolic Lusztig q-analogues*, arXiv:1002.3715.

[10] M. Okado, *$X = M$ conjecture*, MSJ Memoirs **17** (2007) 43–73.

[11] M. Okado, *Existence of crystal bases for Kirillov-Reshetikhin modules of type D*, Publ. RIMS **43** (2007) 977–1004.

[12] M. Okado and R. Sakamoto, *Stable rigged configurations for quantum affine algebras of nonexceptional types*, arXiv:1008.0460.

[13] M. Okado and A. Schilling, *Existence of Kirillov-Reshetikhin crystals for nonexceptional types*, Representation Theory **12** (2008) 186–207.

[14] M. Shimozono, *On the $X = M = K$ conjecture*, arXiv:math.CO/0501353.

[15] M. Shimozono and M. Zabrocki, *Deformed universal characters for classical and affine algebras*, J. of Algebra **299** (2006) 33–61.