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Bilinear Formalism on Gauss's Recurrence of Period 5

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Abstract

Hirota's τ-variables and bilinear formalism are discussed on the equation called "Gauss's recurrence of period 5." Combination of some geometric viewpoints naturally makes the solution written in determinant form. Particularly, a lattice on Möbius band helps understanding the structure of τ-variables. The same lattice is also available to the Bäcklund transformation for the Gauss's periodic recurrence. Moreover, a higher order resemblance and its Bäcklund transformation are referred to.

§1 Introduction

The equation $Q_{n+1}Q_{n-1} = Q_n + 1$, called "Gauss's recurrence of period 5 (GRP5)", is well-known as the nontrivial easiest one of 2nd order nonlinear "periodic recurrences."[1]-[7] However, it's quite mysterious why GRP5 is 5-periodic, so that we want to give an explanation for it.

We see that such periodicity occurs by the combination of some geometric concepts: "rhombic rule and cross-ratio,"[8, 9, 11, 12] "Ptolemy's theorem, Euler formula and Plücker relation,"[10, 11, 12] "lattice on Möbius band,"[12] etc.

Since these concepts naturally combine by the (analogous) method of "Hirota's τ-variables and bilinear formalism,"[5, 13, 14] we can even consider Bäcklund transformations, higher order versions, etc., for GRP5.

Notation: Hereafter, the letter 'τ' almost shows 'length of line segment.'

§2 Bäcklund transformation for GRP5

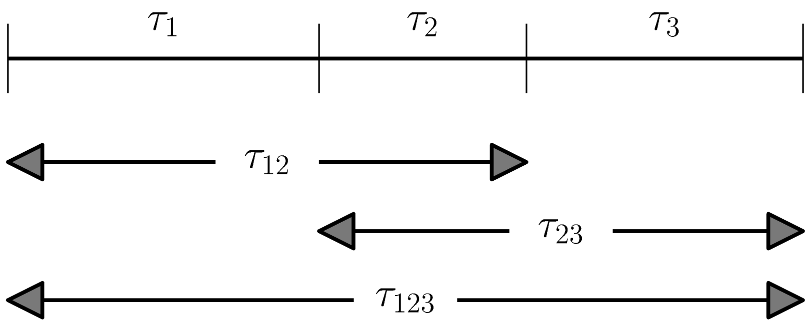
Let us start from a "Bäcklund transformation (BT)"[13, 14] for GRP5. Generally to say, if an integrable equation is given, BT means what transforms its solution into another one.

As a preliminary, "Euler formula" is necessary.

Euler formula:

$$\underline{\tau_{123} \cdot \tau_2 + \tau_1 \cdot \tau_3 = \tau_{12} \cdot \tau_{23}},$$

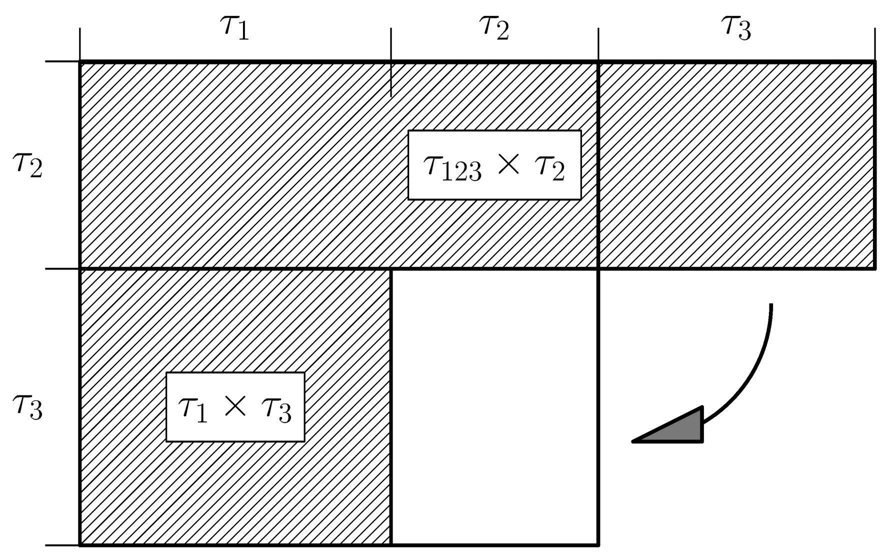
for arbitrary line segment having three components of partition as the figure below,



setting notation of length as

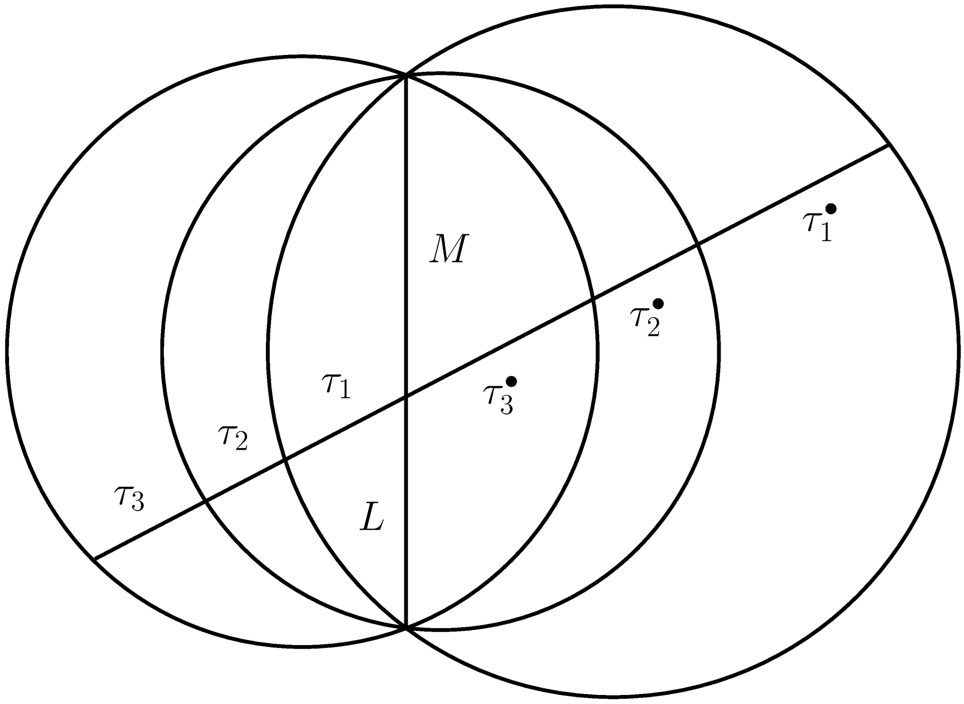
$$\begin{cases} \tau_{12} := \tau_1 + \tau_2, \\ \tau_{23} := \tau_2 + \tau_3, \\ \tau_{123} := \tau_1 + \tau_2 + \tau_3. \end{cases}$$

Proof: Geometrically, consider the gross area below.



Algebraic proof may be easier but it may spoil some interest. So, QED

The following figure shows a bilinear form of BT for GRP5, for arbitrary parameters L and M.



From the above, let us derive a map $\varphi : (\tau_1 : \tau_2 : \tau_3) \mapsto (\tau_1^\bullet : \tau_2^\bullet : \tau_3^\bullet)$.

By the power theorem,

$$\begin{cases} L \cdot M &= \tau_1 \cdot (\tau_1^\bullet + \tau_2^\bullet + \tau_3^\bullet), \\ L \cdot M &= (\tau_1 + \tau_2) \cdot (\tau_2^\bullet + \tau_3^\bullet), \\ L \cdot M &= (\tau_1 + \tau_2 + \tau_3) \cdot \tau_3^\bullet, \end{cases}$$

so we have a system of bilinear equations,

$$\tau_1 \cdot (\tau_1^\bullet + \tau_2^\bullet + \tau_3^\bullet) = (\tau_1 + \tau_2) \cdot (\tau_2^\bullet + \tau_3^\bullet) = (\tau_1 + \tau_2 + \tau_3) \cdot \tau_3^\bullet,$$

or, in short notation, likely in the explanation of Euler formula,

$$\tau_1 \cdot \tau_{123}^\bullet = \tau_{12} \cdot \tau_{23}^\bullet = \tau_{123} \cdot \tau_3^\bullet.$$

We can solve the above as an explicit map

$$\begin{aligned} \varphi : (\tau_1 : \tau_2 : \tau_3) &\mapsto (\tau_1^\bullet : \tau_2^\bullet : \tau_3^\bullet), \\ (\tau_1^\bullet : \tau_2^\bullet : \tau_3^\bullet) &= (\tau_{123}\tau_2 : \tau_1\tau_3 : \tau_1\tau_{12}). \end{aligned}$$

It's easy to check the above by substitution.

Here, by Euler formula, we have

$$\begin{aligned}(\tau_1^{\bullet} : \tau_2^{\bullet} : \underline{\tau_{12}^{\bullet}}) &= (\tau_{123}\tau_2 : \tau_1\tau_3 : \underline{\tau_1^{\bullet} + \tau_2^{\bullet}}) \\ &= (\tau_{123}\tau_2 : \tau_1\tau_3 : \underline{\tau_{12}\tau_{23}}).\end{aligned}$$

And we have

$$\begin{aligned}(\tau_2^{\bullet} : \tau_3^{\bullet} : \tau_{23}^{\bullet}) &= (\tau_1\tau_3 : \tau_1\tau_{12} : \tau_2^{\bullet} + \tau_3^{\bullet}) \\ &= (\tau_1\tau_3 : \tau_1\tau_{12} : \tau_1\tau_{123}),\end{aligned}$$

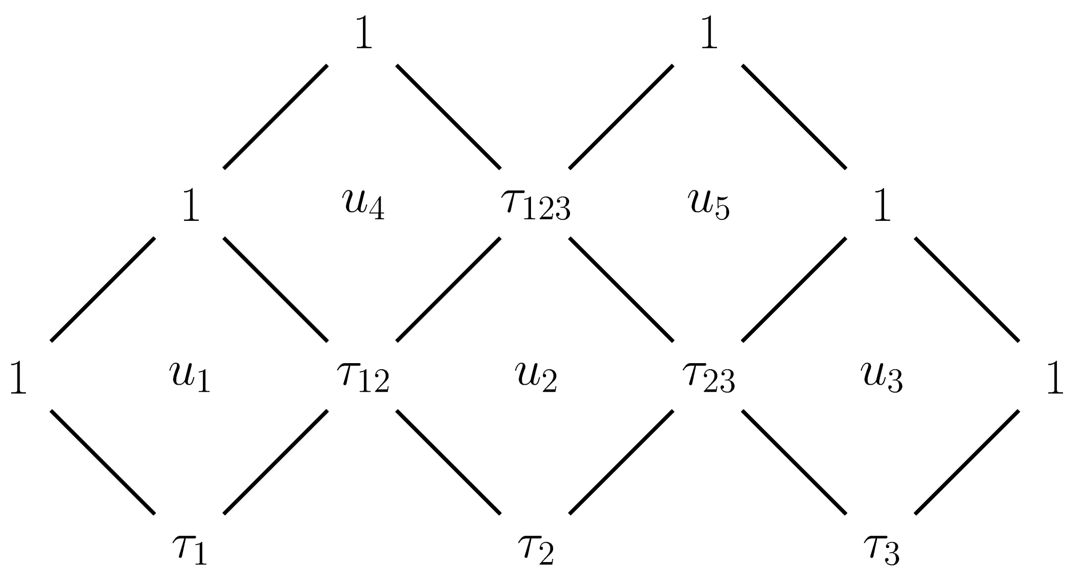
and

$$\begin{aligned}(\tau_1^{\bullet} : \tau_{23}^{\bullet} : \tau_{123}^{\bullet}) &= (\tau_{123}\tau_2 : \tau_1\tau_{123} : \tau_1^{\bullet} + \tau_{23}^{\bullet}) \\ &= (\tau_{123}\tau_2 : \tau_1\tau_{123} : \tau_{12}\tau_{123}).\end{aligned}$$

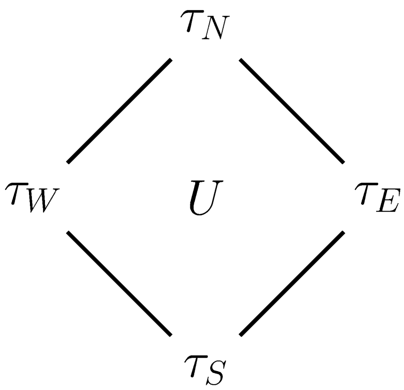
Thus, we obtain the following extended form of the map φ :

$$\begin{aligned} & \left(\tau_1^{\bullet} : \tau_2^{\bullet} : \tau_3^{\bullet} : \tau_{12}^{\bullet} : \tau_{23}^{\bullet} : \tau_{123}^{\bullet} \right) \\ = & \left(\tau_{123}\tau_2 : \tau_1\tau_3 : \tau_1\tau_{12} : \tau_{12}\tau_{23} : \tau_1\tau_{123} : \tau_{12}\tau_{123} \right).\end{aligned}$$

Furthermore, in order to analyze the map above, let us introduce new variables $u_1, u_2, \ldots u_5$, by the following lattice,



looking as "rhombic rule",

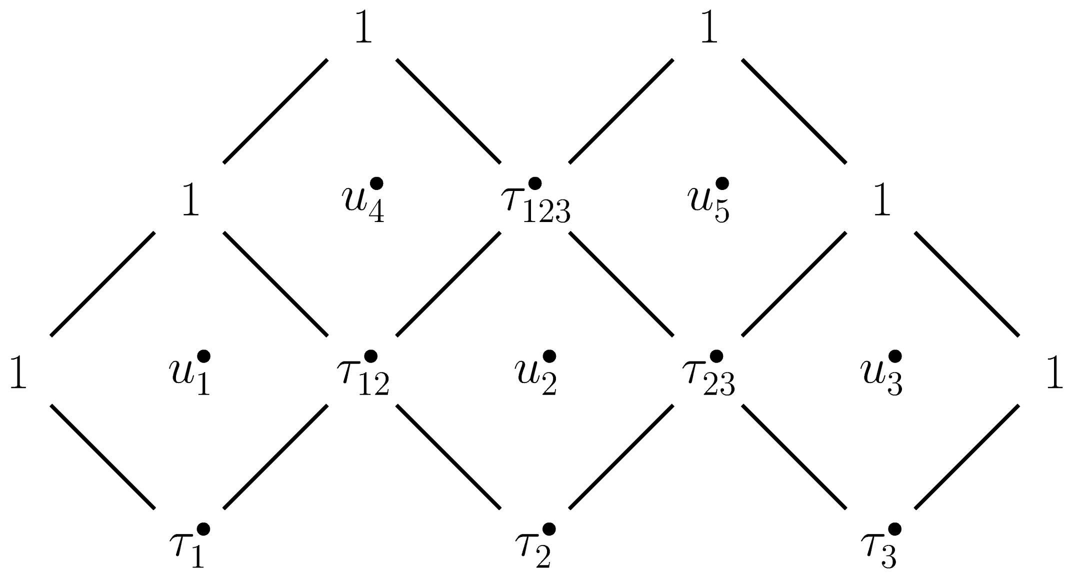


$$\Longleftrightarrow \qquad U = \frac{\tau_W \cdot \tau_E}{\tau_N \cdot \tau_S} \; .$$

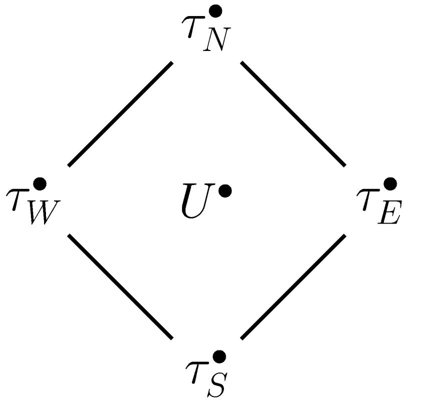
Then, we have a map

$$f : (\tau_1 : \tau_2 : \tau_3 : \tau_{12} : \tau_{23} : \tau_{123}) \mapsto (u_1, u_2, u_3, u_4, u_5),$$
$$\begin{cases} u_1 = \dfrac{\tau_{12}}{\tau_1}, \quad u_2 = \dfrac{\tau_{12} \cdot \tau_{23}}{\tau_{123} \cdot \tau_2} \text{ (cross-ratio)} , \quad u_3 = \dfrac{\tau_{23}}{\tau_3}, \\ u_4 = \dfrac{\tau_{123}}{\tau_{12}}, \; u_5 = \dfrac{\tau_{123}}{\tau_{23}}. \end{cases}$$

In the same way, let us introduce $u_1^\bullet, u_2^\bullet, \dots u_5^\bullet$, by



looking as "rhombic rule",



$$\Longleftrightarrow \quad U^\bullet = \frac{\tau^\bullet_W \cdot \tau^\bullet_E}{\tau^\bullet_N \cdot \tau^\bullet_S} \ .$$

Then, we have

$$u^\bullet_1 = \frac{\tau^\bullet_{12}}{\tau^\bullet_1},\ u^\bullet_2 = \frac{\tau^\bullet_{12} \cdot \tau^\bullet_{23}}{\tau^\bullet_{123} \cdot \tau^\bullet_2},\ u^\bullet_3 = \frac{\tau^\bullet_{23}}{\tau^\bullet_3},\ u^\bullet_4 = \frac{\tau^\bullet_{123}}{\tau^\bullet_{12}},\ u^\bullet_5 = \frac{\tau^\bullet_{123}}{\tau^\bullet_{23}}.$$

Thus, we can obtain the following results:

- We can make the inverse map of f,

$$f^{-1} : (u^\bullet_1, u^\bullet_2, u^\bullet_3, u^\bullet_4, u^\bullet_5) \mapsto (\tau^\bullet_1 : \tau^\bullet_2 : \tau^\bullet_3 : \tau^\bullet_{12} : \tau^\bullet_{23} : \tau^\bullet_{123})$$

using "rhombic rule" $U^\bullet = \dfrac{\tau^\bullet_W \cdot \tau^\bullet_E}{\tau^\bullet_N \cdot \tau^\bullet_S}$ on the lattice. Actually, if $u^\bullet_s$ are given, from the lattice in the preceding page, we can derive the continued ratio of $\tau^\bullet_s$.
(Here, 'τ_S' and 'τ_s' are confusing, and please read 'τ_s' as "taus" etc., which is convenient convention not to mention details of indexes.)

- It's easy to check

$$u^\bullet_1 = u_2,\ u^\bullet_2 = u_3,\ u^\bullet_3 = u_4,\ u^\bullet_4 = u_5,\ u^\bullet_5 = u_1,$$

using the "rhombic rule" and the extended φ,

$$\begin{aligned} & \left(\tau^\bullet_1 \ : \ \tau^\bullet_2 \ : \ \tau^\bullet_3 \ : \ \tau^\bullet_{12} \ : \ \tau^\bullet_{23} \ : \ \tau^\bullet_{123} \right) \\ = \quad & \left(\tau_{123}\tau_2 \ : \ \tau_1\tau_3 \ : \ \tau_1\tau_{12} \ : \ \tau_{12}\tau_{23} \ : \ \tau_1\tau_{123} \ : \ \tau_{12}\tau_{123} \right). \end{aligned}$$

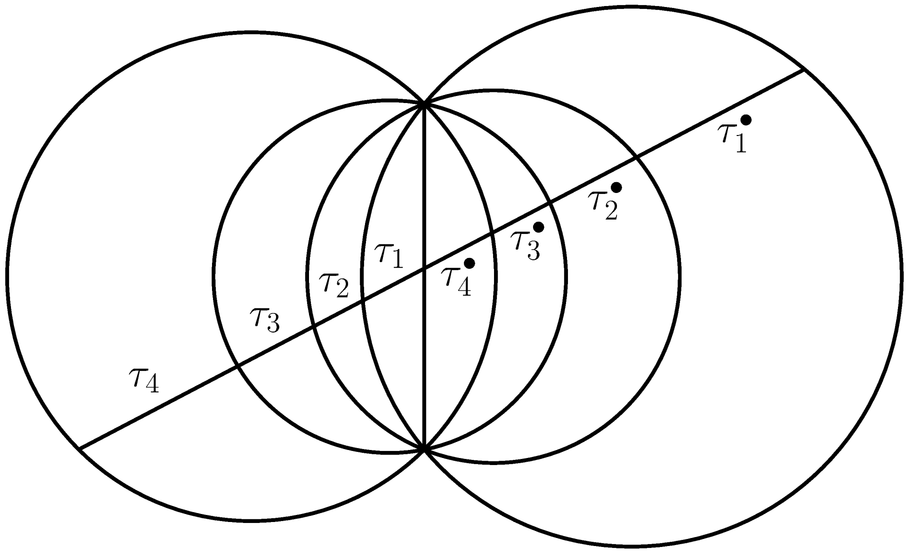
So, $\psi : (u_1, u_2, u_3, u_4, u_5) \mapsto (u^\bullet_1, u^\bullet_2, u^\bullet_3, u^\bullet_4, u^\bullet_5)$ satisfies $\psi^5 = I$.

- By the combination of the results above, we have 'fine' decomposition

$$\varphi : \tau_s \mapsto u_s \mapsto u_s^\bullet \mapsto \tau_s^\bullet .$$

Hence, $\varphi = f^{-1} \circ \psi \circ f$ satisfies $\varphi^5 = I$. In §3, we find that the map $\psi : u_s \mapsto u_s^\bullet$ becomes to the obvious BT of "shift type" for GRP5.

Comment for §2: Higher order version can be considered and the rough sketch of 6-periodic version is given below.



Bilinear form: $\tau_1 \cdot \tau_{1234}^\bullet = \tau_{12} \cdot \tau_{234}^\bullet = \tau_{123} \cdot \tau_{34}^\bullet = \tau_{1234} \cdot \tau_4^\bullet$.

$$\begin{array}{ccccccccccccccccc}
&&&&&&1&&&&1&&&&&& \\
&&&&&/&&\backslash&&/&&\backslash&&&&& \\
&&&&1&&u_5&&\tau_{1234}&&u_6&&1&&&& \\
&&&/&&\backslash&&/&&\backslash&&/&&\backslash&&& \\
&&1&&v_1&&\tau_{123}&&v_2&&\tau_{234}&&v_3&&1&& \\
&/&&\backslash&&/&&\backslash&&/&&\backslash&&/&&\backslash& \\
1&&u_1&&\tau_{12}&&u_2&&\tau_{23}&&u_3&&\tau_{34}&&u_4&&1 \\
&\backslash&&/&&\backslash&&/&&\backslash&&/&&\backslash&&/& \\
&&\tau_1&&&&\tau_2&&&&\tau_3&&&&\tau_4&&
\end{array}$$

Periodicity: $u_1^\bullet = u_2$, $u_2^\bullet = u_3$, $u_3^\bullet = u_4$,
$u_4^\bullet = u_5$, $u_5^\bullet = u_6$, $u_6^\bullet = u_1$.
Middle row: $v_1^\bullet = v_2$, $v_2^\bullet = v_3$, $v_3^\bullet = v_1$.

Furthermore, the author confirmed the 7-periodic version is successful, which he talked about at:

"Spectrum Resolution of Ultradiscrete Periodic Mappings,"
'Nonlinear Waves, Theory and Applications',
Beijing, China (June 9, 2008, Olympic year).

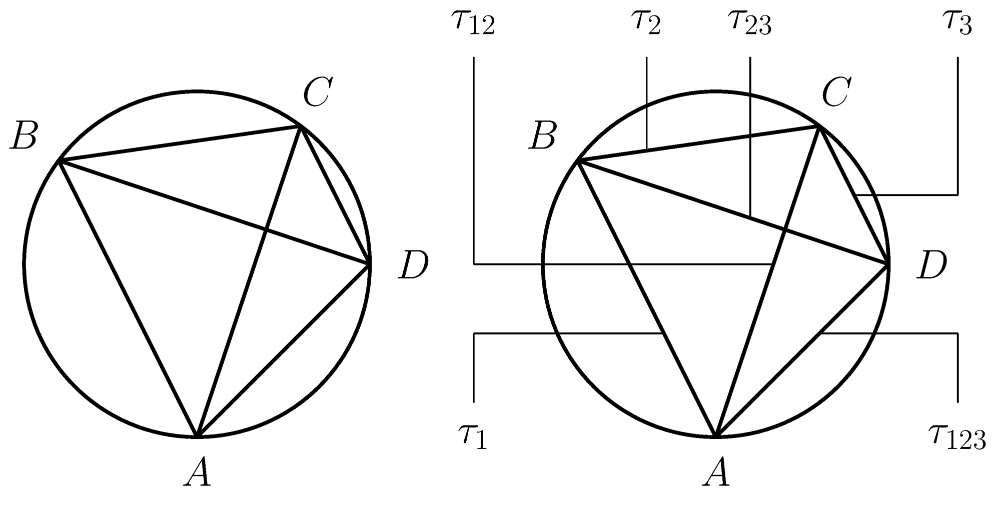
§3 Geometric solution to GRP5

In this section, we show that the solution to GRP5, $Q_{n+1}Q_{n-1} = Q_n + 1$, has a geometric meaning. Preliminary is the following.

Ptolemy's theorem:

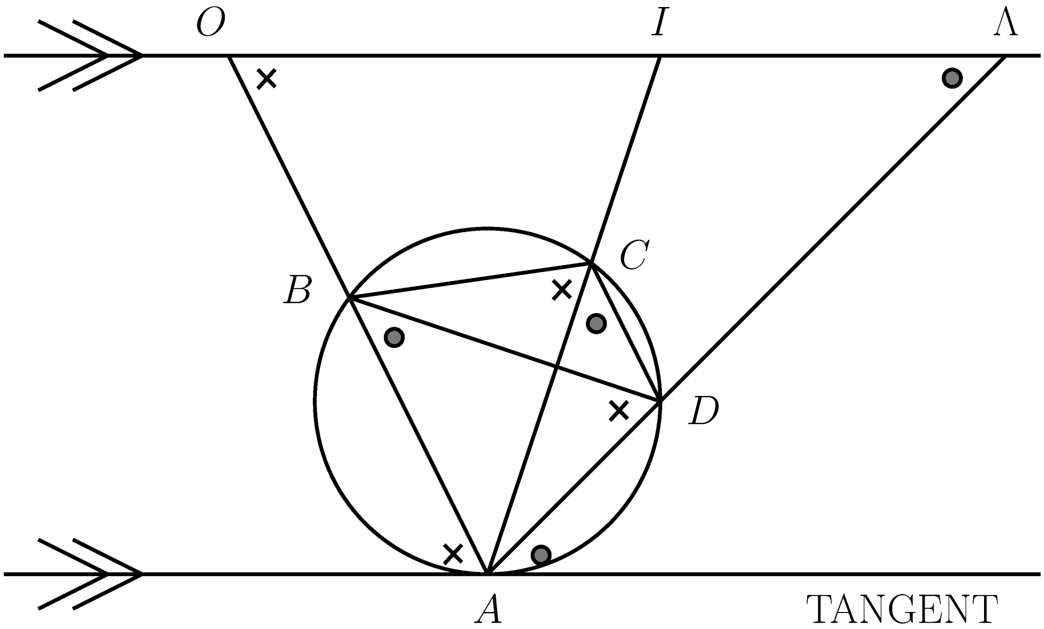
$$\underline{AD \cdot BC + AB \cdot CD = AC \cdot BD},$$

for the 6 chords among 4 points A, B, C, and D on the circle below left.



Or, with τ-notation on the above right, $\underline{\tau_{123} \cdot \tau_2 + \tau_1 \cdot \tau_3 = \tau_{12} \cdot \tau_{23}}$, which goes to "Euler formula" as the radius tends to ∞.

Proof: The theorem is equivalent to an identity $OI + I\Lambda = O\Lambda$ in the following figure:



Because, from the above, we can derive the similarities,

$$\begin{cases} \Delta ABC \sim \Delta AIO, \\ \Delta ABD \sim \Delta A\Lambda O, \\ \Delta ACD \sim \Delta A\Lambda I, \end{cases}$$

thus we can obtain

$$(OI : O\Lambda : I\Lambda) = (AD \cdot BC : AC \cdot BD : AB \cdot CD).$$

(The author has not found any literature which carries this proof, yet. Why does no one draw this figure for this theorem?) QED

Remark: "Cross-ratio" is defined as

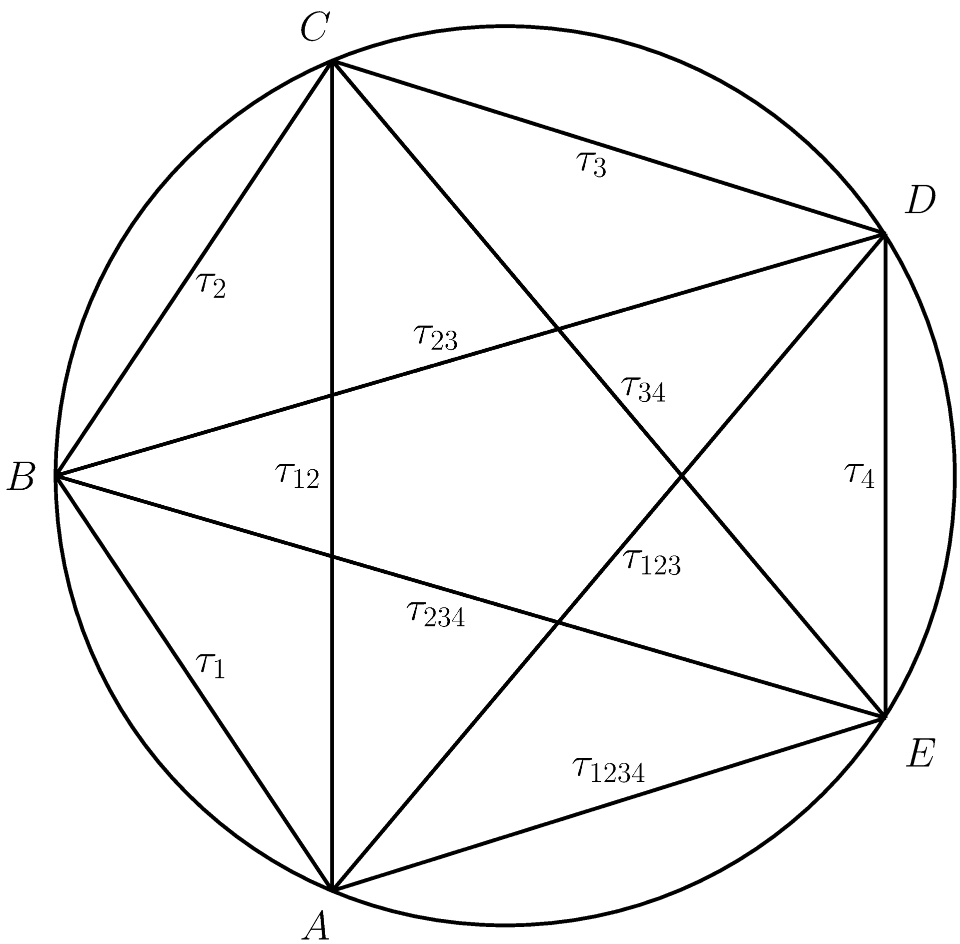
$$\lambda := \frac{AC \cdot BD}{AD \cdot BC} = \frac{O\Lambda}{OI},$$

when both vectors $\overrightarrow{OI}$ and $\overrightarrow{O\Lambda}$ are in same direction. (Details, in §4.) This is the reason why the cross-ratio should be used as a coordinate of projective line.

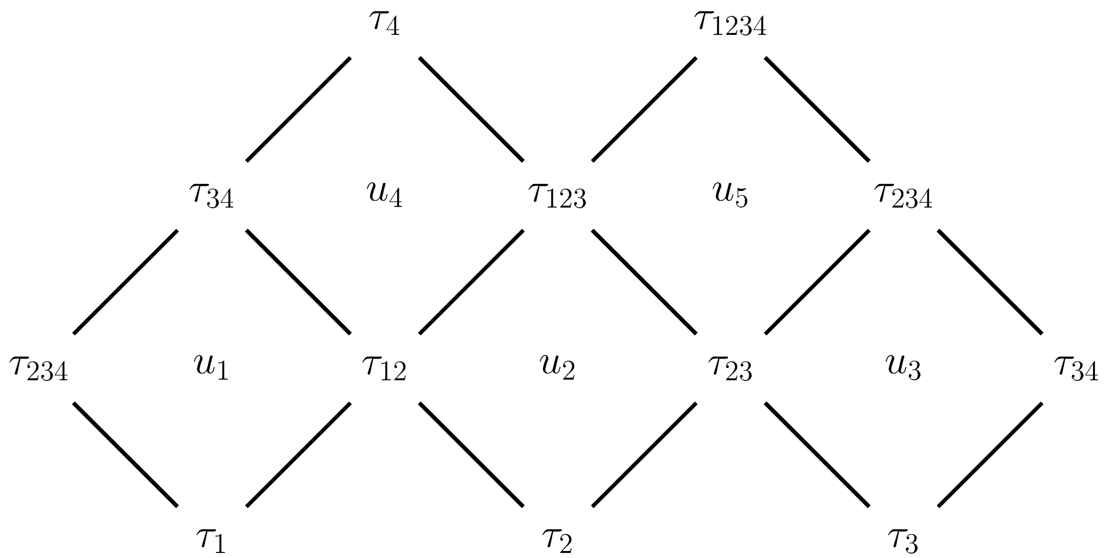
Let us adopt ‘temporary’ τ-variables,

$$\tau_1 := AB,\ \tau_2 := BC,\ \tau_{12} = AC,\ \text{etc.},$$

as follows (‘Improved’ τ_s appear in §4.):



And, let us consider the following lattice:



Here, we note this lattice is on Möbius band, because the rightward edge

'$\tau_{234} - \tau_{34}$' is equal to the leftward upside-down. Namely, the following cyclic lattice is the same one:

$$
\begin{array}{ccccccccccc}
BC & & CD & & DE & & EA & & AB & & BC \\
u_2 & BD & u_3 & CE & u_4 & DA & u_5 & EB & u_1 & AC & u_2 \\
DA & u_5 & EB & u_1 & AC & u_2 & BD & u_3 & CE & u_4 & DA \\
& EA & & AB & & BC & & CD & & DE &
\end{array}
$$

By the "rhombic rule" as same as previous one, we have a transformation $\tau_s \mapsto u_s$ (cross-ratios):

$$
\begin{cases}
u_1 := \dfrac{EB \cdot AC}{EC \cdot AB}, \quad u_2 := \dfrac{AC \cdot BD}{AD \cdot BC}, \quad u_3 := \dfrac{BD \cdot CE}{BE \cdot CD}, \\
u_4 := \dfrac{CE \cdot DA}{CA \cdot DE}, \quad u_5 := \dfrac{DA \cdot EB}{DB \cdot EA},
\end{cases}
$$

(Note: This transformation $\tau_s \mapsto u_s$ is not invertible, whereas the previous one is.)

By Ptolemy's theorem, we have relations,

$$
\begin{cases}
EC \cdot AB + EA \cdot BC = EB \cdot AC, \\
\underline{AD \cdot BC + AB \cdot CD = AC \cdot BD}, \\
BE \cdot CD + BC \cdot DE = BD \cdot CE, \\
CA \cdot DE + CD \cdot EA = CE \cdot DA, \\
DB \cdot EA + DE \cdot AB = DA \cdot EB.
\end{cases}
$$

The typical 2nd relation above is rewritten into

$$
\frac{AD \cdot BC}{AC \cdot BD} + \frac{AB \cdot CD}{AC \cdot BD} = 1,
$$

where, the LHS is transformed into

$$
\frac{1}{u_2} + \frac{1}{u_1 \cdot u_3}.
$$

Thus, we have

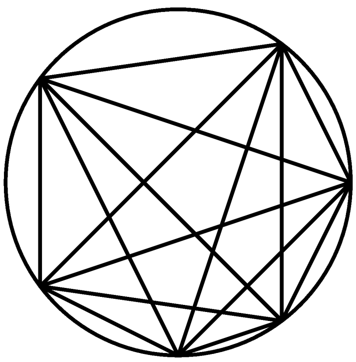
$$\frac{1}{u_n} + \frac{1}{u_{n-1} \cdot u_{n+1}} = 1,$$

for the index n : mod 5.

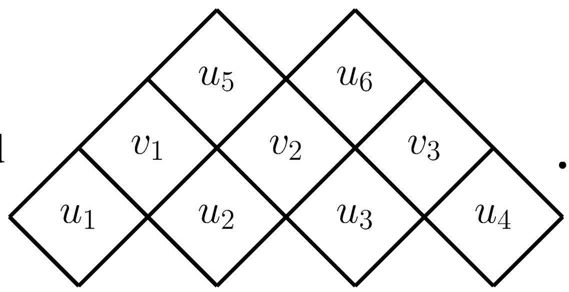
Now, the following results are obvious.

- A 5-periodic solution to the equation
$$\frac{1}{u_n} + \frac{1}{u_{n-1} \cdot u_{n+1}} = 1$$
is made from 'temporary' τ-variables.
- Clearly, shifted variables $u_n^{\bullet} := u_{n+1}$ (n : mod 5) also solves above, which is obvious BT. (Nontrivial BTs for the above appear in §5.)
- Transformation $Q_n := \dfrac{-1}{u_n}$ solves GRP5, $Q_{n+1}Q_{n-1} = Q_n + 1$. We remark that, because $0 < \dfrac{1}{u_n} < 1$, this solution to GRP5 is negative ($-1 < Q_n < 0$). But, in ultra-discretization, positive one is necessary. This problem is resolved in §4.

Comment for §3: Higher order version can be considered and the rough sketch of 6-periodic version is given below.
Consider



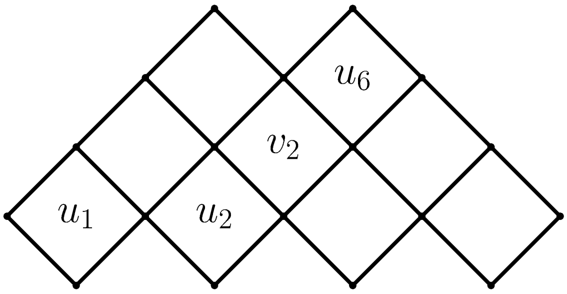
and



.

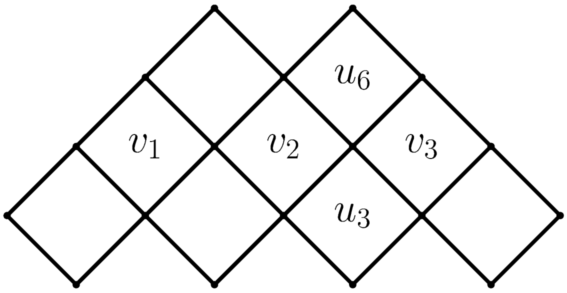
Then, Ptolemy's theorem yields $\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \dfrac{6!}{4!2!} = 15$ relations among 9 variables $u_1, u_2, u_3, u_4, u_5, u_6, v_1, v_2$, and v_3. After brief thinking, we find that these relations are classified into 3 types as follows:

Type A: 6 ways appear and the typical one is,



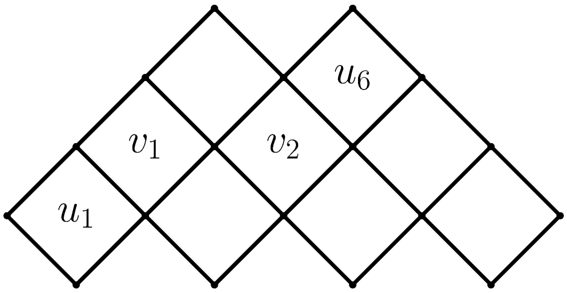
$$\frac{1}{u_1}+\frac{1}{u_2v_2u_6}=1.$$

Type B: 3 ways, typical is,



$$\frac{1}{v_1}+\frac{1}{u_3v_2v_3u_6}=1.$$

Type C: 6 ways, typical is,



$$\frac{1}{u_1v_1}+\frac{1}{v_2u_6}=1.$$

And, transformation $Q_n = \dfrac{-1}{u_n}$, $P_n = \dfrac{-1}{v_n}$ gives polynomial relations $f_1 = f_2 = \cdots = f_{15} = 0$ among $Q_1, Q_2, \ldots, P_3$, where

$$f_1 := 1 + Q_1 + Q_2P_2Q_6,\ \ f_2 := 1 + Q_2 + Q_3P_3Q_1,\ \ \text{etc.}$$

Here, MATHEMATICA® 5 says, roughly:

In[1] := GroebnerBasis$[\{f_1, \ldots, f_{15}\}, \{P_3, P_2, P_1, Q_6, Q_5, Q_4\}]$

Out[1] = $\{-(1 + Q_2)Q_4 - Q_1(1 + Q_3 + Q_4)$, etc.$\}$

So, we have a Gauss's recurrence of period 6 version (GRP6),

$$(1+Q_{n+1})Q_{n+3}+Q_n(1+Q_{n+2}+Q_{n+3})=0.$$

Further, we can confirm the following remarkable facts:

- Let $R_n := -1-Q_n$, then

$$Q_{n+1}Q_{n-1}=Q_n+1 \text{ (GRP5)}$$
$$\Leftrightarrow \begin{vmatrix} 1 & R_{n-1} & 0 & 0 \\ -1 & 1 & R_n & 0 \\ 0 & -1 & 1 & R_{n+1} \\ 0 & 0 & -1 & 1 \end{vmatrix} = 0.$$

- Let $R_n := -1-Q_n$, then

$$(1+Q_{n+1})Q_{n+3}+Q_n(1+Q_{n+2}+Q_{n+3})=0 \text{ (GRP6)}$$
$$\Leftrightarrow \begin{vmatrix} 1 & R_n & 0 & 0 & 0 \\ -1 & 1 & R_{n+1} & 0 & 0 \\ 0 & -1 & 1 & R_{n+2} & 0 \\ 0 & 0 & -1 & 1 & R_{n+3} \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} = 0.$$

- The difference equation of period 5

$$\begin{vmatrix} 1 & R_{n-1} & 0 & 0 \\ -1 & 1 & R_n & 0 \\ 0 & -1 & 1 & R_{n+1} \\ 0 & 0 & -1 & 1 \end{vmatrix} = 0$$

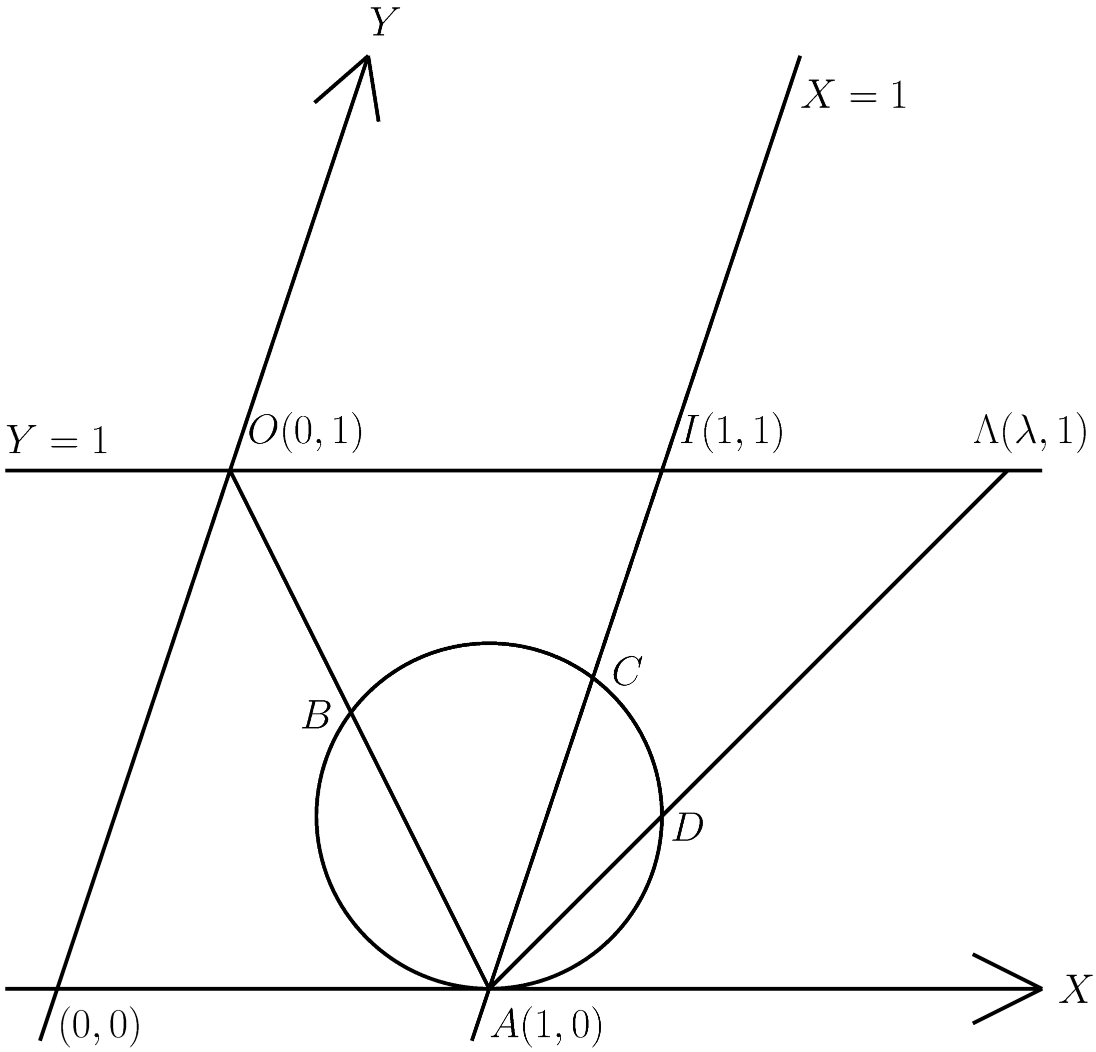
conserves the LHS determinant of the equation of 6-periodic version,

$$\begin{vmatrix} 1 & R_{n+1} & 0 & 0 & 0 \\ -1 & 1 & R_{n+2} & 0 & 0 \\ 0 & -1 & 1 & R_{n+3} & 0 \\ 0 & 0 & -1 & 1 & R_{n+4} \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & R_n & 0 & 0 & 0 \\ -1 & 1 & R_{n+1} & 0 & 0 \\ 0 & -1 & 1 & R_{n+2} & 0 \\ 0 & 0 & -1 & 1 & R_{n+3} \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix}.$$

§4 Determinant solution to GRP5

As a preliminary, "cross-ratio" and "Plücker relation" are necessary to obtain determinant solution to GRP5.

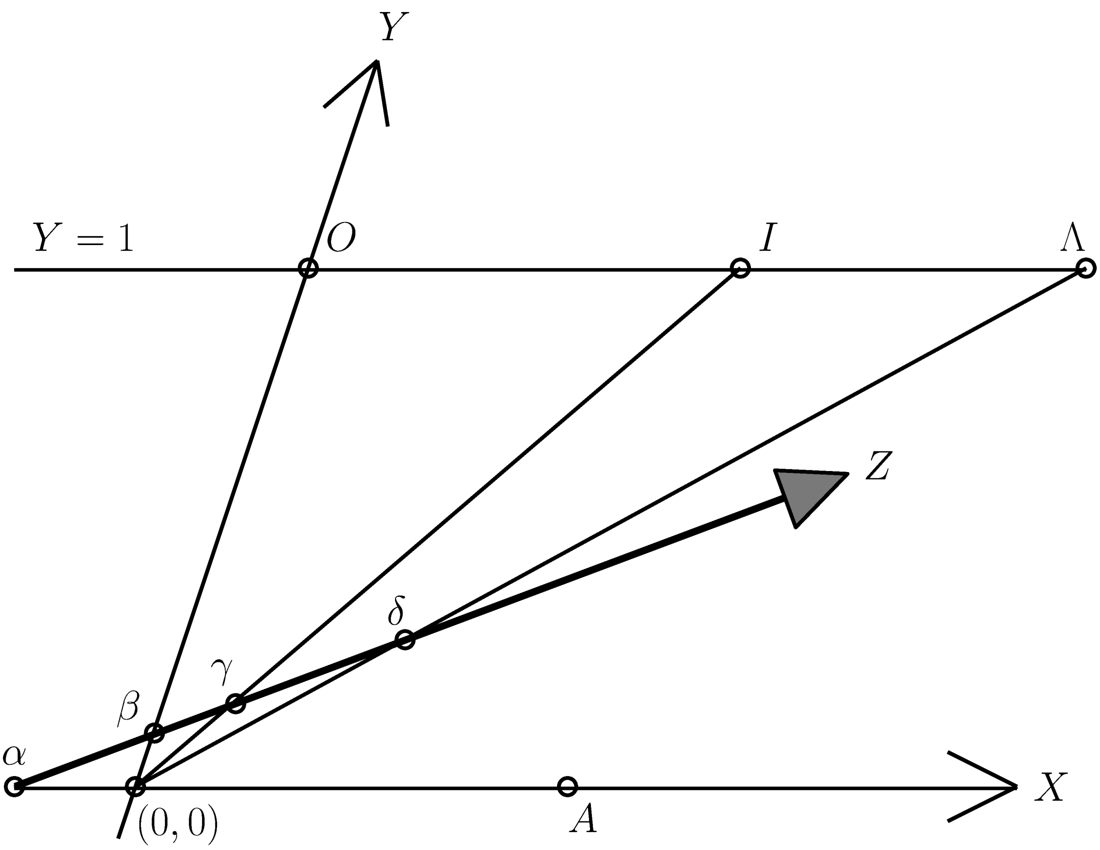
Let us recall the previous figure which explains "Ptolemy's theorem" and "cross-ratio" in the preceding section, and, by the fixed 3 points A, B, and C, let us set (X, Y) coordinate system as follows:



Further, let us introduce another projective line having a homogeneous coordinate

$$Z = (Z_1 : Z_2)$$

than the projective line $(Y = 1) \cup \{A\}$ with the homogeneous coordinate $(X : Y)$, as follows:



Then, by the projection from the origin

$$(X, Y) = (0, 0),$$

the 4 points A, O, I, and Λ correspond to respective 4 points on Z, as

$$\begin{cases} (A \mapsto) & A \mapsto \alpha = (\alpha_1 : \alpha_2), \\ (B \mapsto) & O \mapsto \beta = (\beta_1 : \beta_2), \\ (C \mapsto) & I \mapsto \gamma = (\gamma_1 : \gamma_2), \\ (D \mapsto) & \Lambda \mapsto \delta = (\delta_1 : \delta_2). \end{cases}$$

Here, it's well-known that "cross-ratio" λ is written in two forms:

1. $\lambda = \pm \dfrac{O\Lambda}{OI} = \pm \dfrac{AC \cdot BD}{AD \cdot BC}$,

   where $\begin{cases} + \ : \ D \in \text{ arc } AB \text{ including } C, \\ - \ : \ D \in \text{ arc } AB \text{ not including } C. \end{cases}$

$$2.\ \lambda = \frac{\det\begin{pmatrix}\alpha_1 & \gamma_1\\ \alpha_2 & \gamma_2\end{pmatrix}\cdot\det\begin{pmatrix}\beta_1 & \delta_1\\ \beta_2 & \delta_2\end{pmatrix}}{\det\begin{pmatrix}\alpha_1 & \delta_1\\ \alpha_2 & \delta_2\end{pmatrix}\cdot\det\begin{pmatrix}\beta_1 & \gamma_1\\ \beta_2 & \gamma_2\end{pmatrix}} = \frac{\left(\dfrac{\alpha_1}{\alpha_2}-\dfrac{\gamma_1}{\gamma_2}\right)\cdot\left(\dfrac{\beta_1}{\beta_2}-\dfrac{\delta_1}{\delta_2}\right)}{\left(\dfrac{\alpha_1}{\alpha_2}-\dfrac{\delta_1}{\delta_2}\right)\cdot\left(\dfrac{\beta_1}{\beta_2}-\dfrac{\gamma_1}{\gamma_2}\right)}.$$

The way to obtain the former form above has been seen already in the preceding section. We can obtain the latter form in determinant expression by appropriate projective transformation p which holds

$$\begin{cases} p((\alpha_1 : \alpha_2)) = (1:0),\\ p((\beta_1 : \beta_2)) = (0:1),\\ p((\gamma_1 : \gamma_2)) = (1:1),\\ p((\delta_1 : \delta_2)) = (\lambda:1),\end{cases}$$

where, actually λ is determined by the following matrix calculation:

$$\begin{pmatrix} 1 & 0\\ 0 & \dfrac{\begin{vmatrix}\gamma_1 & \beta_1\\ \gamma_2 & \beta_2\end{vmatrix}}{\begin{vmatrix}\gamma_1 & \alpha_1\\ \gamma_2 & \alpha_2\end{vmatrix}}\end{pmatrix}\begin{pmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{pmatrix}^{-1}\begin{pmatrix}\alpha_1 & \beta_1 & \gamma_1 & \delta_1\\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0 & 1 & \lambda\\ 0 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \dfrac{\begin{vmatrix}\gamma_1 & \beta_1\\ \gamma_2 & \beta_2\end{vmatrix}}{\begin{vmatrix}\gamma_1 & \alpha_1\\ \gamma_2 & \alpha_2\end{vmatrix}} & 0 & 0\\ 0 & 0 & \dfrac{\begin{vmatrix}\gamma_1 & \beta_1\\ \gamma_2 & \beta_2\end{vmatrix}}{\begin{vmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{vmatrix}} & 0\\ 0 & 0 & 0 & \dfrac{\begin{vmatrix}\gamma_1 & \beta_1\\ \gamma_2 & \beta_2\end{vmatrix}\cdot\begin{vmatrix}\delta_1 & \alpha_1\\ \delta_2 & \alpha_2\end{vmatrix}}{\begin{vmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{vmatrix}\cdot\begin{vmatrix}\gamma_1 & \alpha_1\\ \gamma_2 & \alpha_2\end{vmatrix}}\end{pmatrix}.$$

On the other hand, by "Euler formula" in §2, we have

$$\left(\frac{\alpha_1}{\alpha_2}-\frac{\delta_1}{\delta_2}\right)\left(\frac{\beta_1}{\beta_2}-\frac{\gamma_1}{\gamma_2}\right)+\left(\frac{\alpha_1}{\alpha_2}-\frac{\beta_1}{\beta_2}\right)\left(\frac{\gamma_1}{\gamma_2}-\frac{\delta_1}{\delta_2}\right)=\left(\frac{\alpha_1}{\alpha_2}-\frac{\gamma_1}{\gamma_2}\right)\left(\frac{\beta_1}{\beta_2}-\frac{\delta_1}{\delta_2}\right),$$

accordingly,

$$\underline{\begin{vmatrix} \alpha_1 & \delta_1 \\ \alpha_2 & \delta_2 \end{vmatrix} \cdot \begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \cdot \begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_2 & \delta_2 \end{vmatrix} = \begin{vmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{vmatrix} \cdot \begin{vmatrix} \beta_1 & \delta_1 \\ \beta_2 & \delta_2 \end{vmatrix}},$$

which is nothing but the "Plücker relation."

Now, let us improve the previous lattice on Möbius band in §3,

$$\begin{array}{ccccccccccccccccccccc}
BC & & & & CD & & & & DE & & & & EA & & & & AB & & & & BC \\
& \ddots & & \iddots & & \ddots & & \diagup & & \diagdown & & \diagup & & \diagdown & & \iddots & & \ddots & & \iddots & \\
u_2 & & BD & & u_3 & & CE & & u_4 & & DA & & u_5 & & EB & & u_1 & & AC & & u_2 \\
& \iddots & & \ddots & & \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup & & \diagdown & & \iddots & & \ddots & \\
DA & & u_5 & & EB & & u_1 & & AC & & u_2 & & BD & & u_3 & & CE & & u_4 & & DA \\
& \ddots & & \iddots & & \diagdown & & \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup & & \ddots & & \iddots & \\
& & EA & & & & AB & & & & BC & & & & CD & & & & DE & &
\end{array}$$

into the renewal one for Plücker relation, as follows:

$$\begin{array}{ccccccccccccccccccccc}
\tau_{\beta\gamma} & & & & \tau_{\gamma\delta} & & & & \tau_{\delta\varepsilon} & & & & \tau_{\varepsilon\alpha} & & & & \tau_{\alpha\beta} & & & & \tau_{\beta\gamma} \\
& \ddots & & \iddots & & \ddots & & \diagup & & \diagdown & & \diagup & & \diagdown & & \iddots & & \ddots & & \iddots & \\
u_2 & & \tau_{\delta\beta} & & u_3 & & \tau_{\varepsilon\gamma} & & u_4 & & \tau_{\alpha\delta} & & u_5 & & \tau_{\beta\varepsilon} & & u_1 & & \tau_{\gamma\alpha} & & u_2 \\
& \iddots & & \ddots & & \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup & & \diagdown & & \iddots & & \ddots & \\
\tau_{\alpha\delta} & & u_5 & & \tau_{\beta\varepsilon} & & u_1 & & \tau_{\gamma\alpha} & & u_2 & & \tau_{\delta\beta} & & u_3 & & \tau_{\varepsilon\gamma} & & u_4 & & \tau_{\alpha\delta} \\
& \ddots & & \iddots & & \diagdown & & \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup & & \ddots & & \iddots & \\
& & \tau_{\varepsilon\alpha} & & & & \tau_{\alpha\beta} & & & & \tau_{\beta\gamma} & & & & \tau_{\gamma\delta} & & & & \tau_{\delta\varepsilon} & &
\end{array}$$

where, the improved τ_s are

$$\tau_{\mu\nu} = \det \begin{pmatrix} \mu_1 & \nu_1 \\ \mu_2 & \nu_2 \end{pmatrix}, \text{ for } \mu, \nu \in \{\alpha, \beta, \gamma, \delta, \varepsilon\}.$$

Then, the Plücker relation says:

$$\begin{cases} \tau_{\alpha\delta} \cdot \tau_{\beta\gamma} + \tau_{\alpha\beta} \cdot \tau_{\gamma\delta} = \tau_{\gamma\alpha} \cdot \tau_{\delta\beta}, \\ \text{and its cyclic permutations } \alpha \to \beta \to \gamma \to \delta \to \varepsilon \to \alpha. \\ \text{(Total, 5 relations)} \end{cases}$$

Thus, we have the following results:

- By the 5 relations above, also from the improved τ_s in determinant form, we obtain

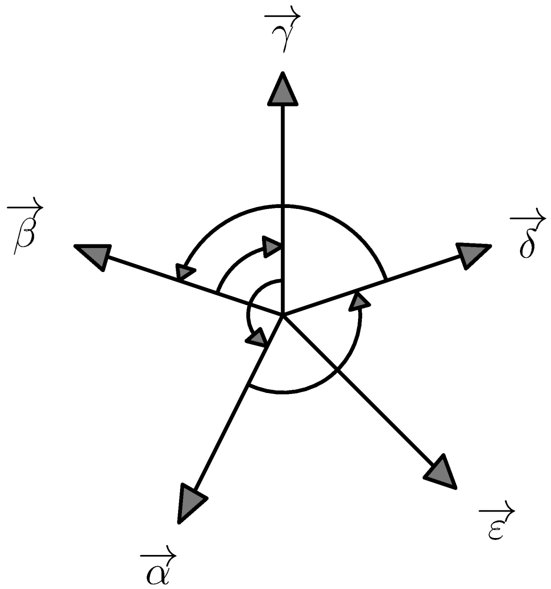
$$\frac{1}{u_n} + \frac{1}{u_{n-1} \cdot u_{n+1}} = 1,$$

for the index n : mod 5, which is the same equation in the preceding section. Therefore, the transformation $Q_n := \dfrac{-1}{u_n}$ solves GRP5, $Q_{n+1}Q_{n-1} = Q_n + 1$.

- The determinant τ_s gives not only negative solution but also positive solution $Q_n > 0$ to GRP5. Actually, using vector notation

$$\overrightarrow{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \overrightarrow{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \ldots, \overrightarrow{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix},$$

the following arrangement gives $Q_n > 0$:



Because, for instance,

$$\begin{aligned} u_2 &= \frac{\det(\overrightarrow{\gamma}, \overrightarrow{\alpha}) \cdot \det(\overrightarrow{\delta}, \overrightarrow{\beta})}{\det(\overrightarrow{\alpha}, \overrightarrow{\delta}) \cdot \det(\overrightarrow{\beta}, \overrightarrow{\gamma})} \\ &= \frac{(\text{ positive }) \cdot (\text{ positive })}{(\text{ positive }) \cdot (\text{ negative })}, \text{ etc.} \end{aligned}$$

§5 Nontrivial Bäcklund transformations for GRP5

Let us use variable u_n, in place of Q_n. Generally, "Bäcklund transformation (BT)" means a permutation of solutions to some given equation having time evolution. Hence, BT $u_n \mapsto u_n^\bullet$ (one-to-one correspondence) shall satisfy

$$\frac{1}{u_n} + \frac{1}{u_{n-1}u_{n+1}} = 1 \quad \Rightarrow \quad \frac{1}{u_n^\bullet} + \frac{1}{u_{n-1}^\bullet u_{n+1}^\bullet} = 1.$$

In §2, we saw

$$\tau_1 \cdot \tau_{123}^\bullet = \tau_{12} \cdot \tau_{23}^\bullet = \tau_{123} \cdot \tau_3^\bullet,$$

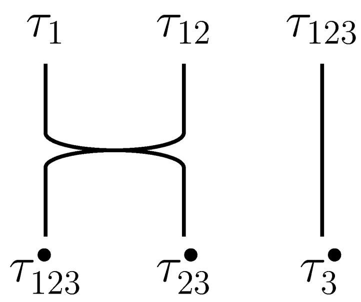
which yields the "shift" $u_n^\bullet = u_{n+1}$. It's a trivial but obvious BT.

Here, let us exchange the pairing among τ_s and $\tau_s^\bullet$. For example, consider

$$\tau_1 \cdot \tau_{23}^\bullet = \tau_{12} \cdot \tau_{123}^\bullet = \tau_{123} \cdot \tau_3^\bullet.$$

Then, by "rhombic rule," we have some transformation $u_n \mapsto u_n^\bullet$. Such a transformation becomes to BT! (But, why?) Let us see details, below.

Case 1: Pairing

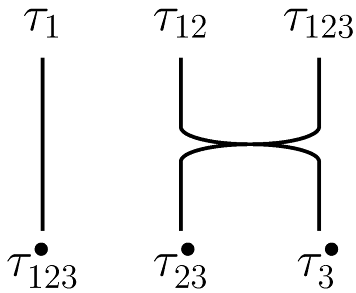


makes $\tau_1 \cdot \tau_{23}^\bullet = \tau_{12} \cdot \tau_{123}^\bullet = \tau_{123} \cdot \tau_3^\bullet$, which becomes to BT of period 4:

$$u_1^\bullet = 1 - u_2, \;\; u_2^\bullet = \frac{1}{u_3}, \;\; u_3^\bullet = \frac{u_5}{u_5 - 1}, \;\; u_4^\bullet = \frac{u_4}{u_4 - 1}, \;\; u_5^\bullet = \frac{1}{u_1}.$$

Here, 'period 4' means $u_n^{\bullet\bullet\bullet\bullet} = u_n$. (Why periodic? It's mysterious.)

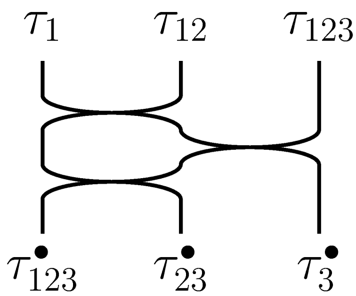
Case 2: Pairing



makes $\tau_1 \cdot \tau^\bullet_{123} = \tau_{12} \cdot \tau^\bullet_3 = \tau_{123} \cdot \tau^\bullet_{23}$, which becomes to $\underline{\text{BT of period 4}}$:

$$u^\bullet_1 = \frac{1}{u_2},\ \ u^\bullet_2 = 1 - u_3,\ \ u^\bullet_3 = \frac{1}{u_4},\ \ u^\bullet_4 = \frac{u_1}{u_1 - 1},\ \ u^\bullet_5 = \frac{u_5}{u_5 - 1}.$$

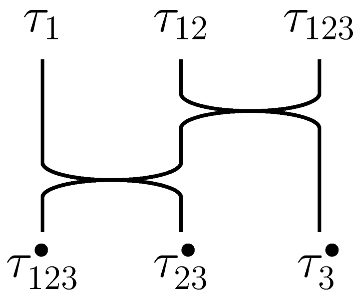
Case 3: Pairing



makes $\tau_1 \cdot \tau^\bullet_3 = \tau_{12} \cdot \tau^\bullet_{23} = \tau_{123} \cdot \tau^\bullet_{123}$, which becomes to $\underline{\text{BT of period 6}}$:

$$u^\bullet_1 = \frac{u_2}{u_2 - 1},\ \ u^\bullet_2 = \frac{u_3}{u_3 - 1},\ \ u^\bullet_3 = \frac{1}{u_1},\ \ u^\bullet_4 = 1 - u_5,\ \ u^\bullet_5 = \frac{1}{u_4}.$$

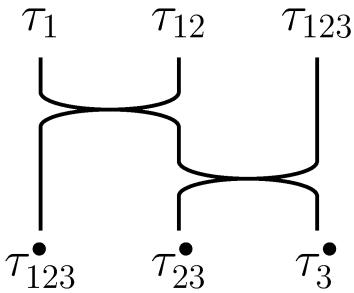
Case 4: Pairing



makes $\tau_1 \cdot \tau^\bullet_{23} = \tau_{12} \cdot \tau^\bullet_3 = \tau_{123} \cdot \tau^\bullet_{123}$, which becomes to $\underline{\text{BT of period 3}}$:

$$u^\bullet_1 = 1 - \frac{1}{u_2},\ \ u^\bullet_2 = \frac{1}{1 - u_3},\ \ u^\bullet_3 = u_1,\ \ u^\bullet_4 = \frac{1}{1 - u_4},\ \ u^\bullet_5 = 1 - \frac{1}{u_5}.$$

Case 5: Pairing



makes $\tau_1 \cdot \tau_3^\bullet = \tau_{12} \cdot \tau_{123}^\bullet = \tau_{123} \cdot \tau_{23}^\bullet$, which becomes to BT of period 5:

$$u_1^\bullet = \frac{1}{1-u_2}, \quad u_2^\bullet = 1 - \frac{1}{u_3}, \quad u_3^\bullet = 1 - \frac{1}{u_5}, \quad u_4^\bullet = \frac{1}{1-u_1}, \quad u_5^\bullet = u_4.$$

Comment for §5: Using so-called "j-invariant,"

$$j(\lambda) := \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2},$$

we can make invariants for BTs. Actually, all symmetric polynomials of $j(u_n)$ $(n = 1, 2, \ldots, 5)$ are invariants for all 6 BTs $u_n \mapsto u_n^\bullet$ we have obtained.

§6 Conclusion

Gauss's recurrence of period 5, $Q_{n+1}Q_{n-1} = Q_n + 1$, has

- geometric solution, determinant solution,
- higher order versions, tri-diagonal determinant form, conserved quantity of tri-diagonal determinant,
- geometric Bäcklund transformation, nontrivial Bäcklund transformations, those higher resemblances,
- etc.

By Euler formula, Ptolemy's theorem, or Plücker relation, local cross-ratios combine Hirota's τ-variables into global lattice on Möbius band, which generates periodicity of the gauss's recurrence.

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