RIMS Kôkyûroku Bessatsu
B30 (2012), 181–208

Higher order Painlevé systems of type A, Drinfeld-Sokolov hierarchies and Fuchsian systems

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Abstract

Recently, higher order generalizations of P_{VI} has been studied from two viewpoints, similarity reductions of infinite dimensional integrable hierarchies and monodromy preserving deformations of Fuchsian systems. The aim of this article is to clarify the relationship between them.

1 Introduction

The connection between the second Painlevé equation and the KdV equation was clarified by Ablowitz and Segur [2]. Since their result, a relationship between (higher order) Painlevé systems and infinite-dimensional integrable hierarchies has been studied. By means of a viewpoint of the Drinfeld-Sokolov hierarchies [3, 6], we list the known connections between Painlevé systems and integrable hierarchies in Table 1 and 2.

Table 1. Painlevé equations and DS hierarchy

Painlevé eq.	P_{II}	P_{IV}		P_{V}		P_{VI}		P_{VI}
Lie alg.	$A_1^{(1)}$	$A_1^{(1)}$	$A_2^{(1)}$	$A_2^{(1)}$	$A_3^{(1)}$	$A_2^{(1)}$	$A_3^{(1)}$	$D_4^{(1)}$
Conj. class	(2)	$(1,1)$	(3)	$(2,1)$	(4)	$(1,1,1)$	$(2,2)$	$(\overline{2},\overline{2})$
Ref.	[2]	[14]	[1]	[13]	[1]	[15]	[5]	[4]

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Table 2. $2n$-th order Painlevé systems and DS hierarchy

Painlevé sys.	Coupled P_{IV}	Coupled P_{V}	Coupled P_{VI}	Coupled P_{VI}
Lie alg.	$A^{(1)}_{2n-1}, A^{(1)}_{2n}$	$A^{(1)}_{2n}, A^{(1)}_{2n+1}$	$A^{(1)}_{2n}, A^{(1)}_{2n+1}$	$D^{(1)}_{2n+2}$
Conj. class	$(2n-1,1)$ $(2n+1)$	$(2n,1)$ $(2n+2)$	$(n,n,1)$ $(n+1,n+1)$	$(\overline{n+1},\overline{n+1})$
Ref.	[5, 18, 20]	[5, 18, 20]	[5, 20]	[4]

Remark 1.1 ([6, 17])**.** *The Drinfeld-Sokolov hierarchies are characterized by the Heisenberg subalgebras of the affine Lie algebras. And the isomorphism classes of the Heisenberg subalgebras are in one-to-one correspondence with the conjugacy classes of the finite Weyl group.*

Remark 1.2 ([24])**.** *The higher order Painlevé system of type $D^{(1)}_{2n+2}$ was first proposed by Sasano as an extension of P_{VI} for the affine Weyl group symmetry with the aid of algebraic geometry for initial value space.*

Remark 1.3 ([25, 26])**.** *Several Painlevé systems are also derived from the UC hierarchy, which is an extension of the KP hierarchy to the universal character.*

On the other hand, a classification of higher order Painlevé systems has been studied from a viewpoint of the monodromy preserving deformations of Fuchsian systems. It is shown in [19] that any irreducible Fuchsian system can be reduced to finite types of systems by using Katz's two operations, addition and middle convolution [11]. It is also shown in [8] that the isomonodromy deformation equation is invariant under Katz's two operations. Based on them, Sakai constructed a classification theory of four-dimensional Painlevé equations [23]. We list the known connections between Painlevé systems and Fuchsian systems in Table 3.

Table 3. $2n$-th order Painlevé systems and Fuchsian systems

Painlevé sys.	Spectral type	Ref.
Coupled P_{VI} of type $A^{(1)}_{2n+1}$	$(n,1),(n,1),(1^{n+1}),(1^{n+1})$	[23, 26]
Coupled P_{VI} of type $D^{(1)}_{2n+2}$	$(2n-1,1),(n^2),(n^2),(1^{2n})$	[9, 23]
Matrix type	$(n^2),(n^2),(n^2),(n,n-1,1)$	[12, 23]

As is seen above, Painlevé systems have two origins, integrable hierarchies and Fuchsian systems. But the relationship between them has not been clarified. In this article, we consider the coupled Painlevé VI system of type $A^{(1)}_{2n+1}$; we denote it by $P_{(n+1,n+1)}$. It is derived from two origins, the Drinfeld-Sokolov hierarchy associated with the conjugacy class $(n+1, n+1)$ and the Fuchsian system with the spectral type $\{(n,1),(n,1),(1^{n+1}),(1^{n+1})\}$. We investigate a relationship between them with the aid of a Laplace transformation for a system of linear differential equations.

The system $P_{(n+1,n+1)}$ was proposed in [20] and [26] independently. It is expressed as a Hamiltonian system of order $2n$ with the coupled Painlevé VI Hamiltonian. It also admits an affine Weyl group symmetry of type $A^{(1)}_{2n+1}$ and a particular solution in terms of the generalized hypergeometric function ${}_{n+1}F_n$ [21, 27]. In a recent work [22], we introduce a system of q-difference equations q-$P_{(n+1,n+1)}$, which has similar properties as $P_{(n+1,n+1)}$ and reduces to $P_{(n+1,n+1)}$ via a continuous limit $q \to 1$. In this article, we investigate such q-difference system. Namely, we show that the Lax form for q-$P_{(n,n)}$ is transformed into a system of linear q-difference equations which reduces to a Fuchsian system via a continuous limit $q \to 1$; we call it *a q-Fuchsian system.*

This article is organized as follows. In Section 2, we review the system q-$P_{(n,n)}$ and its several properties. In section 3, a q-Fuchsian system is derived from the Lax form for q-$P_{(n,n)}$. In section 4, we investigate Lax forms for four-dimensional Painlevé differential equations derived from the Drinfeld-Sokolov hierarchy of type A.

Remark 1.4 ([15])**.** *The Garnier system in m-variables is an extension of P_{VI} to independent variables of number m. It is also derived from two origins, the Drinfeld-Sokolov hierarchy associated with the conjugacy class (1^{m+2}) and the Fuchsian system with the spectral type $\{(1^2)\times(m+3)\}$. The relation between them has been already clarified by Kakei and Kikuchi.*

Remark 1.5 ([26])**.** *An extension of the Garnier system was proposed by Tsuda. He considered a similarity reduction of a UC hierarchy, whose Lax form is equivalent to a Fuchsian system with the spectral type $\{(n,1)\times(m+1),(1^{n+1})\times 2\}$, and derived a Hamiltonian system of order $2mn$. In the case $m=1$, the Hamiltonian system is equivalent to $P_{(n+1,n+1)}$.*

2 Higher order q-Painlevé system

The main object of this article is the higher order q-Painlevé system q-$P_{(n,n)}$ proposed in [22]. It is described as

$$\begin{aligned} x_i(t)-x_{i-1}(t)&=\frac{a_ix_i(qt)}{1+x_i(qt)y_{i-1}(t)}-\frac{b_{i-1}x_{i-1}(qt)}{1+x_{i-1}(qt)y_{i-1}(t)},\\ y_i(qt)-y_{i-1}(qt)&=\frac{b_iy_i(t)}{1+x_i(qt)y_i(t)}-\frac{a_iy_{i-1}(t)}{1+x_i(qt)y_{i-1}(t)}, \end{aligned} \tag{2.1}$$

for $i=1,\dots,n$, where

$$b_0=q^{-1}b_n,\quad x_0(t)=tx_n(t),\quad y_0(t)=q^{-1}t^{-1}y_n(t),$$

with relations

$$\prod_{i=1}^n a_i\frac{1+x_i(qt)y_i(t)}{1+x_i(qt)y_{i-1}(t)}=q^{(n-1)/2}.$$

The system q-$P_{(2,2)}$ coincides with q-$P_{\rm VI}$ proposed by Jimbo and Sakai in [10]. Hence we can regard q-$P_{(n,n)}$ as a generalization of q-$P_{\rm VI}$.

The system q-$P_{(n,n)}$ admits the affine Weyl group symmetry of type $A^{(1)}_{2n-1}$. Let r_j $(j=0,\dots,2n-1)$ be birational transformations defined by

$$\begin{aligned} &r_{2j-2}(a_j)=b_{j-1},\quad r_{2j-2}(b_{j-1})=a_j,\\ &r_{2j-2}(x_{j-1}(t))=x_{j-1}(t),\quad r_{2j-2}(y_{j-1}(y))=y_{j-1}(t)+\frac{b_{j-1}-a_j}{x_j(t)-x_{j-1}(t)},\\ &r_{2j-2}(a_i)=a_i,\quad r_{2j-2}(b_{i-1})=b_{i-1},\\ &r_{2j-2}(x_{i-1}(t))=x_{i-1}(t),\quad r_{2j-2}(y_{i-1}(y))=y_{i-1}(t)\quad (i\neq j), \end{aligned}$$

and

$$\begin{aligned} &r_{2j-1}(a_j)=b_j\quad r_{2j-1}(b_j)=a_j,\\ &r_{2j-1}(x_j(t))=x_j(t)+\frac{a_j-b_j}{y_j(t)-y_{j-1}(t)},\quad r_{2j-1}(y_j(t))=y_j(t),\\ &r_{2j-1}(a_i)=a_i,\quad r_{2j-1}(b_i)=b_i,\\ &r_{2j-1}(x_i(t))=x_i(t),\quad r_{2j-1}(y_i(y))=y_i(t)\quad (i\neq j). \end{aligned}$$

Then we have

Theorem 2.1 ([22])**.** *The system q-$P_{(n,n)}$ is invariant under actions of the transformations $r_0,\dots,r_{2n-1}$. And the group of symmetries $\langle r_0,\dots,r_{2n-1}\rangle$ is isomorphic to the affine Weyl group of type $A^{(1)}_{2n-1}$.*

The system q-$P_{(n,n)}$ also admits a particular solution in terms of the q-hypergeometric function ${}_n\phi_{n-1}$.

Theorem 2.2 ([22]). *Under the system q-$P_{(n,n)}$, we consider a specialization*

$$y_j(t) = 0 \quad (j = 1, \dots, n), \quad \prod_{j=1}^{n} a_j = q^{(n-1)/2}.$$

Then a vector of the variables $\mathbf{x}(t) = {}^t[x_1(t), \dots, x_n(t)]$ satisfies a system of linear q-difference equations

$$\mathbf{x}(q^{-1}t) = \left(A_0 + \frac{A_1}{1 - q^{-1}t}\right)\mathbf{x}(t),$$

with $n \times n$ matrices

$$A_0 = \sum_{j=1}^{n} b_j E_{j,j} + \sum_{i=1}^{n}\sum_{j=i+1}^{n}(b_j - a_j)E_{i,j}, \quad A_1 = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_j - b_j)E_{i,j}.$$

The aim of this article is to investigate Lax forms for q-$P_{(n,n)}$. Let $M_0(z,t)$ and $B_0(z,t)$ be $2n \times 2n$ matrices defined by

$$M_0(z,t) = \begin{bmatrix} \kappa_1 & \varphi_1 & -1 & & & & & \\ & \kappa_2 & \varphi_2 & -1 & & & & \\ & & & \ddots & & & & \\ & & & & \varphi_{2n-3} & -1 & & \\ & & & & \kappa_{2n-2} & \varphi_{2n-2} & -1 & \\ -tz & & & & & \kappa_{2n-1} & \varphi_{2n-1} & \\ \varphi_0 z & -z & & & & & \kappa_{2n} & \end{bmatrix},$$

and

$$B_0(z,t) = \begin{bmatrix} u_1 & v_1 & -1 & & & & & & \\ & u_2 & v_2 & 0 & & & & & \\ & & u_3 & v_3 & -1 & & & & \\ & & & u_4 & v_4 & 0 & & & \\ & & & & & \ddots & & & \\ & & & & & & v_{2n-3} & -1 & \\ & & & & & & u_{2n-2} & v_{2n-2} & 0 \\ -tz & & & & & & & u_{2n-1} & v_{2n-1} \\ v_0 z & 0 & & & & & & & u_{2n} \end{bmatrix},$$

where

$$\begin{aligned}
&\kappa_{2i-1} = a_i, \quad \kappa_{2i} = b_i \quad (i = 1, \dots, n),\\
&\varphi_{2i-2} = x_i(t) - x_{i-1}(t), \quad \varphi_{2i-1} = y_i(t) - y_{i-1}(t) \quad (i = 1, \dots, n),\\
&u_{2i-1} = \frac{a_i}{1 + x_i(qt)y_{i-1}(t)}, \quad u_{2i} = 1 + x_i(qt)y_i(t) \quad (i = 1, \dots, n)\\
&v_0 = -tx_n(qt), \quad v_{2i} = -x_i(qt) \quad (i = 1, \dots, n-1),\\
&v_{2i-1} = y_i(t) \quad (i = 1, \dots, n).
\end{aligned}$$

Then we have

Theorem 2.3 ([22])**.** *The system q-$P_{(n,n)}$ is given as the compatibility conditon of a system of linear q-difference equations*

$$\Psi_0(qz, t) = M_0(z, t)\Psi_0(z, t), \quad \Psi_0(z, qt) = B_0(z, t)\Psi_0(z, t). \tag{2.2}$$

In the next section, we show that the system (2.2) reduces to a q-Fuchsian system with $n \times n$ matrices; see Theorem 3.6. The following lemma is needed to prove the main theorem.

Lemma 2.4. *We have*

$$\begin{aligned}
\det M_0(z, t) &= (q^{(n-1)/2} - tz)(q^{-(n-1)/2}\kappa_1 \dots \kappa_{2n} - z),\\
\det B_0(z, t) &= q^{(n-1)/2} - tz.
\end{aligned}$$

Proof. The determinant of the matrix $B_0(z, t)$ is obtained as follows:

$$\det B_0(z, t) = \prod_{i=1}^{2n} u_i - z(v_{2n-1}v_0 + tu_{2n}) \prod_{i=1}^{n-1}(v_{2i-1}v_{2i} + u_{2i}) = q^{(n-1)/2} - tz.$$

On the other hand, we set $B_0^c(z, t) = M_0(z, t)B_0^{-1}(z, t)$. It is given explicitly by

$$B_0^c(z, t) = \begin{bmatrix}
u_1^c & v_1^c & 0 & & & & & & \\
 & u_2^c & v_2^c & -1 & & & & & \\
 & & u_3^c & v_3^c & 0 & & & & \\
 & & & u_4^c & v_4^c & -1 & & & \\
 & & & & & \ddots & & & \\
 & & & & & & v_{2n-3}^c & 0 & \\
 & & & & & & u_{2n-2}^c & v_{2n-2}^c & -1\\
0 & & & & & & & u_{2n-1}^c & v_{2n-1}^c\\
v_0^c z & -z & & & & & & & u_{2n}
\end{bmatrix},$$

where

$$u^c_{2i-1} = 1 + x_i(qt)y_{i-1}(t), \quad u^c_{2i} = \frac{b_i}{1 + x_i(qt)y_i(t)} \quad (i = 1, \dots, n)$$
$$v^c_{2i-2} = x_i(qt), \quad v^c_{2i-1} = -y_{i-1}(t) \quad (i = 1, \dots, n).$$

Similary as above, we obtain

$$\det B^c_0(z,t) = q^{-(n-1)/2}\kappa_1 \dots \kappa_{2n} - z.$$

Hence this lemma is proved. □

3 q-Fuchsian system

In this section, we show that the Lax form (2.2) is transformed into a q-Fuchsian system with the aid of q-Laplace transformations (cf. [7, 16]). In order to achive it, we propose Lax forms

$$(L_j): \quad \Psi_j(qz,t) = M_j(z,t)\Psi_j(z,t), \quad \Psi_j(z,qt) = B_j(z,t)\Psi_j(z,t),$$

with $(2n-j)\times(2n-j)$ matrices for $j = 0, 1, \dots, n-1$ and

$$(L_n): \quad \Psi_n(q^{-1}z,t) = M_n(z,t)\Psi_n(z,t), \quad \Psi_n(z,qt) = B_n(z,t)\Psi_n(z,t),$$

with $n \times n$ matrix. Then the system L_n is equivalent to our purpose.

The systems $L_1, \dots, L_{n-1}$ are obtained as follows. For each $j = 1, \dots, n-1$, we can take a function $\tau_j(z,t)$ satisfying

$$\frac{\tau_j(qz,t)}{\tau_j(z,t)} = \frac{\kappa_{j-1}}{q\kappa_j}, \quad \frac{\tau_j(z,qt)}{\tau_j(z,t)} = \frac{u_{j-1}}{u_j}, \quad \kappa_0 = u_0 = 1,$$

under the system L_{j-1}. By using it, we consider a gauge transformation

$$\Psi_{j-1}(z,t) \to \frac{1}{\tau_j(z,t)}\Psi^*_{j-1}(z,t).$$

We next consider a q-Laplace transformation

$$z\Psi^*_{j-1}(z,t) \to \frac{\Phi_{j-1}(\zeta,t) - \Phi_{j-1}(q^{-1}\zeta,t)}{\varepsilon\zeta},$$
$$\Psi^*_{j-1}(qz,t) \to q^{-1}\Phi_{j-1}(q^{-1}\zeta,t).$$

where $\varepsilon = 1 - q$. Via a Möbius transformation $\zeta \to z^{-1}$, the system L_{j-1} is transformed into

$$\Phi_{j-1}(qz,t) = N_{j-1}(z,t)\Phi_{j-1}(z,t), \quad \Phi_{j-1}(z,qt) = C_{j-1}(z,t)\Phi_{j-1}(z,t), \tag{3.1}$$

with

$$\begin{aligned}N_{j-1}(z,t) &= (I+\varepsilon^{-1}qzM^*_{j-1,1}(t))^{-1}(qM^*_{j-1,0}(t)+\varepsilon^{-1}qzM^*_{j-1,1}(t))\\ &= qM^*_{j-1,0}(t)+\varepsilon^{-1}qzM^*_{j-1,1}(t)(I-qM^*_{j-1,0}(t)),\\ C_{j-1}(z,t) &= B^*_{j-1,0}(t)+\varepsilon^{-1}zB^*_{j-1,1}(t)(I-N_{j-1}(z,t))\\ &= B^*_{j-1,0}(t)+\varepsilon^{-1}zB^*_{j-1,1}(t)(I-qM^*_{j-1,0}(t)),\end{aligned}$$

where

$$\begin{aligned}\frac{\kappa_{j-1}}{q\kappa_j}M_{j-1}(z,t) &= M^*_{j-1,0}(t)+zM^*_{j-1,1}(t),\\ \frac{u_{j-1}}{u_j}B_{j-1}(z,t) &= B^*_{j-1,0}(t)+zB^*_{j-1,1}(t).\end{aligned}$$

Note that

$$(M^*_{j-1,1}(t))^2=O,\quad B^*_{j-1,1}(t)M^*_{j-1,1}(t)=O.$$

For each of the matrices $N_{j-1}(z,t)$ and $C_{j-1}(z,t)$, the first column is equivalent to the fundamental vector ${}^t[1,0,\dots,0]$. Hence the system (3.1) can be restricted to L_j.

The coefficient matrices $M_j(z,t)=M_{j,0}(t)+zM_{j,1}(t)$ are of the form

$$M_{j,0}(t)=\frac{1}{\kappa_j}\begin{bmatrix}\kappa_{j+1} & \varphi_{j+1} & -1 & & & O\\ & \kappa_{j+2} & \varphi_{j+2} & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \kappa_{2n-2} & \varphi_{2n-2} & -1\\ & & & & \kappa_{2n-1} & \varphi_{2n-1}\\ O & & & & & \kappa_{2n}\end{bmatrix},$$

and

$$M_{j,1}(t)=\frac{1}{\varepsilon^j\kappa_1\dots\kappa_{j-1}\kappa_j^2}\begin{bmatrix} & & & 0 & & & \\ & O & & \vdots & & O & \\ & & & 0 & & & \\ m^{(j)}_{1,1} & \dots & m^{(j)}_{1,j+1} & 0 & 0 & \dots & 0\\ m^{(j)}_{2,1} & \dots & m^{(j)}_{2,j+1} & m^{(j)}_{2,j+2} & 0 & \dots & 0\end{bmatrix},$$

for $j=1,\dots,n-1$. The components are given by the following recurrence relations:

$$\begin{aligned}&m^{(j)}_{1,i}=m^{(j-1)}_{1,i-1}-\varphi_{j+i-1}m^{(j-1)}_{1,i}+(\kappa_j-\kappa_{j+i})m^{(j-1)}_{1,i+1}\quad(i=1,\dots,j+1),\\ &m^{(0)}_{1,1}=-t,\quad m^{(j-1)}_{1,0}=m^{(j-1)}_{1,j+1}=m^{(j-1)}_{1,j+2}=0,\end{aligned}$$

and

$$m_{2,i}^{(j)} = m_{2,i-1}^{(j-1)} - \varphi_{j+i-1} m_{2,i}^{(j-1)} + (\kappa_j - \kappa_{j+i}) m_{2,i+1}^{(j-1)} \quad (i = 1, \ldots, j+2),$$
$$m_{2,1}^{(0)} = \varphi_0, \quad m_{2,2}^{(0)} = -1, \quad m_{2,0}^{(j-1)} = m_{2,j+2}^{(j-1)} = m_{2,j+3}^{(j-1)} = 0,$$

for $j = 1, \ldots, n-1$. Note that $m_{1,j+1}^{(j)} = -t$ and $m_{2,j+2}^{(j)} = -1$. The coefficient matrices $B_j(z,t) = B_{j,0}(t) + zB_{j,1}(t)$ are of the form

$$B_{j,0}(t) = \frac{1}{u_j} \begin{bmatrix} u_{j+1} & v_{j+1} & \frac{-1+(-1)^{j+1}}{2} & & & & O \\ & u_{j+2} & v_{j+2} & \frac{-1+(-1)^{j+2}}{2} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & u_{2n-3} & v_{2n-3} & -1 & \\ & & & & u_{2n-2} & v_{2n-2} & 0 \\ & & & & & u_{2n-1} & v_{2n-1} \\ O & & & & & & u_{2n} \end{bmatrix},$$

and

$$B_{j,1}(t) = \frac{1}{\varepsilon^j \kappa_1 \ldots \kappa_j u_j} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -t^{-1} v_0 \end{bmatrix} \begin{bmatrix} m_{1,1}^{(j)} & \ldots & m_{1,j+1}^{(j)} & 0 \ldots & 0 \end{bmatrix},$$

for $j = 1, \ldots, n-1$.

Lemma 3.1. *We have*

$$\det M_j(z,t) = \frac{(q^{(n-1)/2} - \varepsilon^{-j} tz)(q^{-(n-1)/2} \kappa_1 \ldots \kappa_{2n} - \varepsilon^{-j} z)}{\kappa_1 \ldots \kappa_{j-1} \kappa_j^{2n-j+1}},$$

for $j = 1, \ldots, n-1$.

Proof. We can prove by using Lemma 2.4 and

$$\det M_j(z,t) = \det N_{j-1}(z,t) = (\frac{\kappa_{j-1}}{\kappa_j})^{2n-j+1} \det M_{j-1}(\varepsilon^{-1} z, t).$$

□

Lemma 3.2. *We have*

$$\det B_j(z,t) = \frac{q^{(n-1)/2} - \varepsilon^{-j} tz}{u_1 \ldots u_{j-1} u_j^{2n-j+1}},$$

for $j = 1, \ldots, n-1$.

Proof. Let

$$\Delta_j = \begin{vmatrix} u_{j+1} & v_{j+1} & \frac{-1+(-1)^{j+1}}{2} & & & O \\ & u_{j+2} & v_{j+2} & \frac{-1+(-1)^{j+2}}{2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & u_{2j+1} & v_{2j+1} & -1 \\ & & & & u_{2j+2} & v_{2j+2} \\ m^{(j)}_{1,1} & m^{(j)}_{1,2} & \dots & \dots & m^{(j)}_{1,j} & m^{(j)}_{1,j+1} \end{vmatrix},$$

for $j = 1, \dots, n-1$. Then we can show that

$$\Delta_1 = -\frac{\kappa_1}{u_1} t, \quad \Delta_j = \frac{\kappa_j}{u_j}\Delta_{j-1} = -\frac{\kappa_1 \dots \kappa_j}{u_1 \dots u_j} t \quad (j = 2, \dots, n-1),$$

by a direct computation. By using it, we obtain

$$\begin{aligned} & u_j^{2n-j} \det B_j(z,t) \\ &= \begin{vmatrix} u_{j+1} & v_{j+1} & \frac{-1+(-1)^{j+1}}{2} & & & & & O \\ & \ddots & \ddots & \ddots & & & & \\ & & u_{2j+1} & v_{2j+1} & -1 & & & \\ & & & u_{2j+2} & v_{2j+2} & 0 & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & u_{2n-2} & v_{2n-2} & 0 \\ & & & & & & u_{2n-1} & \frac{t}{v_0} \\ -\frac{v_0 m^{(j)}_{1,1} z}{\varepsilon^j \kappa_1 \dots \kappa_j t} & \dots & -\frac{v_0 m^{(j)}_{1,j+1} z}{\varepsilon^j \kappa_1 \dots \kappa_j t} & 0 & \dots & \dots & 0 & u_{2n} \end{vmatrix} \\ &= u_{j+1} \dots u_{2n} + \frac{z}{\varepsilon^j \kappa_1 \dots \kappa_j} \Delta_j \prod_{i=j+1}^{n-2} (v_{2i+1} v_{2i+2} + u_{2i+2}) \\ &= \frac{q^{(n-1)/2} - \varepsilon^{-j} t z}{u_1 \dots u_{j-1} u_j}. \end{aligned}$$

□

The system L_n is obtained in a similar way. We can take a function $\tau_n(z,t)$ satisfying

$$\frac{\tau_n(qz,t)}{\tau_n(z,t)} = \frac{\kappa_{n-1}}{q\kappa_n}, \quad \frac{\tau_n(z,qt)}{\tau_n(z,t)} = \frac{u_{n-1}}{u_n},$$

under the system L_{n-1}. By using it, we consider a gauge transformation

$$\Psi_{n-1}(z,t) \to \frac{1}{\tau_n(z,t)} \Psi^*_{n-1}(z,t).$$

We also consider a q-Laplace transformation

$$z\Psi^*_{n-1}(z,t) \to \frac{\Phi_{n-1}(\zeta,t)-\Phi_{n-1}(q^{-1}\zeta,t)}{\varepsilon\zeta},$$
$$\Psi^*_{n-1}(qz,t) \to q^{-1}\Phi_{n-1}(q^{-1}\zeta,t).$$

Then the system L_{n-1} is transformed into

$$\Phi_{n-1}(q^{-1}\zeta,t) = N_{n-1}(\zeta,t)\Phi_{n-1}(\zeta,t), \quad \Phi_{n-1}(\zeta,qt) = C_{n-1}(\zeta,t)\Phi_{n-1}(\zeta,t), \tag{3.2}$$

with

$$\begin{aligned} N_{n-1}(\zeta,t) &= (I+\varepsilon^{-1}q\zeta^{-1}M^*_{n-1,1}(t))^{-1}(qM^*_{n-1,0}(t)+\varepsilon^{-1}q\zeta^{-1}M^*_{n-1,1}(t)) \\ &= I+(I+\varepsilon^{-1}q\zeta^{-1}M^*_{n-1,1}(t))^{-1}(qM^*_{n-1,0}(t)-I), \\ C_{n-1}(\zeta,t) &= B^*_{n-1,0}(t)+\varepsilon^{-1}\zeta^{-1}B^*_{n-1,1}(t)(I-N_{n-1}(\zeta,t)), \end{aligned}$$

where

$$\frac{\kappa_{n-1}}{q\kappa_n}M_{n-1}(z,t) = M^*_{n-1,0}(t)+zM^*_{n-1,1}(t),$$
$$\frac{u_{n-1}}{u_n}B_{n-1}(z,t) = B^*_{n-1,0}(t)+zB^*_{n-1,1}(t).$$

For each of the matrices $N_{n-1}(z,t)$ and $C_{n-1}(z,t)$, the first column is equivalent to the fundamental vector ${}^t[1,0,\dots,0]$. Hence the system (3.2) can be restricted to L_n via a transformation of independent variable

$$\zeta \to \frac{z}{\varepsilon^n\kappa_1\dots\kappa_n}.$$

Note that

$$\begin{aligned} &(I+\varepsilon^{-1}q\zeta^{-1}M^*_{n-1,1}(t))^{-1}\Big|_{\zeta\to\frac{z}{\varepsilon^n\kappa_1\dots\kappa_n}} \\ &= I+\frac{1}{z-t}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ \frac{m^{(n-1)}_{2,n}}{t-1} \end{bmatrix}\begin{bmatrix} m^{(n-1)}_{1,1} & \dots & m^{(n-1)}_{1,n} & 0 \end{bmatrix} \\ &\quad +\frac{1}{z-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}\begin{bmatrix} m^{(n-1)}_{2,1}-\frac{m^{(n-1)}_{1,1}m^{(n-1)}_{2,n}}{1-t} & \dots & m^{(n-1)}_{2,n}-\frac{m^{(n-1)}_{1,n}m^{(n-1)}_{2,n}}{1-t} & m^{(n-1)}_{2,n+1} \end{bmatrix}, \end{aligned}$$

and

$$
\begin{aligned}
&qM^*_{n-1,0}(t)-I \\
&=\frac{1}{\kappa_n}\begin{bmatrix}
0 & \varphi_n & -1 & & & & O \\
 & \kappa_{n+1}-\kappa_n & \varphi_{n+1} & -1 & & & \\
 & & \ddots & \ddots & \ddots & & \\
 & & & \kappa_{2n-2}-\kappa_n & \varphi_{2n-2} & -1 \\
 & & & & \kappa_{2n-1}-\kappa_n & \varphi_{2n-1} \\
O & & & & & \kappa_{2n}-\kappa_n
\end{bmatrix}.
\end{aligned}
$$

We also recall that $m^{(n-1)}_{1,n}=-t$ and $m^{(n-1)}_{2,n+1}=-1$.

The coefficient matrix $M_n(z,t)$ is of the form

$$
M_n(z,t)=M_{n,\infty}(t)+\frac{M_{n,1}(t)}{z-1}+\frac{M_{n,t}(t)}{z-t},
$$

where

$$
\begin{aligned}
M_{n,\infty}(t)&=\frac{1}{\kappa_n}\begin{bmatrix}
\kappa_{n+1} & \varphi_{n+1} & -1 & & & O \\
 & \kappa_{n+2} & \varphi_{n+2} & -1 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & \kappa_{2n-2} & \varphi_{2n-2} & -1 \\
 & & & & \kappa_{2n-1} & \varphi_{2n-1} \\
O & & & & & \kappa_{2n}
\end{bmatrix}, \\
M_{n,t}(t)&=\frac{1}{\kappa_n}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ \frac{m^{(n-1)}_{2,n}}{t-1}\end{bmatrix}\begin{bmatrix} m^{(n,t)}_1 & \dots & m^{(n,t)}_n\end{bmatrix}, \\
M_{n,1}(t)&=\frac{1}{\kappa_n}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1\end{bmatrix}\begin{bmatrix} m^{(n,1)}_1 & \dots & m^{(n,1)}_n\end{bmatrix}.
\end{aligned}
$$

The components are given by

$$
m^{(n,\xi)}_i=m^{(n-1,\xi)}_{i-1}-\varphi_{n+i-1}m^{(n-1,\xi)}_i+(\kappa_n-\kappa_{n+i})m^{(n-1,\xi)}_{i+1}\quad (i=1,\dots,n),
$$

for $\xi = t, 1$, where

$$m_0^{(n-1,t)} = 0, \quad m_0^{(n-1,1)} = 0,$$
$$m_i^{(n-1,t)} = m_{1,i}^{(n-1)}, \quad m_i^{(n-1,1)} = m_{2,i}^{(n-1)} - \frac{m_{1,i}^{(n-1)} m_{2,n}^{(n-1)}}{1-t} \quad (i = 1, \dots, n),$$
$$m_{n+1}^{(n-1,t)} = 0, \quad m_{n+1}^{(n-1,1)} = m_{2,n+1}^{(n-1)}.$$

The coefficient matrix $B_n(z,t)$ is of the form

$$B_n(z,t) = B_{n,\infty}(t) + \frac{B_{n,t}(t)}{z-t},$$

where

$$B_{n,\infty}(t) = \frac{1}{u_n} \begin{bmatrix} u_{n+1} & v_{n+1} & \frac{-1+(-1)^{n+1}}{2} & & & & O \\ & u_{n+2} & v_{n+2} & \frac{-1+(-1)^{n+2}}{2} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & u_{2n-3} & v_{2n-3} & -1 & \\ & & & & u_{2n-2} & v_{2n-2} & 0 \\ & & & & & u_{2n-1} & v_{2n-1} \\ O & & & & & & u_{2n} \end{bmatrix},$$

and

$$B_{n,t}(t) = \frac{1}{u_n} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -t^{-1} v_0 \end{bmatrix} \begin{bmatrix} m_1^{(n,t)} & \dots & m_n^{(n,t)} \end{bmatrix}.$$

Lemma 3.3. *We have*

$$\det M_n(z,t) = \frac{(z - q^{-(n-1)/2} \kappa_1 \dots \kappa_n t)(\kappa_{n+1} \dots \kappa_{2n} z - q^{(n-1)/2})}{\kappa_n^n (z-1)(z-t)}.$$

Proof. We can prove by using Lemma 3.1 and

$$\begin{aligned} \det M_n(z,t) &= \det N_{n-1}(z,t) \\ &= \frac{z^2}{(z-t)(z-1)} (\frac{\kappa_{n-1}}{\kappa_n})^{n+1} \det M_{n-1}(\varepsilon^{n-1} \kappa_1 \dots \kappa_n z^{-1}, t). \end{aligned}$$

□

Lemma 3.4. *The eigenvalues of $\kappa_n M_n(0,t)$ are given by $\kappa_1, \dots, \kappa_n$.*

Proof. We can prove by using

$$\det \kappa_n M_n(0,t) = \kappa_1, \ldots, \kappa_n,$$

and

$$\kappa_n M_n(0,t) \Big|_{\kappa_i \to \kappa_i - \lambda} = \kappa_n M_n(0,t) - \lambda I.$$

□

Lemma 3.5. *We have*

$$\det B_n(z,t) = \frac{u_{n+1} \ldots u_{2n}(z - q^{-(n-1)/2}\kappa_1 \ldots \kappa_n t)}{u_n^n (z-t)}.$$

Proof. Let

$$\Delta_n = \begin{vmatrix} u_{n+1} & v_{n+1} & \frac{-1+(-1)^{n+1}}{2} & & & O \\ & u_{n+2} & v_{n+2} & \frac{-1+(-1)^{n+2}}{2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & u_{2n-1} & v_{2n-1} & -1 \\ & & & & u_{2n} & \frac{v_0}{t} \\ m_1^{(n,t)} & m_2^{(n,t)} & \ldots & \ldots & m_n^{(n,t)} & -t \end{vmatrix}.$$

Then we can show that

$$\Delta_n = \frac{\kappa_n}{u_n}\Delta_{n-1} = -\frac{\kappa_1 \ldots \kappa_n}{u_1 \ldots u_n} t,$$

by a direct computation. By using it, we obtain

$$
\begin{aligned}
&u_n^n \det B_n(z,t)\\
&= \begin{vmatrix}
u_{n+1} & v_{n+1} & \frac{-1+(-1)^{n+1}}{2} & & & O \\
 & u_{n+2} & v_{n+2} & \frac{-1+(-1)^{n+2}}{2} & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & u_{2n-2} & v_{2n-2} & 0 \\
 & & & & u_{2n-1} & \frac{t}{v_0} \\
-\frac{v_0 m_1^{(n,t)}}{t(z-t)} & -\frac{v_0 m_2^{(n,t)}}{t(z-t)} & \dots & \dots & -\frac{v_0 m_{n-1}^{(n,t)}}{t(z-t)} & u_{2n} - \frac{v_0 m_n^{(n,t)}}{t(z-t)}
\end{vmatrix}\\
&= u_{n+1}\dots u_{2n-1}u_{2n} - \frac{u_{n+1}\dots u_{2n-1}v_0 m_n^{(n,t)}}{t(z-t)}\\
&\quad + \frac{1}{z-t}\begin{vmatrix}
u_{n+1} & v_{n+1} & \frac{-1+(-1)^{n+1}}{2} & & & O \\
 & u_{n+2} & v_{n+2} & \frac{-1+(-1)^{n+2}}{2} & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & u_{2n-3} & v_{2n-3} & -1 \\
 & & & & u_{2n-2} & v_{2n-2} \\
m_1^{(n,t)} & m_2^{(n,t)} & \dots & \dots & m_{n-2}^{(n,t)} & m_{n-1}^{(n,t)}
\end{vmatrix}\\
&= \frac{u_{n+1}\dots u_{2n}z + \Delta_n}{z-t}\\
&= \frac{u_{n+1}\dots u_{2n}(z - q^{-(n-1)/2}\kappa_1\dots\kappa_n t)}{z-t}.
\end{aligned}
$$

□

We consider a gauge transformation

$$
Y(z,t) = \frac{(q^{-1}tz^{-1};q^{-1})_\infty (qz;q)_\infty^2}{(q^{-1}z^{-1};q^{-1})_\infty}\begin{bmatrix} \tau_{n+1}(z,t) & & O \\ & \ddots & \\ O & & \tau_{2n}(z,t)\end{bmatrix} P(t)\Psi_n(z,t).
$$

where $\tau_{n+1}(z,t),\dots,\tau_{2n}(z,t)$ are functions such that

$$
\frac{\tau_j(q^{-1}z,t)}{\tau_j(z,t)} = \kappa_n, \quad \frac{\tau_j(z,qt)}{\tau_j(z,t)} = \frac{u_n}{u_j} \quad (j = n+1,\dots,2n),
$$

and $P(t)$ is a $n\times n$ matrix such that

$$
P(t)M_{n,\infty}(t)P(t)^{-1} = \frac{1}{\kappa_n}\begin{bmatrix} \kappa_{n+1} & & O \\ & \ddots & \\ O & & \kappa_{2n}\end{bmatrix}.
$$

Note that $P(t)$ is an upper triangular matrix whose diagonal elements are all 1. Then the system L_n is transformed into

$$Y(q^{-1}z,t)=\mathcal{M}(z,t)Y(z,t),\quad Y(z,qt)=\frac{\mathcal{B}(z,t)}{z}Y(z,t). \tag{3.3}$$

Theorem 3.6. *The coefficient matrices satisfy*

$$\mathcal{M}(z,t)=\mathcal{M}_0(t)+z\mathcal{M}_1(t)+z^2\mathcal{M}_2,$$
$$\mathcal{M}_2=\begin{bmatrix}\kappa_{n+1} & & O\\ & \ddots & \\ O & & \kappa_{2n}\end{bmatrix},\quad \mathcal{M}_0(t) \text{ has eigenvalues } t\kappa_1,\dots,t\kappa_n,$$
$$\det\mathcal{M}(z,t)=(z-t)^{n-1}(z-1)^{n-1}(z-\frac{\kappa_1\dots\kappa_n}{q^{(n-1)/2}}t)(\kappa_{n+1}\dots\kappa_{2n}z-q^{(n-1)/2}),$$

and

$$\mathcal{B}(z,t)=\mathcal{B}_0(t)+zI,\quad \det\mathcal{B}(z,t)=(z-t)^{n-1}(z-\frac{\kappa_1\dots\kappa_n}{q^{(n-1)/2}}t).$$

We can show it by a direct computation with Lemma 3.3, 3.4, 3.5 and the following lemma.

Lemma 3.7. *The matrix $P(qt)B_{n,\infty}(t)P(t)^{-1}$ turns to be a diagonal matrix.*

Proof. The system (L_n) implies

$$M_{n,\infty}(qt)B_{n,\infty}(t)=B_{n,\infty}(t)M_{n,\infty}(t),$$

from which we obtain

$$\mathcal{M}_2P(qt)B_{n,\infty}(t)P(t)^{-1}=P(qt)B_{n,\infty}(t)P(t)^{-1}\mathcal{M}_2.$$

The matrix $\mathcal{M}_2$ is a diagonal one whose components are mutually distinct. Therefore $P(qt)B_{n,\infty}(t)P(t)^{-1}$ is also diagonal. □

Remark 3.8. *Via a continuous limit $q\to 1$, the system* (3.3) *reduces to a Fuchsian system with a spectral type $\{(n-1,1),(n-1,1),(1^n),(1^n)\}$.*

4 Fourth order Painlevé systems and laplace transformations

Three types of fourth order Painlevé type ordinary differential equations, whom we denote by $\mathcal{P}_{(5)},\mathcal{P}_{(6)},\mathcal{P}_{(3,3)}$, have been studied from a viewpoint of affine Lie algebra of type A [5, 18]. They are expressed as Hamiltonian systems

$$\mathcal{P}_\lambda : \quad \frac{dq_i}{dt} = \frac{\partial H_\lambda}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H_\lambda}{\partial q_i} \quad (i = 1, 2),$$

with the coupled Hamiltonians

$$\begin{aligned}
H_{(5)} &= H_{\mathrm{IV}}(q_1, p_1; \alpha_2, \alpha_1) + H_{\mathrm{IV}}(q_2, p_2; \alpha_4, \alpha_1 + \alpha_3) + 2q_1 p_1 p_2, \\
tH_{(6)} &= H_{\mathrm{V}}(q_1, p_1; \alpha_2, \alpha_1, \alpha_1 + \alpha_3) \\
&\quad + H_{\mathrm{V}}(q_2, p_2; \alpha_4, \alpha_1 + \alpha_3, \alpha_1 + \alpha_3) + 2q_1 p_1 (q_2 - 1) p_2, \\
t(t-1)H_{(3,3)} &= H_{\mathrm{VI}}(q_1, p_1; \alpha_2, 1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_5, \alpha_3 + \alpha_5 - \eta, \eta\alpha_1) \\
&\quad + H_{\mathrm{VI}}(q_2, p_2; 1 - \alpha_1 - \alpha_3 - \alpha_4 - \alpha_5, \alpha_4, \alpha_1 + \alpha_3 - \eta, \eta\alpha_5) \\
&\quad + (q_1 - t)(q_2 - 1)\left\{(q_1 p_1 + \alpha_1) p_2 + p_1 (p_2 q_2 + \alpha_5)\right\},
\end{aligned}$$

where

$$\begin{aligned}
H_{\mathrm{IV}}(q, p; a, b) &= qp(p - q - t) - aq - bp, \\
H_{\mathrm{V}}(q, p; a, b, c) &= q(q-1)p(p+t) + atq + bp - cqp, \\
H_{\mathrm{VI}}(q, p; a, b, c, d) &= q(q-1)(q-t)p^2 - \{(a-1)q(q-1) \\
&\qquad + bq(q-t) + c(q-1)(q-t)\}p + dq.
\end{aligned}$$

Note that the symbol λ corresponds to a partition of a natural number; see [5].

The system $\mathcal{P}_{(3,3)}$ is obtained as the compatibility conditions of Lax pairs of two types, Borel type [5] and Fuchsian type [23, 26]. In this section, we clarify the relations between them with the aid of Laplace transformations. We also consider the same for the other two systems.

4.1 Coupled Painlevé VI system

The system $\mathcal{P}_{(3,3)}$ is derived from a Lax pair

$$z\frac{\partial}{\partial z}\Psi_6(z,t) = M_6(z,t)\Psi_6(z,t), \quad \frac{\partial}{\partial t}\Psi_6(z,t) = B_6(z,t)\Psi_6(z,t), \tag{4.1}$$

with 6×6 matrices

$$M_6(z,t) = \begin{bmatrix}
-1 & -p_2 & 1 & 0 & 0 & 0 \\
0 & \kappa_2 & q_2 - 1 & 1 & 0 & 0 \\
0 & 0 & \kappa_3 & q_1 p_1 + q_2 p_2 + \eta & 1 & 0 \\
0 & 0 & 0 & \kappa_4 & -\frac{q_1 - t}{t} & 1 \\
tz & 0 & 0 & 0 & \kappa_5 & -tp_1 \\
(q_1 - q_2)z & z & 0 & 0 & 0 & \kappa_6
\end{bmatrix},$$

where

$$\kappa_i = -1 + \sum_{j=1}^{i-1} \alpha_{6-j} \quad (i = 2, \dots, 6),$$

and

$$B_6(z,t) = \frac{1}{t}\begin{bmatrix} 0 & v_1 & 1 & 0 & 0 & 0 \\ 0 & u_2 & v_2 & 0 & 0 & 0 \\ 0 & 0 & u_3 & v_3 & 1 & 0 \\ 0 & 0 & 0 & u_4 & v_4 & 0 \\ tz & 0 & 0 & 0 & u_5 & v_5 \\ v_0 z & 0 & 0 & 0 & 0 & u_6 \end{bmatrix}.$$

The components $u_2, \dots, u_6$ and $v_0, \dots, v_5$ are polynomial in q_1, q_2, p_1, p_2; we do not give its explicit formulas here.

Via a Laplace transformation

$$\frac{\partial}{\partial z}\Psi_6(z,t) \to \zeta \Phi_6(\zeta,t), \quad z\Psi_6(z,t) \to -\frac{\partial}{\partial \zeta}\Phi_6(\zeta,t),$$

and a Möbius transformation $\zeta \to z^{-1}$, we obtain

$$z\frac{\partial}{\partial z}\Phi_6(z,t) = N_6(z,t)\Phi_6(z,t), \quad \frac{\partial}{\partial t}\Phi_6(z,t) = C_6(z,t)\Phi_6(z,t), \tag{4.2}$$

with

$$\begin{aligned} N_6(z,t) &= (I - zM_{6,0}(t))^{-1}(I + M_{6,0}(t)), \\ C_6(z,t) &= B_{6,0}(t) + zB_{6,1}(t)(I - zM_{6,1}(t))^{-1}(I + M_{6,0}(t)), \end{aligned}$$

where

$$M_6(z,t) = M_{6,0}(t) + zM_{6,1}(t), \quad B_6(z,t) = B_{6,0}(t) + zB_{6,1}(t).$$

Then the first columns of $N_6(z,t)$ and $B_6(z,t)$ are both equivalent to the zero vectors. Hence we can reduce the Lax form (4.2) to the one with 5×5 matrices

$$z\frac{\partial}{\partial z}\Psi_5(z,t) = M_5(z,t)\Psi_5(z,t), \quad \frac{\partial}{\partial t}\Psi_5(z,t) = B_5(z,t)\Psi_5(z,t), \tag{4.3}$$

with

$$M_5(z,t) = \begin{bmatrix} \kappa_2 & q_2 - 1 & 1 & 0 & 0 \\ 0 & \kappa_3 & q_1p_1 + q_2p_2 + \eta & 1 & 0 \\ 0 & 0 & \kappa_4 & -\frac{q_1 - t}{t} & 1 \\ m_{1,1}^{(5)} z & tz & 0 & \kappa_5 & -tp_1 \\ m_{2,1}^{(5)} z & m_{2,2}^{(5)} z & z & 0 & \kappa_6 \end{bmatrix},$$

where

$$m^{(5)}_{1,1} = -tp_2, \quad m^{(5)}_{2,1} = (q_2 - q_1)p_2 + \alpha_5, \quad m^{(5)}_{2,2} = q_1 - 1.$$

and

$$B_5(z,t) = \frac{1}{t}\begin{bmatrix} u_2 & v_2 & 0 & 0 & 0 \\ 0 & u_3 & v_3 & 1 & 0 \\ 0 & 0 & u_4 & v_4 & 0 \\ -tp_2 z & tz & 0 & u_5 & v_5 \\ -p_2 v_0 z & v_0 z & 0 & 0 & u_6 \end{bmatrix}.$$

Remark 4.1. *The Lax form* (4.3) *is equivalent to the one derived from the Drinfeld-Sokolov hierarchy of type* $A^{(1)}_4$ *for a partition* $(2,2,1)$ *given in* [5].

Here we can take a function $\tau_5(z,t)$ satisfying

$$z\frac{d}{dz}\log\tau_5(z,t) = \kappa_2 + 1, \quad \frac{d}{dt}\log\tau_5(z,t) = u_2.$$

By using it, we consider a gauge transformation

$$\Psi_5(z,t) \to \tau_5(z,t)\Psi^*_5(z,t).$$

We also consider a Laplace transformation

$$\frac{\partial}{\partial z}\Psi^*_5(z,t) \to \zeta\Phi_5(\zeta,t), \quad z\Psi^*_5(z,t) \to -\frac{\partial}{\partial\zeta}\Phi_5(\zeta,t),$$

and a Möbius transformation $\zeta \to z^{-1}$. Then, similarly as above, we can reduce the Lax form (4.3) to the one with 4×4 matrices

$$z\frac{\partial}{\partial z}\Psi_4(z,t) = M_4(z,t)\Psi_4(z,t), \quad \frac{\partial}{\partial t}\Psi_4(z,t) = B_4(z,t)\Psi_4(z,t), \tag{4.4}$$

with

$$M_4(z,t) = \begin{bmatrix} \kappa_3 & q_1p_1 + q_2p_2 + \eta & 1 & 0 \\ 0 & \kappa_4 & -\frac{q_1 - t}{t} & 1 \\ m^{(4)}_{1,1}z & m^{(4)}_{1,2}z & tz + \kappa_5 & -tp_1 \\ m^{(4)}_{2,1}z & m^{(4)}_{2,2}z & m^{(4)}_{2,3}z & z + \kappa_6 \end{bmatrix},$$

where

$$m^{(4)}_{1,1} = -t\{(q_2 - 1)p_2 - \alpha_4\}, \quad m^{(4)}_{1,2} = t\{q_1p_1 + (q_2 - 1)p_2 + \eta\},$$
$$m^{(4)}_{2,1} = (q_2 - 1)\{(q_2 - q_1)p_2 + \alpha_5\} + \alpha_4(q_1 - 1),$$
$$m^{(4)}_{2,2} = (q_1 - 1)(q_1p_1 + \eta) + q_1(q_2 - 1)p_2 + \alpha_3 + \alpha_4 + \alpha_5, \quad m^{(4)}_{2,3} = \frac{t-1}{t}q_1,$$

and

$$B_4(z,t)=\frac{1}{t}\begin{bmatrix} u_3 & v_3 & 1 & 0 \\ 0 & u_4 & v_4 & 0 \\ tb_1^{(4)}z & tb_2^{(4)}z & tz+u_5 & v_5 \\ v_0b_1^{(4)}z & v_0b_2^{(4)}z & v_0z & u_6 \end{bmatrix},$$

where

$$b_1^{(4)}=-(q_2-1)p_2+\alpha_4,\quad b_2^{(4)}=q_1p_1+(q_2-1)p_2+\eta.$$

Remark 4.2. *The Lax form* (4.4) *is equivalent to the one derived from the Drinfeld-Sokolov hierarchy of type* $A_3^{(1)}$ *for a partition* $(1,1,1,1)$.

Via a gauge transformation

$$\Psi_4(z,t)\to\tau_4(z,t)\Psi_4^*(z,t),\quad z\frac{d}{dz}\log\tau_4(z,t)=\kappa_3+1,\quad \frac{d}{dt}\log\tau_4(z,t)=u_3,$$

a Laplace transformation

$$\frac{\partial}{\partial z}\Psi_4^*(z,t)\to\zeta\Phi_4(\zeta,t),\quad z\Psi_4^*(z,t)\to-\frac{\partial}{\partial\zeta}\Phi_4(\zeta,t),$$

and a replacement $\zeta\to z$, we can reduce the Lax form (4.4) to the one with 3×3 matrices

$$z\frac{\partial}{\partial z}\Psi_3(z,t)=M_3(z,t)\Psi_3(z,t),\quad \frac{\partial}{\partial t}\Psi_3(z,t)=B_3(z,t)\Psi_3(z,t). \tag{4.5}$$

The matrix $M_3(z,t)$ is of the form

$$M_3(z,t)=-M_{3,\infty}(t)+\frac{M_{3,1}(t)}{z-1}+\frac{M_{3,t}(t)}{z-t},$$

with

$$M_{3,1}(t)=\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}m_1^{(3,1)} & m_2^{(3,1)} & m_3^{(3,1)}\end{bmatrix},$$

$$M_{3,t}(t)=\begin{bmatrix}0\\t\\q_1\end{bmatrix}\begin{bmatrix}m_1^{(3,t)} & m_2^{(3,t)} & m_3^{(3,t)}\end{bmatrix},\quad M_{3,\infty}(t)=\begin{bmatrix}\kappa_4 & -\frac{q_1-t}{t} & 1\\ 0 & \kappa_5 & -tp_1\\ 0 & 0 & \kappa_6\end{bmatrix},$$

where

$$\begin{aligned}
m_1^{(3,1)} &= -(q_1p_1+q_2p_2+\eta)\{(q_2-1)(q_2p_2+\alpha_5)-\alpha_3-\alpha_4\} \\
&\quad - \alpha_3(q_2p_2+\alpha_3+\alpha_4+\alpha_5), \\
m_2^{(3,1)} &= -\frac{(q_1-t)(q_1p_1+\eta-\alpha_2-2\alpha_3-\alpha_4-\alpha_5)}{t} \\
&\quad - (q_2-1)(q_2p_2+\alpha_5)+\alpha_2+\alpha_3+\alpha_4, \\
m_3^{(3,1)} &= \eta-\alpha_1-\alpha_2-2\alpha_3-\alpha_4-\alpha_5, \\
m_1^{(3,t)} &= (q_1p_1+q_2p_2+\eta)\{(q_2-1)p_2-\alpha_3-\alpha_4\}+\alpha_3p_2, \\
m_2^{(3,t)} &= \frac{(q_1-t)(q_1p_1+\eta)}{t}+\frac{q_1(q_2-1)p_2}{t}-\alpha_2-\alpha_3-\alpha_4, \\
m_3^{(3,t)} &= -(q_1-t)p_1-(q_2-1)p_2-\eta.
\end{aligned}$$

Note that the matrix $M_3(0,t)$ has eigenvalues α_4, $\alpha_4+\alpha_5$ and 0. The matrix $B_3(z,t)$ is of the form

$$B_3(z,t) = \frac{1}{t}\begin{bmatrix} u_4 & v_4 & 0 \\ 0 & u_5 & v_5 \\ 0 & 0 & u_6 \end{bmatrix} - \frac{M_{3,t}(t)}{z-t}.$$

Theorem 4.3. *The system* (4.5) *is equivalent to the Fuchsian system with the spectral type* (21, 21, 111, 111).

4.2 Coupled Painlevé V system

The system $\mathcal{P}_{(6)}$ is derived from a Lax pair

$$z\frac{\partial}{\partial z}\Psi_6(z,t) = M_6(z,t)\Psi_6(z,t), \quad \frac{\partial}{\partial t}\Psi_6(z,t) = B_6(z,t)\Psi_6(z,t), \tag{4.6}$$

with 6×6 matrices

$$M_6(z,t) = \begin{bmatrix} -1 & \frac{p_2}{\sqrt{t}} & -1 & 0 & 0 & 0 \\ 0 & \kappa_2 & \sqrt{t}(q_2-1) & -1 & 0 & 0 \\ 0 & 0 & \kappa_3 & -\frac{t+p_1+p_2}{\sqrt{t}} & -1 & 0 \\ 0 & 0 & 0 & \kappa_4 & -\sqrt{t}q_1 & -1 \\ -z & 0 & 0 & 0 & \kappa_5 & \frac{p_1}{\sqrt{t}} \\ \sqrt{t}(q_1-q_2)z & -z & 0 & 0 & 0 & \kappa_6 \end{bmatrix},$$

where

$$\kappa_i = -1+\sum_{j=1}^{i-1}\alpha_{j+3} \quad (i=2,3), \quad \kappa_i = -\sum_{j=1}^{7-i}\alpha_{4-j} \quad (i=4,5,6),$$

and

$$B_6(z,t) = \frac{1}{\sqrt{t}} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & u_2 & -1 & 0 & 0 & 0 \\ 0 & 0 & u_3 & -1 & 0 & 0 \\ 0 & 0 & 0 & u_4 & -1 & 0 \\ 0 & 0 & 0 & 0 & u_5 & -1 \\ -z & 0 & 0 & 0 & 0 & u_6 \end{bmatrix}.$$

The components $u_2, \dots, u_6$ are polynomial in q_1, q_2, p_1, p_2; we do not give its explicit formulas here.

Via a Laplace transformation

$$\frac{\partial}{\partial z}\Psi_6(z,t) \to \zeta\Phi_6(\zeta,t), \quad z\Psi_6(z,t) \to -\frac{\partial}{\partial\zeta}\Phi_6(\zeta,t),$$

and a Möbius transformation $\zeta \to z^{-1}$, we can reduce the Lax form (4.6) to the one with 5×5 matrices

$$z\frac{\partial}{\partial z}\Psi_5(z,t) = M_5(z,t)\Psi_5(z,t), \quad \frac{\partial}{\partial t}\Psi_5(z,t) = B_5(z,t)\Psi_5(z,t), \tag{4.7}$$

with

$$M_5(z,t) = \begin{bmatrix} \kappa_2 & \sqrt{t}(q_2-1) & -1 & 0 & 0 \\ 0 & \kappa_3 & -\frac{t+p_1+p_2}{\sqrt{t}} & -1 & 0 \\ 0 & 0 & \kappa_4 & -\sqrt{t}q_1 & -1 \\ m_{1,1}^{(5)}z & z & 0 & \kappa_5 & \frac{p_1}{\sqrt{t}} \\ m_{2,1}^{(5)}z & m_{2,2}^{(5)}z & z & 0 & \kappa_6 \end{bmatrix},$$

where

$$m_{1,1}^{(5)} = -\frac{p_2}{\sqrt{t}}, \quad m_{2,1}^{(5)} = (q_1-q_2)p_2 - \alpha_4, \quad m_{2,2}^{(5)} = -\sqrt{t}(q_1-1).$$

and

$$B_5(z,t) = \frac{1}{\sqrt{t}} \begin{bmatrix} u_2 & -1 & 0 & 0 & 0 \\ 0 & u_3 & -1 & 0 & 0 \\ 0 & 0 & u_4 & -1 & 0 \\ 0 & 0 & 0 & u_5 & -1 \\ -\frac{p_2}{\sqrt{t}}z & z & 0 & 0 & u_6 \end{bmatrix}.$$

Remark 4.4. *The Lax form* (4.7) *is equivalent to the one derived from the Drinfeld-Sokolov hierarchy of type* $A_4^{(1)}$ *for a partition* $(4,1)$ *given in* [5].

Via a gauge transformation

$$\Psi_5(z,t) \to \tau_5(z,t)\Psi_5^*(z,t), \quad z\frac{d}{dz}\log\tau_5(z,t) = \kappa_2 + 1, \quad \frac{d}{dt}\log\tau_5(z,t) = u_2,$$

a Laplace transformation

$$\frac{\partial}{\partial z}\Psi_5^*(z,t) \to \zeta\Phi_5(\zeta,t), \quad z\Psi_5^*(z,t) \to -\frac{\partial}{\partial\zeta}\Phi_5(\zeta,t),$$

and a Möbius transformation $\zeta \to z^{-1}$, we can reduce the Lax form (4.7) to the one with 4×4 matrices

$$z\frac{\partial}{\partial z}\Psi_4(z,t) = M_4(z,t)\Psi_4(z,t), \quad \frac{\partial}{\partial t}\Psi_4(z,t) = B_4(z,t)\Psi_4(z,t), \tag{4.8}$$

with

$$M_4(z,t) = \begin{bmatrix} \kappa_3 & -\frac{t+p_1+p_2}{\sqrt{t}} & -1 & 0 \\ 0 & \kappa_4 & -\sqrt{t}q_1 & -1 \\ m_{1,1}^{(4)}z & m_{1,2}^{(4)}z & -z+\kappa_5 & \frac{p_1}{\sqrt{t}} \\ m_{2,1}^{(4)}z & m_{2,2}^{(4)}z & m_{2,3}^{(4)}z & -z+\kappa_6 \end{bmatrix},$$

where

$$\begin{aligned} &m_{1,1}^{(4)} = -(q_2-1)p_2+\alpha_5, \quad m_{1,2}^{(4)} = -\frac{p_1+t}{\sqrt{t}}, \\ &m_{2,1}^{(4)} = \sqrt{t}(q_2-1)\{(q_1-q_2)p_2-\alpha_4\} - \alpha_5\sqrt{t}(q_1-1), \\ &m_{2,2}^{(4)} = (q_1-1)(p_1+t)+(q_2-1)p_2-\alpha_1-\alpha_2-\alpha_3+1, \quad m_{2,3}^{(4)} = -\sqrt{t}, \end{aligned}$$

and

$$B_4(z,t) = \frac{1}{\sqrt{t}}\begin{bmatrix} u_3 & -1 & 0 & 0 \\ 0 & u_4 & -1 & 0 \\ 0 & 0 & u_5 & -1 \\ b_1^{(4)}z & b_2^{(4)}z & -z & u_6 \end{bmatrix},$$

where

$$b_1^{(4)} = -(q_2-1)p_2+\alpha_5, \quad b_2^{(4)} = -\frac{p_1+t}{\sqrt{t}}.$$

Remark 4.5. *The Lax form* (4.8) *is equivalent to the one derived from the Drinfeld-Sokolov hierarchy of type* $A_3^{(1)}$ *for a partition* $(2,1,1)$.

Via a gauge transformation

$$\Psi_4(z,t) \to \tau_4(z,t)\Psi_4^*(z,t), \quad z\frac{d}{dz}\log\tau_4(z,t) = \kappa_3+1, \quad \frac{d}{dt}\log\tau_4(z,t) = u_3,$$

a Laplace transformation

$$\frac{\partial}{\partial z}\Psi_4^*(z,t) \to \zeta\Phi_4(\zeta,t), \quad z\Psi_4^*(z,t) \to -\frac{\partial}{\partial\zeta}\Phi_4(\zeta,t),$$

and a Möbius transformation $\zeta \to -z^{-1}$, we can reduce the Lax form (4.8) to the one with 3×3 matrices

$$z\frac{\partial}{\partial z}\Psi_3(z,t) = M_3(z,t)\Psi_3(z,t), \quad \frac{\partial}{\partial t}\Psi_3(z,t) = B_3(z,t)\Psi_3(z,t). \tag{4.9}$$

The matrix $M_3(z,t)$ is of the form

$$M_3(z,t) = M_{3,0}(t) + \frac{M_{3,1}(t)z}{z-1} + \frac{M_{3,2}(t)z}{(z-1)^2},$$

with

$$M_{3,0}(t) = \begin{bmatrix} \kappa_4 & -\sqrt{t}q_1 & -1 \\ 0 & \kappa_5 & \frac{p_1}{\sqrt{t}} \\ 0 & 0 & \kappa_6 \end{bmatrix}, \quad M_{3,1}(t) = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{m_1^{(3,2)}}{\sqrt{t}} & -\frac{m_2^{(3,2)}}{\sqrt{t}} & \sqrt{t} \\ m_1^{(3,1)} & m_2^{(3,1)} & m_3^{(3,1)} \end{bmatrix},$$

$$M_{3,2}(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} m_1^{(3,2)} & m_2^{(3,2)} & -t \end{bmatrix},$$

where

$$\begin{aligned} m_1^{(3,1)} &= -(q_1 - q_2 + 1)(p_1 + t)\{(q_2 - 1)p_2 + \alpha_4\} \\ &\quad - (q_1 - q_2 + 1)p_2\{(q_2 - 1)p_2 - \alpha_5\} \\ &\quad - (\alpha_1 + \alpha_2 + \alpha_3 - 1)\{q_1(p_1 + t) + (q_2 - 1)p_2\} \\ &\quad + (\alpha_1 + \alpha_2 + \alpha_3 - 1)(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 - 1), \\ m_2^{(3,1)} &= -\sqrt{t}q_1^2(p_1 + t) - \sqrt{t}(2q_1 - q_2 + 1)(q_2 - 1)p_2 \\ &\quad + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 - 1)\sqrt{t}q_1 + \alpha_4\sqrt{t}(q_2 - 1), \\ m_3^{(3,1)} &= -q_1(p_1 + t) - (q_2 - 1)p_2 + \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 - 2, \\ m_1^{(3,2)} &= -(p_1 + t)\{(q_2 - 1)p_2 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 1\} \\ &\quad - \{(q_2 - 1)p_2 - \alpha_5\}p_2, \\ m_2^{(3,2)} &= -\sqrt{t}\{q_1(p_1 + t) + (q_2 - 1)p_2 + \alpha_2 + \alpha_3 + \alpha_4 - 1\}. \end{aligned}$$

The matrix $B_3(z,t)$ is of the form

$$B_3(z,t) = \frac{1}{\sqrt{t}}\begin{bmatrix} u_4 & -1 & 0 \\ 0 & u_5 & -1 \\ 0 & 0 & u_6 \end{bmatrix} + \frac{M_{3,2}(t)z}{t(z-1)}.$$

4.3 Coupled Painlevé IV system

The system $\mathcal{P}_{(5)}$ is derived from a Lax pair

$$z\frac{\partial}{\partial z}\Psi_5(z,t) = M_5(z,t)\Psi_5(z,t), \quad \frac{\partial}{\partial t}\Psi_5(z,t) = B_5(z,t)\Psi_5(z,t), \tag{4.10}$$

with 5×5 matrices

$$M_5(z,t) = \begin{bmatrix} -1 & q_1 & 1 & 0 & 0 \\ 0 & \kappa_2 & p_1+p_2-q_2-t & 1 & 0 \\ 0 & 0 & \kappa_3 & -p_2 & 1 \\ z & 0 & 0 & \kappa_4 & -q_1+q_2 \\ -p_1 z & z & 0 & 0 & \kappa_5 \end{bmatrix},$$

where

$$\kappa_2 = -1+\alpha_1, \quad \kappa_i = -\sum_{j=1}^{6-i}\alpha_{j+1} \quad (i=3,4,5),$$

and

$$B_5(z,t) = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & u_2 & -1 & 0 & 0 \\ 0 & 0 & u_3 & -1 & 0 \\ 0 & 0 & 0 & u_4 & -1 \\ -z & 0 & 0 & 0 & u_5 \end{bmatrix}.$$

The components $u_2,\dots,u_5$ are polynomial in q_1,q_2,p_1,p_2; we do not give its explicit formulas here.

Via a Laplace transformation

$$\frac{\partial}{\partial z}\Psi_5(z,t) \to \zeta\Phi_5(\zeta,t), \quad z\Psi_5(z,t) \to -\frac{\partial}{\partial \zeta}\Phi_5(\zeta,t),$$

and a Möbius transformation $\zeta \to z^{-1}$, we can reduce the Lax form (4.10) to the one with 4×4 matrices

$$z\frac{\partial}{\partial z}\Psi_4(z,t) = M_4(z,t)\Psi_4(z,t), \quad \frac{\partial}{\partial t}\Psi_4(z,t) = B_4(z,t)\Psi_4(z,t), \tag{4.11}$$

with

$$M_4(z,t) = \begin{bmatrix} \kappa_2 & p_1+p_2-q_2-t & 1 & 0 \\ 0 & \kappa_3 & -p_2 & 1 \\ q_1 z & z & \kappa_4 & -q_1+q_2 \\ -(q_1p_1-\alpha_1)z & (p_2-q_2-t)z & z & \kappa_5 \end{bmatrix},$$

and

$$B_4(z,t) = \begin{bmatrix} u_2 & -1 & 0 & 0 \\ 0 & u_3 & -1 & 0 \\ 0 & 0 & u_4 & -1 \\ -q_1 z & -z & 0 & u_5 \end{bmatrix}.$$

Remark 4.6. *The Lax form* (4.11) *is equivalent to the one derived from the Drinfeld-Sokolov hierarchy of type* $A_3^{(1)}$ *for a partition* $(3,1)$.

Via a gauge transformation

$$\Psi_4(z,t) \to \tau_4(z,t)\Psi_4^*(z,t), \quad z\frac{d}{dz}\log\tau_4(z,t) = \kappa_2 + 1, \quad \frac{d}{dt}\log\tau_4(z,t) = u_2,$$

a Laplace transformation

$$\frac{\partial}{\partial z}\Psi_4^*(z,t) \to \zeta\Phi_4(\zeta,t), \quad z\Psi_4^*(z,t) \to -\frac{\partial}{\partial \zeta}\Phi_4(\zeta,t),$$

and a Möbius transformation $\zeta \to z^{-1}$, we can reduce the Lax form (4.11) to the one with 3×3 matrices

$$z\frac{\partial}{\partial z}\Psi_3(z,t) = M_3(z,t)\Psi_3(z,t), \quad \frac{\partial}{\partial t}\Psi_3(z,t) = B_3(z,t)\Psi_3(z,t). \tag{4.12}$$

The matrix $M_3(z,t)$ is of the form

$$M_3(z,t) = M_{3,0}(t) + M_{3,1}(t)z + M_{3,2}(t)z^2,$$

with

$$M_{3,0}(t) = \begin{bmatrix} \kappa_3 & -p_2 & 1 \\ 0 & \kappa_4 & -q_1+q_2 \\ 0 & 0 & \kappa_5 \end{bmatrix}, \quad M_{3,1}(t) = \begin{bmatrix} 0 & 0 & 0 \\ m_1^{(3,2)} & m_2^{(3,2)} & 1 \\ m_1^{(3,1)} & m_2^{(3,1)} & m_3^{(3,1)} \end{bmatrix},$$

$$M_{3,2}(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} m_1^{(3,2)} & m_2^{(3,2)} & 1 \end{bmatrix},$$

where

$$m_1^{(3,1)} = -\{q_1(p_1+p_2-q_2-t)-\alpha_1\}p_1 - (\alpha_2+\alpha_3+\alpha_4-1)(p_2-q_2-t),$$
$$m_2^{(3,1)} = -q_1p_1 - (p_2-q_2-t)p_2 - \alpha_2 - \alpha_3 + 1, \quad m_3^{(3,1)} = p_2 - q_1 - t,$$
$$m_1^{(3,2)} = q_1(p_1+p_2-q_2-t) - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 + 1, \quad m_2^{(3,2)} = -p_2 + q_1,$$

The matrix $B_3(z,t)$ is of the form

$$B_3(z,t) = \begin{bmatrix} u_3 & -1 & 0 \\ 0 & u_4 & -1 \\ 0 & 0 & u_5 \end{bmatrix} - M_{3,2}(t)z.$$

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