RIMS Kôkyûroku Bessatsu B3
(2007), 059–082

On stable patterns for reaction-diffusion equations and systems

By

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Abstract

In this article, we survey results on the shape of stable steady states of reaction-diffusion equations and systems. Specifically, we survey results on instability criteria which can be determined by the shape of steady states. In particular, instability criteria of steady states to shadow reaction-diffusion systems of activator-inhibitor type are investigated. In the appendix, we explain a method analyzing an eigenvalue problem related to the stability of steady states to shadow systems of activator-inhibitor type.

§ 1. Introduction

In this article, we survey instability criteria of steady states to reaction-diffusion equations and systems with the Neumann boundary condition in homogeneous media which can be determined by the shape of steady states. In particular, we study instability criteria for steady states to shadow systems of activator-inhibitor type.

All the mathematical results in the article are already known except slight improvements. However, investigating the technique that has been used before seems to be useful to develop new technique.

This article consists of three sections. Section 1 has two subsections. In Subsection 1.1, we explain the motivation: Why do we consider instability criteria? We state known results for scalar equations. In Subsection 1.2, we state known results for systems with a special structure. Section 2 has three subsections. In Subsection 2.1, we state assumptions on the non-linear terms. In Subsection 2.2, we state abstract instability criteria of steady states of shadow systems. According to the abstract instability criteria, if a steady state of shadow systems of activator-inhibitor type is stable, then the

2000 Mathematics Subject Classification(s): Primary 35B35, 35K57; Secondary 35J60, 35P15.
Key Words: Instability criteria, activator-inhibitor system, second eigenvalue.
Supported by JSPS research fellowship for young scientists.
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	Scalar equations	(Shadow) systems
1D intervals	[Ch75]	[N94, NPY01, FR01]
N dimensional domains ($N \geq 2$)	[CH78, Ma79]	[KW85, JM94, Lo96] [Y02a, Mi06b, Mi06c]

Table 1. Instability results in the case of convex domains.

Morse index of a solution of the first equation should be 1. In Subsection 2.3, we give several necessary conditions that solutions to elliptic equations have Morse index 1. We explain a conjecture (Conjecture 2.10) in Subsection 2.3 which is related to the shape of stable patterns. Section 3 is an appendix. In the appendix, we explain a method analyzing the spectrum of a linear operator.

§ 1.1. Motivation and instability criteria for scalar equations

One of the main concerns in the non-linear analysis is pattern formation. Finding all the stable patterns is a way to understand patterns. We study reaction-diffusion equations and systems in this article. If the domain is a one-dimensional interval, then there are cases such that all the steady states can be found, since we can use phase plane (or space) arguments. However, in the case of high dimensional domains, it is difficult in general to find all the steady states. Hence we change the setting of our problem. If the steady state is stable, then what shape is it? In other words, we want to know the shape of all the stable steady states.

Our strategy is the following: We find necessary conditions for steady states to be stable. In order to obtain the necessary condition, we find sufficient conditions for steady states to be unstable. Then the contrapositive of the sufficient condition becomes a desired necessary condition. Therefore we want sufficient conditions that capture many unstable steady states, because the contrapositive narrows the candidates of the stable steady states.

We divide the problem into four cases as described in Table 1. The reason for treating independently the case of a one-dimensional interval is because we can use phase plane (or space) arguments and the Sturm-Liouville theory.

1.1.1. Scalar equations in intervals N. Chafee [Ch75, Theorem 6.2] has proven the following under some technical condition:

Theorem 1.1 (RD equations in intervals). *Let I be an interval. All the non-constant steady states to the problem*

$$u_t = Du_{xx} + f(u) \ \text{ in } \ I, \qquad u_x = 0 \ \text{ on } \ \partial I \tag{1.1}$$

are unstable. Thus the contrapositive is the following: If a steady state is stable, then it is constant.

Note that we do not impose assumption on f except the regularity.

Fortunately, this instability criterion captures all the non-constant steady states. Hence only constant functions can be stable.

Proof. Let u be a non-constant steady state of (1.1). Let λ_1 denote the first eigenvalue of L with the Neumann boundary condition, where $L := D\Delta + f'(u)$. We define $\mathcal{H}[\cdot]$ by

$$\mathcal{H}[\psi] := \int_I \{-D(\psi_x)^2 + f'(u)\psi^2\}\, dx. \tag{1.2}$$

We have

$$\begin{aligned}\mathcal{H}[u_x] &= \int_I \{-D(u_{xx})^2 + f'(u)u_x^2\}\, dx \\ &= \int_I u_x\,(Du_{xxx} + f'(u)u_x)\, dx - [Du_x u_{xx}]_0^1 = 0,\end{aligned}$$

because $Du_{xxx} + f'(u)u_x = 0$. We have

$$\lambda_1 = \sup_{\psi\in H^1} \frac{\mathcal{H}[\psi]}{\|\psi\|_2^2} \geq \frac{\mathcal{H}[u_x]}{\|u_x\|_2^2} = 0,$$

where $\|\cdot\|_2$ denotes the usual L^2-norm. We show that $\lambda_1 > 0$. Suppose the contrary, i.e., $\lambda_1 = 0$. Then u_x is an eigenfunction corresponding to λ_1, and the boundary condition is satisfied: $u_{xx} = 0$ at $x = 0,\ 1$. Since $u_{xxx} + f(u)u_x = 0$ in I, u_x is constant, where we use the uniqueness of ODEs. Since u satisfies the Neumann boundary condition, u is also constant. We obtain a contradiction. □

1.1.2. Scalar equations in multi-dimensional convex domains R. Casten and C. Holland [CH78] and H. Matano [Ma79] independently have shown the same type instability criterion as Theorem 1.1 in the case of high dimensional bounded convex domains.

Theorem 1.2 (RD equations in convex domins of $\mathbb{R}^N$). *Let $\Omega \subset \mathbb{R}^N$ be a bounded and convex domain with smooth boundary. Then all the non-constant steady states to the problem*

$$u_t = D\Delta u + f(u) \ \text{ in } \Omega, \qquad \partial_\nu u = 0 \ \text{ on } \partial\Omega$$

are unstable. The contrapositive is the following: If a steady state is stable, then it is constant.

The proofs of [CH78] and [Ma79] are essentially the same. They use the fact that

$$\partial_\nu |\nabla u|^2 \leq 0 \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

provided that Ω is convex. See [CH78] for the proof of (1.3).

Proof. We use essentially the same method as one used in the proof of Theorem 1.1. First, we have

$$\begin{aligned}\mathcal{H}\left[u_{x_j}\right] &= \int_\Omega \left(-D\left|\nabla u_{x_j}\right|^2 + f'(u)u_{x_j}^2\right) dx \\ &= \int_\Omega u_{x_j}\left(D\Delta u_{x_j} + f'(u)u_{x_j}\right) dx - D\int_{\partial\Omega} u_{x_j}\partial_\nu u_{x_j} d\sigma.\end{aligned}$$

Since $D\Delta u_{x_j} + f'(u)u_{x_j} = 0$, we have

$$\sum_{j=1}^N \mathcal{H}\left[u_{x_j}\right] = -\frac{D}{2}\int_{\partial\Omega} \partial_\nu |\nabla u|^2 d\sigma. \tag{1.4}$$

We use (1.3). Since the right-hand side of (1.4) is not negative, there is $k \in \{1, 2, \ldots, N\}$ such that $\mathcal{H}[u_{x_k}] \geq 0$. Let λ_1 denote the first eigenvalue of $L := D\Delta + f'(u)$ with the Neumann boundary condition. Using a variational characterization of λ_1, we have

$$\lambda_1 = \sup_{\psi \in H^1} \frac{\mathcal{H}[\psi]}{\|\psi\|_2^2} \geq \frac{\mathcal{H}[u_{x_k}]}{\|u_{x_k}\|_2^2} \geq 0.$$

We show that $\lambda_1 > 0$. Suppose the contrary, i.e., $\lambda_1 = 0$. The function u_{x_k} attains $\sup_{\psi \in H^1} \mathcal{H}[\psi]/\|\psi\|_2$. Any function that attains the supremum does not vanish in $\overline{\Omega}$ (see [KW75, p. 570]). However, there is a point on $\partial\Omega$ such that u_{x_k} vanishes, because Ω is convex. This is a contradiction. □

In the proof, the positiveness of

$$-\int_{\partial\Omega} u_{x_j}\partial_\nu u_{x_j} d\sigma$$

is a key. Therefore the analysis of u on the boundary is important for proving that the Morse index is larger than 1.

§ 1.2. Instability criteria for systems with a special structure

We consider a reaction-diffusion system

$$\text{(FS)} \quad \begin{aligned} u_t &= D_u\Delta u + f(u,v) \;\text{ in }\; \Omega, & \tau v_t &= D_v\Delta v + g(u,v) \;\text{ in }\; \Omega, \\ \partial_\nu u &= 0 \;\text{ on }\; \partial\Omega, & \partial_\nu v &= 0 \;\text{ on }\; \partial\Omega, \end{aligned}$$

where $u = u(x,t)$, $v = v(x,t)$. We also consider the shadow limit ($D_v \to +\infty$)

$$\text{(SS)} \qquad u_t = D_u \Delta u + f(u,\xi) \ \text{ in } \ \Omega, \qquad \tau\xi_t = \frac{1}{|\Omega|}\int_\Omega g(u,\xi)dx \ \text{ in } \ \Omega, \qquad \partial_\nu u = 0 \ \text{ on } \ \partial\Omega,$$

where $u = u(x,t)$, $\xi = \xi(t)$. We call (SS) the shadow system of (FS), following [N82]. Not only the shapes of steady states of (FS) and (SS) but also the dynamics of those are close to each other [Mi06a].

The stability properties of steady states to reaction-diffusion systems (or shadow systems) are different from those of scalar equations. In order to see this, we consider a specific system called the Gierer-Meinhardt system [GM72]

$$\text{(GM)} \qquad \begin{aligned} u_t &= D_u \Delta u - u + \frac{u^p}{v^q} \ \text{ in } \ \Omega, & \tau v_t &= D_v \Delta v - v + \frac{u^r}{v^s} \ \text{ in } \ \Omega, \\ \partial_\nu u &= 0 \ \text{ on } \ \partial\Omega, & \partial_\nu v &= 0 \ \text{ on } \ \partial\Omega, \end{aligned}$$

and its shadow system

$$\text{(SGM)} \qquad u_t = D_u \Delta u - u + \frac{u^p}{v^q} \ \text{ in } \ \Omega, \qquad \tau\xi_t = -\xi + \frac{1}{|\Omega|\,\xi^s}\int_\Omega u^r dx \ \text{ in } \ \Omega, \qquad \partial_\nu u = 0 \ \text{ on } \ \partial\Omega.$$

When D_u and τ are small and D_v is large, the system (GM) has an inhomogeneous stable steady state even if the domain is convex (For the existence, see [W97]. For the stability, see [DK02] for 1D intervals, [NTY01] for 2D balls and [Mi05] for general domains. See [Li01, LT01, NT91, NT93] for the shape of steady states). The system (SGM) also has an inhomogeneous stable steady state under the same assumptions except that D_v is large. These inhomogeneous stable steady states are called *a boundary one-spike layer*. From the existence of stable inhomogeneous steady states in convex domains, several questions naturally arise.

Problem 1.3. (i) There is a reaction-diffusion system having inhomogeneous stable steady states. Therefore in order to obtain instability criteria for all the inhomogeneous steady states, we have to restrict the class of non-linear terms. Under what conditions on the nonlinearity can we obtain this type of instability criteria?
(ii) Under what conditions on the nonlinearity does the system have inhomogeneous stable steady states? In that case, what shape is an inhomogeneous stable steady state?
(iii) In the case of scalar equations, the stability property does not change with respect to the time constant τ. However, in the case of systems, the stability property may change. Clarify the relation between the stability and the time constant τ of the second (or first) equation.

Hereafter in this section, we state known results in the research direction of Problem 1.3 (i).

S. Jimbo and Y. Morita [JM94] have obtained an instability result. They consider the gradient system. See also [Lo96].

Theorem 1.4 (Gradient systems). *Let $\Omega \subset \mathbb{R}^N$ be a bounded and convex domain with smooth boundary. Let $u := (u^{(1)}, u^{(2)}, \dots, u^{(k)})$ be a steady state to*

$$u_t^{(j)} = D\Delta u^{(j)} + \frac{\partial F}{\partial u^{(j)}}(u^{(1)}, u^{(2)}, \dots, u^{(k)}) \ \text{ in } \ \Omega \quad \text{for } j = 1, 2, \dots, k,$$
$$\partial_\nu u^{(j)} = 0 \ \text{ on } \ \partial\Omega \quad \text{for } j = 1, 2, \dots, k.$$

If u is not constant, then u is unstable.

Note that the time constant τ is fixed to 1 in Theorem 1.4 and that all the diffusion coefficients are equal.

Proof. We omit the proof. See [JM94]. □

E. Yanagida [Y02a] constructs a general theory of skew-gradient systems. In particular, the stability of steady states is investigated. He considers $2n$-components systems. However, we treat 2-components systems for simplicity. One of the main results is the following instability criterion:

Theorem 1.5 (Skew-gradient systems). *Let $\Omega \subset \mathbb{R}^N$ be a bounded and convex domain with smooth boundary. Let (u, v) be a steady state to*

$$\begin{aligned} u_t &= D_u\Delta u + \frac{\partial F}{\partial u}(u, v) \ \text{ in } \ \Omega, & \tau v_t &= D_v\Delta v - \frac{\partial F}{\partial v}(u, v) \ \text{ in } \ \Omega, \\ \partial_\nu u &= 0 \ \text{ on } \ \partial\Omega, & \partial_\nu v &= 0 \ \text{ on } \ \partial\Omega. \end{aligned} \tag{1.5}$$

If (u, v) is not constant, then (u, v) is unstable for large $\tau > 0$. A similar statement holds in the case of the shadow system with skew-gradient structure.

Proof. We omit the proof. See [Y02a]. □

Several instability criteria are known for steady states of skew-gradient systems. See [Y02b, KY03, K05].

The skew-gradient system includes a special case of the Gierer-Meinhardt system and a reaction-diffusion system with FitzHugh-Nagumo type nonlinearity. Therefore, Theorem 1.5 seems to contradict the existence of the stable boundary one-spike layer. However, this theorem holds provided that $\tau > 0$ is large, hence an inhomogeneous stable steady state may exist for small $\tau > 0$.

K. Kishimoto and H. Weinberger [KW85] have obtained an instability criterion for cooperation-diffusion systems

Theorem 1.6 (Coopration-diffusion systems). *Let $\Omega \subset \mathbb{R}^N$ be a bounded and convex domain with smooth boundary. Let $u := (u^{(1)}, u^{(2)}, \dots, u^{(k)})$ be a steady state to*

$$u_t^{(j)} = D_j \Delta u^{(j)} + f_j(u^{(1)}, u^{(2)}, \dots, u^{(k)}) \text{ in } \Omega \quad \text{for } j = 1, 2, \dots, k,$$
$$\partial_\nu u^{(j)} = 0 \text{ on } \partial\Omega \quad \text{for } j = 1, 2, \dots, k.$$

Suppose that

$$\frac{\partial f_j}{\partial u_l} > 0 \text{ for } j \neq l.$$

Then every inhomogeneous steady state u is unstable. Thus if u is stable, then it is constant.

Proof. See [KW85]. We omit the proof. □

¿From the change of variables $u \mapsto u$, $v \mapsto -v$ we immediately obtain a result for a two-species competition-diffusion system.

§ 2. Stable patterns for activator inhibitor systems

§ 2.1. Assumptions on the nonlinearity

We explain an activator-inhibitor system, and state assumption on the non-linear terms f and g in (SS).

We consider (SS). In theoretical biology, u and ξ stand for the concentrations of biochemicals called *the short range activator* and *the long range inhibitor*, respectively. The activator activates the production rate of the inhibitor ($g_u > 0$), and the inhibitor suppresses the production rate of the activator ($f_v < 0$). the production rate of the inhibitor decreases as the inhibitor increases ($g_v < 0$). However, we do not impose a monotonicity assumption of f in u, because the activator may react autocatalytically and f may not be monotone in u. We call (SS) the activator-inhibitor system if f and g satisfy

$$\text{(AI)} \qquad f_\xi < 0, \quad g_u > 0, \quad \text{and} \quad g_\xi < 0.$$

The time constant of the inhibitor τ, which appears in the second equation of (SS), means the ratio of the reaction speeds between the activator and the inhibitor. If τ is large, then the inhibitor reacts slowly in time, and the system behaves like a scalar reaction-diffusion equation. In this case, we can expect and show that, if the domain is convex, then every inhomogeneous steady state is unstable for large $\tau > 0$ (See Corollary 2.4 below). On the contrary, if τ is small, then the inhibitor reacts quickly, and the system tends to be stable. Hence, an inhomogeneous stable steady state can exist. There is a possibility that a steady state that is unstable for large $\tau > 0$ is stable when $\tau > 0$ is small. (A Hopf bifurcation occurs as τ increases. See [NTY01, WW03] for the case of the shadow Gierer-Meinhardt system.) Therefore, it is important to obtain a sufficient condition, which can be determined by the shape, for steady states to be *unstable* not only in the case for large $\tau > 0$ but also in the case for *all* $\tau > 0$, because the contrapositive of the sufficient condition becomes a necessary condition for steady states to be stable for *some* $\tau > 0$. In other words, we know the shape of all the stable steady states.

Hereafter, we assume that f and g satisfy

$$\text{(N)} \qquad f_\xi < 0, \quad g_\xi < 0, \quad \text{there is } k(\xi) < 0 \text{ such that } g_u(u,\xi) = k(\xi) f_\xi(u,\xi).$$

The classes (AI) and (N) include several important systems.

Example 2.1. The Gierer-Meinhardt system [GM72] is (GM), where (p,q,r,s) satisfy $p > 1$, $q > 0$, $r > 0$, $s \geq 0$ and $0 < (p-1)/q < r/(s+1)$. The assumption on (p,q,r,s) comes from a biological reason. (AI) always holds. If $p = r - 1$, then (N) holds. This system is a model describing the head formation of a hydra, which is a small creature. Specifically, [GM72] shows experimentally that the head appears at the point where the activator u attains the local maximum. It is known that this system has steady states having various shapes (see [NT91, NT93, GW00, MM02] for example).

Example 2.2. The shadow system with the FitzHugh-Nagumo type nonlinearity [Fi61, NAY62] is the following:

$$\text{(FHN)} \qquad u_t = D_u \Delta u + f_0(u) - \alpha\xi \quad \text{and} \quad \tau\xi_t = \frac{1}{|\Omega|}\iint_\Omega (\beta u - \gamma\xi)\, dxdy,$$

where α, β and γ are positive constants and $f_0(u)$ is the so-called cubic-like function. A typical example of f_0 is $u(1-u)(u-\delta)$ $(0 < \delta < 1)$. (AI) and (N) hold. When $\tau = 0$, it is known that the full system of (FHN) has an inhomogeneous stable steady state [O03].

§ 2.2. Abstract instability results

In this subsection, we study instability criteria of steady states to the shadow system (SS). The main result in this subsection is the following:

Lemma 2.3 (Abstract instability criteria). *Let (u,ξ) be a steady state of* (SS), *and let μ_2 denote the second eigenvalue of the eigenvalue problem*

$$\text{(2.1)} \qquad D_u \Delta\phi + f_u(u,\xi)\phi = \mu\phi \ \text{ in } \ \Omega, \qquad \partial_\nu \phi = 0 \ \text{ on } \ \partial\Omega.$$

(i) *Assume that* (N) *holds. If* $\mu_2 > 0$, *then* (u, ξ) *is unstable for* $\tau > 0$.
(ii) *Assume that* (AI) *holds. If* $\mu_1 > 0$, *then there is* $\tau_0 > 0$ *such that* (u, ξ) *is unstable for* $\tau > \tau_0$.

(ii) of Lemma 2.3 is slightly improved than Lemma 3.2 (ii) of [Mi06b].

In the decade, the stability of steady states to various shadow systems including Gierer-Meinhardt system has attracted great attention. In studying an eigenvalue problem which is (2.2) below, the main technical difficulty is the eigenvalue analysis of partial differential operators with non-local term (see (2.3) below). In order to overcome the difficulty, authors develop several methods. Some of the methods are closely related. We want to clarify the relation among them. The method analyzing the eigenvalue problems (2.2) used in this subsection is based on [Mi05, Mi06b].

Proof. Let (u, ξ) be a steady state of (SS), and let $\langle \cdot , \cdot \rangle$ denote the usual inner product of L^2. We consider the linearized eigenvalue problem

$$\begin{pmatrix} L & f_\xi \\ \langle g_u, \cdot \rangle & \langle g_\xi, 1 \rangle \end{pmatrix} \begin{pmatrix} \phi \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} \phi \\ \tau|\Omega|\eta \end{pmatrix}, \tag{2.2}$$

where $L := D_u \Delta + f_u$ and $(\phi, \eta) \in H^2_N(\Omega) \times \mathbb{R}$. ¿From the second equality of (2.2) we have

$$\eta(\lambda\tau |\Omega| - \langle g_\xi, 1 \rangle) = \langle g_u, \phi \rangle .$$

Hereafter, we assume that $\lambda \neq \langle g_\xi, 1 \rangle / (\tau|\Omega|)$. Then we have

$$\eta = \frac{\langle g_u, \phi \rangle}{\lambda\tau |\Omega| - \langle g_\xi, 1 \rangle}.$$

Substituting this equality into the first equation of (2.2), we have

$$(L - \lambda)\phi + \frac{\langle g_u, \phi \rangle f_\xi}{\lambda\tau |\Omega| - \langle g_\xi, 1 \rangle} = 0. \tag{2.3}$$

This is an eigenvalue problem with non-local term. Moreover, this is not a standard eigenvalue problem, because λ appears in the second term. This derivation of (2.3) is essentially the same as that in [W99, NTY01]. However, they study only Gierer-Meinhardt system. [W99] studies the case that $\tau = 0$.

We establish instability criteria. It is enough, if we show that (2.3) has an eigenpair (ϕ, λ) such that $\lambda \in \mathbb{R}$ and $\lambda > 0$. We suppose that $\lambda \notin \sigma(L)$, where $\sigma(L)$ denotes the set of the eigenvalues of L with the Neumann boundary condition. Substituting $\phi = (L - \lambda)^{-1} [f_\xi]$ into (2.3), we have

$$\left(1 + \frac{\langle g_u, (L - \lambda)^{-1} [f_\xi] \rangle}{\lambda\tau |\Omega| - \langle g_\xi, 1 \rangle} \right) f_\xi = 0. \tag{2.4}$$

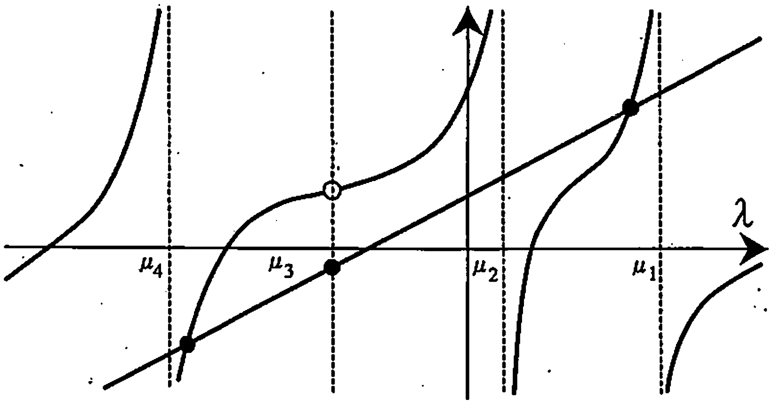

Figure 1. The graphs of the both sides of (2.6) in the case that $k(\xi) < 0$ and $\langle f_\xi, \psi_3 \rangle = 0$.

We want to prove the existence of $\lambda(> 0)$ satisfying (2.4). Let $\{(\psi_n, \mu_n)\}_{n\geq 1}$ denote the set of the eigenpairs of L. Then

$$(L-\lambda)^{-1}[\,\cdot\,] = \sum_{n\geq 1} \frac{\langle\,\cdot\,, \psi_n\rangle}{\mu_n - \lambda}\psi_n. \tag{2.5}$$

Using (2.5), we can write (2.4) as

$$1 + \frac{1}{\lambda\tau|\Omega| - \langle g_\xi, 1\rangle} \sum_{n\geq 1} \frac{\langle f_\xi, \psi_n\rangle \langle g_u, \psi_n\rangle}{\mu_n - \lambda} = 0.$$

Therefore, we will find intersections of the following two functions:

$$\lambda\tau|\Omega| - \langle g_\xi, 1\rangle = \sum_{n\geq 1} \frac{a_n}{\lambda - \mu_n} (=: h(\lambda)), \quad \text{where } a_n = \langle f_\xi, \psi_n\rangle \langle g_u, \psi_n\rangle. \tag{2.6}$$

See Figure 1. We assume that (AI) holds. Then $a_1 \neq 0$, because ψ_1 does not change the sign. Therefore, when τ is large, there is $\lambda_0 > 0$ such that (2.6) is satisfied at $\lambda = \lambda_0$, because $\mu_1 > 0$. Thus we obtain Lemma 2.3 (ii). When $f_\xi = -g_u$, (SS) has the skew-gradient structure. In this case, Lemma 2.3 (i) and (ii) are already obtained by [Y02c] and [Y02a] respectively. When $f_\xi = g_u$, (SS) may have the gradient structure. In this case, [JM94] proves that every non-constant steady state is unstable.

We assume that (N) holds. The case that $k(\xi) = 0$ is trivial. We assume that $k(\xi) \neq 0$. If $\langle g_u, \psi_2\rangle = 0$, then (ψ_2, μ_2) is an eigenpair of (2.3). Thus Lemma 2.3 (i) holds. If $\langle g_u, \psi_2\rangle \neq 0$, then a_1 and a_2 are the same sign, and one of the following holds:

(1) $\lim_{\lambda\to\mu_1-} h(\lambda) = +\infty, \quad \lim_{\lambda\to\mu_2+} h(\lambda) = -\infty, \quad h(\lambda) \in C^0((\mu_2, \mu_1)),$

(2) $\lim_{\lambda\to\mu_1-} h(\lambda) = -\infty, \quad \lim_{\lambda\to\mu_2+} h(\lambda) = +\infty, \quad h(\lambda) \in C^0((\mu_2, \mu_1)).$

Therefore even if which case occurs, (2.6) has a positive root provided that $\mu_2 > 0$. Lemma 2.3 (i) is proven. □

When we want instability criteria, it is enough to show that there is an eigenvalue with a positive real part. On the contrary, when we want the stability of a steady state, we have to prove the non-existence of the spectrum on the right half plane. We consider this problem in the appendix.

The results and the proofs of [Fr94a, Fr94b, Y02c] are easily understood if one see Figure 1.

Corollary 2.4. *Assume that Ω is bounded and convex and that* (AI) *holds. Let (u, ξ) be a steady state of* (SS). *If u is not constant, then there is $\tau_0 > 0$ such that (u, ξ) is unstable for $\tau > \tau_0$.*

Proof. Theorems 1.1 and 1.2 say that $\mu_1 > 0$ if u is not constant. The statement immediately follows from Lemma 2.3 (ii). □

In the statement of Corollary 2.4, the largeness of τ is needed. However, the case that τ is large is trivial in some sense, as stated in an intuitive discussion in Subsection 2.1. We want to know instability criteria in the case that τ is small. We can obtain such criteria, using Lemma 2.3 (i). In order to use Lemma 2.3 (i), we need the sign of μ_2, which is the second eigenvalue of (2.1).

Let us compare (FS) and (SS). If $v(x)$ is fixed, then the mapping $u \mapsto f(u, v)$ does depend on x explicitly. However, in the case of the shadow system, the mapping $u \mapsto f(u, \xi)$ does not depend on x explicitly, and the first equation of (SS) can be treated as a scalar equation in homogeneous media provided that ξ is fixed. This fact makes it easier to know the sign of μ_2.

We consider the second eigenvalue of (2.1). As stated above, (2.1) can be treated as a usual eigenvalue problem of scalar reaction-diffusion equations in homogeneous media. Here u is a solution of an elliptic equation in homogeneous media

$$\Delta u + N(u) = 0 \ \text{ in } \ \Omega, \qquad \partial_\nu u = 0 \ \text{ on } \ \partial\Omega, \tag{2.7}$$

and the second eigenvalue means the second eigenvalue of the eigenvalue problem

$$\Delta \phi + N'(u)\phi = \kappa\phi \ \text{ in } \ \Omega, \qquad \partial_\nu \phi = 0 \ \text{ on } \ \partial\Omega. \tag{2.8}$$

§ 2.3. Analysis of the second eigenvalue

In this subsection, we state instability criteria in the research direction of Problem 1.3 (ii) and (iii). Specifically, we study sufficient conditions that $\kappa_2 > 0$.

2.3.1. Shadow systems in intervals Y. Nishiura [N94] has obtained an instability criterion for steady states to shadow systems in one-dimensional intervals (The same result has been obtained by P. Freitas and C. Rocha [FR01] for (FHN)). This result is generalized by Ni-Polacik-Yanagida [NPY01]. They consider the case that $f = f(u, \xi, t)$ satisfies $f(u, \xi, t+T) = f(u, \xi, t)$ for some $T > 0$.

Theorem 2.5 (Shadow systems in intervals). *Let I be an interval. Let (u, ξ) be a steady state of the problem*

$$u_t = D_u u_{xx} + f(u, \xi) \text{ in } I, \qquad \tau\xi_t = \frac{1}{|I|}\int_I g(u, \xi)dx, \tag{2.9}$$
$$u_x = 0 \text{ on } \partial I.$$

Suppose that u is non-constant, non-monotone increasing, and non-monotone decreasing.
(i) *Suppose that* (N) *holds. Then (u, ξ) is unstable for all $\tau > 0$.*
(ii) *Suppose that* (AI) *holds. Then there is $\tau_0 > 0$ such that (u, ξ) is unstable for $\tau > \tau_0$.*

The contrapositive of Theorem 2.5 (i) is the following:

Corollary 2.6. *Suppose that* (N) *holds. If the steady state (u, ξ) of* (2.9) *is stable for some $\tau > 0$, then u is constant, monotone increasing, or monotone decreasing.*

Proof of Theorem 2.5 (i). Suppose that u is not constant, monotone increasing, or monotone decreasing. Then there is $x_0 \in I\backslash\partial I$ such that $u_x(x_0) = 0$, $u_x(x) \neq 0$ in $(0, x_0)$. Let (ϕ, λ) be an eigenpair of (2.1) such that $\lambda \leq 0$. We consider the case that $u_x(x) > 0$ in $(0, x_0)$. We assume that $\phi > 0$ in $(0, x_0)$. Then we have

$$\begin{aligned} 0 \geq \lambda \int_0^{x_0} u_x \phi dx &= \int_0^{x_0} u_x L\phi dx \\ &= \int_0^{x_0} Lu_x \phi dx + D_u(\phi(0)u_{xx}(0) - \phi(x_0)u_{xx}(x_0)). \end{aligned}$$

Since $Lu_x = 0$, $u_{xx}(0) > 0$ and $u_{xx}(x_0) < 0$, the right-hand side is positive. We obtain a contradiction. Therefore ϕ has at least one zero in $[0, x_0]$. We see that $\phi(0) \neq 0$, using uniqueness of ODEs. Moreover, from the inequality we easily see that $\phi(x_0) \neq 0$. Thus the zeros of ϕ are in $(0, x_0)$. Using the same argument, we can show that ϕ has at least one zero in $(0, x_0)$ in other cases. We see by induction that ϕ has at least one zero in the interior set of any interval of non-zero level set of u_x.

Because of the assumption of the lemma, $\{u_x \neq 0\}$ consists of at least two intervals. Therefore if $\lambda \leq 0$, then the corresponding eigenfunction ϕ has at least two zeros in I. From the Sturm-Liouville theory we see that the second eigenfunction has exactly one zero in I and the second eigenvalue cannot be 0 or negative. We obtain the desirable result from Lemma 2.3. □

In the proof, the positiveness of the second eigenvalue is a key. In order to show this, we use the zero number of u_x (the lap-number of u in the sense of [Ma82]) and eigenfunctions.

2.3.2. Shadow systems in 2D domains In the case of one-dimensional intervals, the number of zeros of u_x plays an important role in determining the Morse index. Analogously, the nodal curves (the zero curves) of u_x, u_y and u_θ play an important role in the case of two-dimensional domains. Note that $\partial_\theta := -y\partial_x + x\partial_y$ and that ∂_x, ∂_y and ∂_θ commute with $\Delta_{(x,y)}$. If the spatial dimension is 3 or larger, the topology of the non-zero level set can be very complicated. However, in the case of $2D$ domains, that is relatively simple (If the spatial dimension is 1, each nodal set should be an interval). Moreover, the Carleman-Hartman-Wintner theory [Ca33, HW53] gives us the information about the nodal curves of u_x, u_y and u_θ. Using these information, we can obtain information about the number of the nodal domain of u_x, u_y and u_θ and prove the positivity of the second eigenvalue if the shape of the domain is not complicated, e.g., a rectangle and a ball. We state several results on the shape of stable steady states. See [Mi06c] for the proofs of the following two theorems:

Theorem 2.7 (Shadow systems in 2D rectangles). *Let Ω be a rectangle R, and let (u, ξ) be a non-constant steady state to* (SS). *Suppose that* (N) *holds. If (u, ξ) is stable for some $\tau > 0$, then either* (i) *or* (ii) *holds.*

(i) *There is a direction which is not parallel to the x-axis and y-axis such that u is strictly monotone with respect to the direction. Moreover u attains its global maximum (minimum) at exactly one point of the corner of R.*

(ii) *u depends only on x or y, and it is strictly monotone in x or y respectively. Therefore u attains its global maximum (minimum) on one side of R.*

Theorem 2.8 (Shadow systems in 2D balls). *Let Ω be a ball B, and let (u, ξ) be a non-constant steady state to* (SS). *Suppose that* (N) *and that*

$$\sup_{(\rho_1,\rho_2)\in\mathbb{R}^2} f_u(\rho_1, \rho_2) < D_u\chi_4, \tag{2.10}$$

where χ_4 is the fourth eigenvalue of the Neumann Laplacian in B. If (u, ξ) is stable for some $\tau > 0$, then B has a diameter PQ such that

(i) *u is symmetric with respect to PQ,*

(ii) *u is strictly monotone in the direction parallel to PQ, i.e., $\partial_\alpha u > 0$ on $B\backslash\{P, Q\}$, where $\partial_\alpha u$ denotes the derivative in the direction,*

(iii) *$u_\theta > 0$ on one side of $B\backslash\overline{PQ}$, $u_\theta < 0$ on the other side, where $\overline{PQ}$ denotes the*

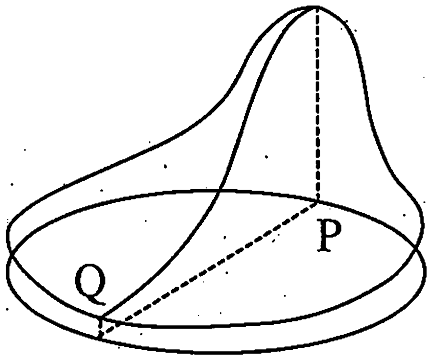

Figure 2. The shape of stable steady states stated in Theorem 2.8 in the case of B

segment whose endpoints are P and Q,
(iv) $u(Q) < u(x,y) < u(P)$ *for* $(x,y) \in B \backslash \{P, Q\}$.

In the case of R and B, the maximum (minimum) of u is attained on the boundary if (u, ξ) is stable for some $\tau > 0$.

In the case of B, the assumption (2.10) seems to be technical. We obtain information about the shape of u on the boundary, even if we do not assume (2.10).

Theorem 2.9 (Shadow systems in 2D balls). *Let Ω be a ball B with radius R, and let (u, ξ) be a non-constant steady state to* (SS). *Suppose that* (N) *holds. If (u, ξ) is stable for some $\tau > 0$, then $\mathcal{Z}[U_\theta(\cdot)] = 2$ or u is constant. Here $U(\theta) := u(R\cos\theta, R\sin\theta)$ and $\mathcal{Z}[\cdot]$ denotes the cardinal number of the zero level set of 2π-periodic functions. Therefore, $\mathcal{Z}[U_\theta(\cdot)]$ is the lap-number of U in the sense of [Ma82].*

See [Mi06b] for details.

Theorems 2.8 and 2.9 suggest that only the steady states whose shape are like a boundary one-spike layer can be stable, even if the diffusion coefficient is not small. See Figure 2.

2.3.3. Conjecture We can expect that a result similar to Theorems 2.7 and 2.8 holds in the case of 2D bounded convex domains. In order to prove that, we have to prove the following:

Conjecture 2.10 ([Y06] Convex domains of $\mathbb{R}^2$). Let Ω be a two-dimensional bounded convex domain with smooth boundary, and let u be a non-constant solution of (2.7). If there is an interior point $(x_0, y_0) \in \mathrm{int}(\Omega)$ such that (x_0, y_0) is a critical point of u, i.e., $u_x(x_0, y_0) = u_y(x_0, y_0) = 0$, then the second eigenvalue of (2.8) is positive.

The contrapositive of Conjecture 2.10 is the following: Non-constant solutions of

Morse index	Linear	Nonlinear
0	The first e.f. of $\Delta_N$is constant.	Theorem 1.2
1	The "hot spots" conjecture.	Conjecture 2.10
$n-1$	The shape of n-th e.f. of Δ_N.	The shape of u.

Table 2. The "hot spots" conjecture of J. Rauch and related results.

	Equation	Domain	Solution
[NT91, NT93]	$\varepsilon^2\Delta u - u + u^p = 0$	any domain (possibly non-convex)	least-energy sol.
Conjecture 2.10	$\Delta u + f(u) = 0$	convex domain	any solution

Table 3. Relations between [NT91, NT93] and Conjecture 2.10.

(2.7) with Morse index 1 do not have critical points in the interior of the domain and attain the maximum (minimum) on the boundary if the domain is convex.

If Conjecture 2.10 holds, then we see that all the stable steady states of (SS) do not have interior spikes or spots in the case of two-dimensional bounded convex domains with smooth boundary.

Conjecture 2.10 is a non-linear version of the "hot spots" conjecture of J. Rauch [R74] ([Y06]). See [BW99, BB99, JN00, B05] for partial answers of the "hot spots" conjecture. The "hot spots" conjecture immediately follows from Conjecture 2.10. See Table 2. There are well-known results obtained by W. M. Ni and I. Takagi [NT91, NT93] which is a sufficient condition for solutions to attain the maximum on the boundary of the domain. Specifically, they have shown that the least-energy-solution of

$$\varepsilon^2\Delta u - u + u^p = 0 \text{ in } \Omega, \qquad \partial_\nu u = 0 \text{ on } \partial\Omega,$$

which is of mountain pass type, is spike-shaped and it attains the maximum at exactly one point on the boundary provided that ε is small. Note that the least-energy-solution has Morse index 1. They also have shown that the peak should be at the point on the boundary where the mean curvature of the boundary attains the global maximum. See Table 3 for relations between [NT91, NT93] and Conjecture 2.10.

In [Mi06c], the above conjecture is proven in the case that $\Omega = R$ and that $\Omega = B$ and (2.10) holds. A key quantity is the lap-number of U, i.e., $\mathcal{Z}[U_\theta(\cdot)]$. This number gives a lower bound of the number of the nodal domains of u_θ. Therefore, $\mathcal{Z}[U_\theta(\cdot)]$ connects the shape of u and the Morse index, and it plays the role similar to the lap-number in one-dimensional cases. See [Mi06b, Mi06c] for details. See [JN00] for connections between $\mathcal{Z}[U_\theta(\cdot)]$ and the "hot spots" conjecture.

Remark. The author has recently proven Conjecture 2.10 in the case of a two-dimensional ball. Therefore, when the domain is a disk, the stable steady states of the shadow system of activator-inhibitor type (N) have exactly one maximum (minimum) point on the boundary, and do not have a critical point in the interior of the disk, even if (2.10) is not assumed.

§ 2.4. Shadow systems in high dimensional domains

In the case that the spatial dimension is 1 or 2, the number of the nodal domains of u_x (u_y and u_θ) plays a critical role, as stated above. In order to obtain the sufficient condition for the second eigenvalue to be positive, we should pay attention to a certain quantity related to the number of the nodal domains. In the case of one-dimensional (resp. two-dimensional) domains, it is the number of zeros of u_x (resp. $\mathcal{Z}[U_\theta(\cdot)]$). We now do not know what the quantity is, when the dimension is 3 or larger. However, it may be related to the shape of u on the boundary.

§ 3. Appendix

Let (u, ξ) be a steady state of (SS), and let $\mathcal{L}$ denote the linearized operator of (SS) at the steady state, i.e.,

$$\mathcal{L} = \begin{pmatrix} L & f_\xi \\ \frac{\langle g_u, \cdot \rangle}{\tau |\Omega|} & \frac{\langle g_\xi, 1 \rangle}{\tau |\Omega|} \end{pmatrix},$$

where $L := D_u \Delta + f_u$. Let $\{\mu_n\}_{n \geq 1}$ denote the eigenvalue of L with the Neumann boundary condition counting multiplicities.

In the appendix, we study the spectrum of $\mathcal{L}$.

§ 3.1. Eigenvalues

It is well-known that the spectrum of $\mathcal{L}$ with the Neumann boundary condition consists only of eigenvalues. We briefly see this fact in this subsection.

Let us consider the eigenvalue problem

$$(\mathcal{L} - \lambda) \begin{pmatrix} \phi \\ \eta \end{pmatrix} = \begin{pmatrix} \Phi \\ Y \end{pmatrix}. \tag{3.1}$$

We easily see that $(\mathcal{L} - \lambda_0)$ has the inverse for some λ_0. Moreover, the inverse is compact. See the form (ϕ, η) in the proof of Proposition 3.2 (ii) below for example. Operating

$(\mathcal{L}-\lambda_0)^{-1}$ on (3.1), we have

$$\{I+(\lambda_0-\lambda)(\mathcal{L}-\lambda_0)^{-1}\}\begin{pmatrix}\phi\\ \eta\end{pmatrix}=(\mathcal{L}-\lambda_0)^{-1}\begin{pmatrix}\Phi\\ Y\end{pmatrix}.$$

The Fredholm alternative says that, if $\{I+(\lambda_0-\lambda)(\mathcal{L}-\lambda_0)^{-1}\}$ does not have the inverse, then there is (ϕ_1,η_1) such that

$$\{I+(\lambda_0-\lambda)(\mathcal{L}-\lambda_0)^{-1}\}\begin{pmatrix}\phi_1\\ \eta_1\end{pmatrix}=0 \iff (\mathcal{L}-\lambda)\begin{pmatrix}\phi_1\\ \eta_1\end{pmatrix}=0.$$

Therefore, λ is an eigenvalue of $\mathcal{L}$.

From now on, we divide the eigenvalues of $\mathcal{L}$ into two sets. One is the set of the non-real eigenvalues and the other is the set of the real eigenvalues.

§ 3.2. Non-real eigenvalues

Proposition 3.1. *Suppose that* (N) *holds.*
(i) *If* $\tau<\langle g_\xi,1\rangle^2/(-2|\Omega|\langle f_\xi,g_u\rangle)$, *then all the non-real eigenvalues are bounded away from the imaginary axis.*
(ii) *If* $\tau<\langle -g_\xi,1\rangle/(\mu_1|\Omega|)$, *then all the non-real eigenvalues are bounded away from the imaginary axis.*

The calculations in the proof are essentially same as those of [W99, Theorem 1.4]. We know by the proposition that a Hopf bifurcation may occur only when

$$\tau\geq\max\left\{\frac{\langle g_\xi,1\rangle^2}{-2|\Omega|\langle f_\xi,g_u\rangle},\frac{\langle -g_\xi,1\rangle}{\mu_1|\Omega|}\right\}.$$

Proof. We consider the eigenvalue problem

$$L[\phi_R+i\phi_I]+(\eta_R+i\eta_I)f_\xi=(\lambda_R+i\lambda_I)(\phi_R+i\phi_I), \tag{3.2}$$

$$\langle g_u,\phi_R+i\phi_I\rangle+(\eta_R+i\eta_I)\langle g_\xi,1\rangle=\tau|\Omega|(\lambda_R+i\lambda_I)(\eta_R+i\eta_I), \tag{3.3}$$

where

$$\|\phi_R\|_2^2+\|\phi_I\|_2^2+\eta_R^2+\eta_I^2=1. \tag{3.4}$$

We show by contradiction that $\phi_R+i\phi_I\neq 0$. Suppose the contrary, namely, $\phi_R+i\phi_I=0$. Then (3.2) becomes $(\eta_R+i\eta_I)f_\xi=0$. Thus $\eta_R+i\eta_I=0$, which contradicts (3.4). We can assume that $\phi_R+i\phi_I\neq 0$. ¿From (3.3) we have

$$\eta_R+i\eta_I=\frac{\langle g_u,\phi_R+i\phi_I\rangle}{(\lambda_R+i\lambda_I)\tau|\Omega|-\langle g_\xi,1\rangle}.$$

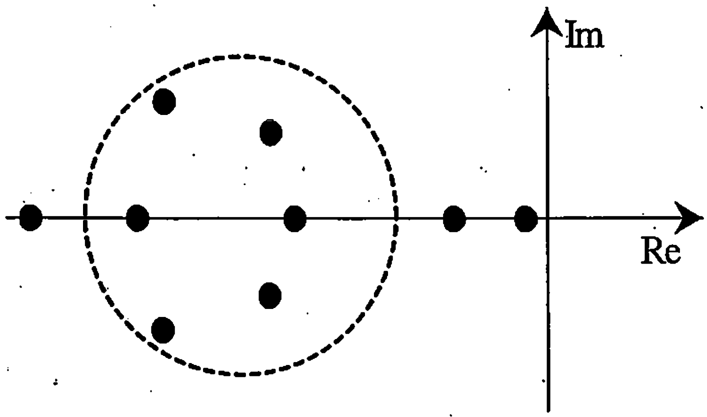

Figure 3. All the non-real eigenvalues of $\mathcal{L}$ are in the ball (3.9).

Substituting this equation into (3.2), we have

$$(L-\lambda_R-i\lambda_I)\left[\phi_R+i\phi_I\right]+\frac{\langle g_u,\phi_R+i\phi_I\rangle}{(\lambda_R+i\lambda_I)\tau\,|\Omega|-\langle g_\xi,1\rangle}=0. \tag{3.5}$$

Taking the real part and the imaginary part of (3.5), we have

$$(L-\lambda_R)\left[\phi_R\right]+\lambda_I\phi_I+\frac{(\tau\,|\Omega|\,\lambda_R-\langle g_\xi,1\rangle)\,\langle g_u,\phi_R\rangle\,f_\xi+\lambda_I\tau\,|\Omega|\,\langle g_u,\phi_I\rangle\,f_\xi}{(\tau\,|\Omega|\,\lambda-\langle g_\xi,1\rangle)^2+(\lambda_I\tau\,|\Omega|)^2}=0, \tag{3.6}$$

$$(L-\lambda_R)\left[\phi_I\right]-\lambda_I\phi_R+\frac{(\tau\,|\Omega|\,\lambda_R-\langle g_\xi,1\rangle)\,\langle g_u,\phi_I\rangle\,f_\xi-\lambda_I\tau\,|\Omega|\,\langle g_u,\phi_R\rangle\,f_\xi}{(\tau\,|\Omega|\,\lambda-\langle g_\xi,1\rangle)^2+(\lambda_I\tau\,|\Omega|)^2}=0. \tag{3.7}$$

Calculating $\langle(3.6),\phi_I\rangle-\langle(3.7),\phi_R\rangle$, we have

$$\lambda_I(\langle\phi_R,\phi_R\rangle+\langle\phi_I,\phi_I\rangle)+\frac{2\lambda_I\tau\,|\Omega|\,k(\langle f_\xi,\phi_R\rangle^2+\langle f_\xi,\phi_I\rangle^2)}{(\tau\,|\Omega|\,\lambda_R-\langle g_\xi,1\rangle)^2+(\lambda_I\tau\,|\Omega|)^2}=0, \tag{3.8}$$

where we use $g_u(u,\xi)=k(\xi)f_\xi(u,\xi)$. Since $\lambda_I\neq 0$ and $\|\phi_R\|_2^2+\|\phi_I\|_2^2\neq 0$, we have

$$\left(\lambda_R-\frac{\langle g_\xi,1\rangle}{\tau\,|\Omega|}\right)^2+\lambda_I^2=\frac{-2k(\langle f_\xi,\phi_R\rangle^2+\langle f_\xi,\phi_I\rangle^2)}{\tau\,|\Omega|\,(\langle\phi_R,\phi_R\rangle^2+\langle\phi_I,\phi_I\rangle^2)}\leq\frac{-2k\,\|f_\xi\|_2^2}{\tau\,|\Omega|}, \tag{3.9}$$

where we use $\langle f_\xi,\phi_R\rangle^2+\langle f_\xi,\phi_I\rangle^2\leq\|f_\xi\|_2^2\,(\|\phi_R\|_2^2+\|\phi_I\|_2^2)$. Hence, all the non-real eigenvalues are in the ball (3.9). Since $-2k\,\|f_\xi\|_2^2=-2\,\langle f_\xi,g_u\rangle$, the ball is on a half plane $\{z;\ \mathrm{Re}(z)\leq\langle g_\xi,1\rangle/(\tau\,|\Omega|)+\sqrt{-2\,\langle f_\xi,g_u\rangle/\tau\,|\Omega|}\}$. See Figure 3. Therefore (i) is proven.

We prove (ii). Calculating $\langle(3.6, \phi_R\rangle + \langle(3.7), \phi_I\rangle$, we have

$$\begin{aligned}(3.10)\quad & \langle(L-\lambda_R)\phi_R, \phi_R\rangle + \langle(L-\lambda_R)\phi_I, \phi_I\rangle \\ & + \frac{2k(\tau|\Omega|\lambda_R - \langle g_\xi, 1\rangle)(\langle f_\xi, \phi_R\rangle^2 + \langle f_\xi, \phi_I\rangle^2)}{(\tau|\Omega|\lambda_R - \langle g_\xi, 1\rangle)^2 + (\lambda_I\tau|\Omega|)^2} = 0,\end{aligned}$$

where we use $g_u(u,\xi) = k(\xi)f_\xi(u,\xi)$. Substituting (3.8) into (3.10), we have

$$\begin{aligned}\frac{\tau|\Omega|\lambda_R - \langle g_\xi, 1\rangle}{\tau|\Omega|}\left(\|\phi_R\|_2^2 + \|\phi_I\|_2^2\right) &= \langle(L-\lambda_R)\phi_R, \phi_R\rangle + \langle(L-\lambda_R)\phi_I, \phi_I\rangle \\ &\leq (\mu_1 - \lambda_R)\left(\|\phi_R\|_2^2 + \|\phi_I\|_2^2\right),\end{aligned}$$

Therefore,

$$\lambda_R \leq \frac{\mu_1}{2} - \frac{\langle -g_\xi, 1\rangle}{2\tau|\Omega|}.$$

The proof of (ii) is complete. □

Eigenvalues do not have a limit point in $\mathbb{C}$ except ∞. Since all the non-real eigenvalues are in the ball (3.9), the number of all the non-real eigenvalues is finite provided that $\tau > 0$ is fixed.

§ 3.3. Real eigenvalues

Proposition 3.2. *Suppose that* (N) *holds.*
(i) $\mathrm{Spec}(\mathcal{L}) \cap \{\lambda \in \mathbb{R};\ \lambda > \mu_1\} = \emptyset$.
(ii) $\mu_1 \notin \mathrm{Spec}(\mathcal{L})$.
(iii) *If*

$$(3.11)\qquad \tau|\Omega|\lambda - \langle g_\xi, 1\rangle < k\sum_{n\geq 1}\frac{\langle f_\xi, \psi_n\rangle^2}{\lambda - \mu_n} \quad \textit{for}\ \ \lambda \in [0, \mu_1),$$

then $\mathrm{Spec}(\mathcal{L})$ *has no eigenvalues in* $\{\lambda \in \mathbb{R};\ 0 \leq \lambda < \mu_1\}$.

The techniques used in the proof of Proposition 3.2 are developed in [Mi05]. See Figure 4.

Proof. We prove (ii). We consider

$$(3.12)\qquad (\mathcal{L} - \lambda)\begin{pmatrix}\phi \\ \eta\end{pmatrix} = \begin{pmatrix}\Phi \\ Y\end{pmatrix},$$

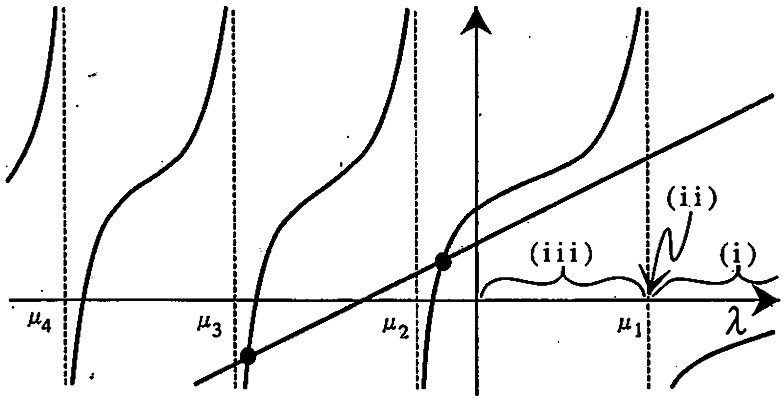

Figure 4. The graphs of the both sides of (3.11) in the case that $k(\xi) < 0$. (i), (ii) and (iii) of Proposition 3.2 cover corresponding parts in the graph.

when $\lambda = \mu_1$. Since μ_1 is a simple eigenvalue of L, $(L-\mu_1)^{-1}\left[\Phi - \frac{\langle\Phi,\psi_1\rangle}{\langle f_\xi,\psi_1\rangle}f_\xi\right]$ exists, where ψ_1 is an eigenfunction corresponding to μ_1. Let (ϕ,η) be

$$\phi = (L-\mu_1)^{-1}\left[\Phi - \frac{\langle\Phi,\psi_1\rangle}{\langle f_\xi,\psi_1\rangle}f_\xi\right] + c_0 f_\xi, \qquad \eta = \frac{\langle\Phi,\psi_1\rangle}{\langle f_\xi,\psi_1\rangle},$$

where

$$c_0 = \frac{1}{\langle g_u,\psi_1\rangle}\left\{\tau|\Omega|Y - \frac{\langle\Phi,\psi_1\rangle}{\langle f_\xi,\psi_1\rangle}(\langle g_\xi,1\rangle - \tau|\Omega|\mu_1)\right.$$
$$\left. - \left\langle g_u, (L-\mu_1)^{-1}\left[\Phi - \frac{\langle\Phi,\psi_1\rangle}{\langle f_\xi,\psi_1\rangle}f_\xi\right]\right\rangle\right\}.$$

Then (ϕ,η) satisfies (3.12), which means that $\mu_1 \notin \mathrm{Spec}(\mathcal{L})$.

We prove (iii). We consider (3.12) in the case when $\lambda \in [0,\mu_1)$. A similar calculation in the proof of Lemma 2.3 derives

$$(L + A_{\lambda,\tau} - \lambda)\phi = \Phi + \frac{\tau|\Omega|Y f_\xi}{\lambda\tau|\Omega| - \langle g_\xi,1\rangle}, \tag{3.13}$$

where

$$A_{\lambda,\tau}\phi := \frac{\langle g_u,\phi\rangle f_\xi}{\lambda\tau|\Omega| - \langle g_\xi,1\rangle}.$$

Here λ appears in $A_{\lambda,\tau}$, hence (3.13) is not a standard eigenvalue problem. Note that $\lambda\tau|\Omega| - \langle g_\xi,1\rangle \neq 0$. Since $A_{\lambda,\tau}$ is a rank-one operator, we see by the Sherman-Morrison formula that

$$(L + A_{\lambda,\tau} - \lambda)^{-1} = \left(1 + \frac{(L-\lambda)^{-1}A_{\lambda,\tau}}{H(\lambda)}\right)(L-\lambda)^{-1},$$

where

$$H(\lambda) := 1 + \frac{k\langle f_\xi, (L-\lambda)^{-1}[f_\xi]\rangle}{\lambda\tau|\Omega| - \langle g_\xi, 1\rangle} = 1 - \frac{k}{\lambda\tau|\Omega| - \langle g_\xi, 1\rangle}\sum_{n\geq 1}\frac{\langle f_\xi, \psi_n\rangle^2}{\lambda - \mu_n}.$$

Because of (3.11) $H(\lambda)$ does not vanish in $[0, \mu_1)$ and $(L + A_{\lambda,\tau} - \lambda)^{-1}$ exists for $\lambda \in [0, \mu_1)$. Hence (3.13) has unique solution ϕ. Using the second equation of (3.12), we have

$$\eta = \frac{\langle g_u, \phi\rangle - \tau|\Omega|Y}{\lambda\tau|\Omega| - \langle g_\xi, 1\rangle}.$$

The pair (ϕ, η) obtained here is a unique solution of (3.12). The proof of (iii) is complete.

We prove (i). We show that (2.6) has no root in $\{\lambda \in \mathbb{R};\ \lambda > \mu_1\}$. Since $\lambda > \mu_1$, the right-hand side of (2.6) is negative. Because of (N), the left-hand side of (2.6) is positive. Hence, (2.3) has no root if $\lambda > \mu_1$, and $H(\lambda)$ also has no root. The problem (3.12) has a unique solution. The proof of (i) is complete. □

When one studies the eigenvalue problem (3.12), eigenvalues of a differential operator with non-local term have to be analyzed. The Sherman-Morrison formula is useful. A brief history of the Sherman-Morrison formula can be seen in [HS81].

Because of Proposition 3.2, we have to check (3.11) when we prove the stability of (u, ξ). A necessary condition is that $\mu_2 \leq 0 < \mu_1$. Hence, the Morse index of u should be 1 if (u, ξ) is stable for some $\tau > 0$. However, checking (3.11) is difficult in general. In the case of the stable boundary one-spike layer of the shadow Gierer-Meinhardt system (SGM), (3.11) holds if $\tau > 0$ is small. See [Mi05].

Corollary 3.3. *Suppose that* (N) *holds. If* $\mu_2 < 0$ *and if*

$$\langle g_\xi, 1\rangle > \langle g_u, L^{-1}[f_\xi]\rangle,$$

then, for small $\tau > 0$*, the steady state* (u, ξ) *is stable.*

Remark. If $0 \notin \mathrm{Spec}(L)$ and if $\langle g_u, L^{-1}[f_\xi]\rangle \neq \langle g_\xi, 1\rangle$, then $0 \notin \mathrm{Spec}(\mathcal{L})$. Hence, if a steady state that is stable for small $\tau > 0$ becomes unstable for large $\tau > 0$, then, as τ increases, eigenvalues do not pass the origin in $\mathbb{C}$, and should pass the imaginary axis. Therefore, a Hopf bifurcation may occur as τ increases.

Acknowledgement. The author would like to thank Professors E. Yanagida and E. N. Dancer for pointing out that Theorem A (ii) and Lemma 3.2 (ii) of [Mi06b] can be improved to Corollary 2.4 and Lemma 2.3 (ii). This work was partially supported by a COE program of Kyoto University.

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