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Wave speeds for pushed fronts in reaction-diffusion equations with cut-offs

Dedicated to Professor Yasumasa Nishiura on the occasion of his sixtieth birthday

By

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Abstract

Cut-off functions have been introduced in scalar reaction-diffusion equations to model regions in which the species concentrations are below a small threshold and hence the reactions are effectively deactivated. This article continues our analysis of cut-offs, in particular of their impact on the propagation of fronts. Our previous analyses were for pulled fronts, see Dumortier, Popovic, and Kaper, Nonlinearity **20** (2007), 855-877, and bistable fronts, see Dumortier, Popovic, and Kaper, PhysicaD **239** (2010), 1984-1999, where we derived explicit asymptotic formulas for the change in the wave speed due to a wide class of cut-offs. In this article, we examine pushed fronts in reaction-diffusion equations with cut-offs, and present rigorous asymptotics for their wave speeds. The principal finding is that the wave speeds are decreased by the cut-offs, in contrast to the case for bistable fronts, and that the decrement scales with a fractional power of the small parameter. Our analysis relies on the method of geometric desingularization. It is presented for pushed fronts in the Nagumo equation with cut-off, and it is applicable to pushed fronts in general scalar reaction-diffusion equations with cut-offs.

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§ 1. Introduction

Cut-offs in reaction-diffusion equations were introduced in [4] as a model in statistical physics. Cut-offs decrease the amplitudes of the reaction terms (or even set them to zero) in regions where the concentrations (or particle densities) are below a small threshold. The pioneering study in [4] was carried out for the fronts in the Fisher-Kolmogorov-Petrowskii-Piscounov (FKPP) equation with cut-off,

$$\phi_t = \phi_{xx} + \phi(1-\phi^2)H(\phi-\epsilon), \tag{1.1}$$

where H denotes the Heaviside cut-off function $H(\phi-\epsilon) = 0$ if $\phi < \epsilon$ and $H(\phi-\epsilon) = 1$ if $\phi > \epsilon$. A principal result was that the wave speed of pulled fronts approaches the FKPP limit ($c_0 = 2$) slowly as $\varepsilon \to 0$, namely

$$c_{FKPP}(\varepsilon) \sim 2 - \frac{\pi^2}{(\ln(\varepsilon))^2}. \tag{1.2}$$

In [7], we proved the existence of traveling front solutions in (1.1). The principal result was that the wave speed for fronts in (1.1) is given by (1.2) for a broad range of cut-off functions, not just H, including smooth and non-smooth functions. This analysis established a conjecture formulated in [4]. The coefficient $-\pi^2$ was shown to be universal within these classes of cut-off functions. A second principal result of [7] was that the logarithmic corrections to the wave speed arise for pulled fronts in general reaction-diffusion equations with cut-offs, not just in the cut-off FKPP equation.

The results in [7] were obtained using the method of geometric desingularization, see [6]. This method is also known as the blow-up method. Cut-off functions make the forward equilibrium state degenerate, in the phase space of the traveling wave system associated to (1.1), and the blow-up method is ideally suited to analyze the system dynamics near degenerate fixed points such as these. Using this method, we were able to carry out a constructive analysis of the shapes, locations, and speeds of the fronts. An alternative approach for front speeds in (1.1) was subsequently developed in [2] using a variational formulation.

In [8], we extended the scope of the above analysis to bistable fronts propagating into metastable states in reaction-diffusion equations with cut-offs. The cut-off functions again cause the forward equilibrium point in the phase space of the traveling wave system to be degenerate, and hence we again employed the blow-up method to construct geometrically the fronts as heteroclinic orbits in the associated phase space. The principal results were explicit, rigorous asymptotics for the speeds of bistable fronts. These revealed that cut-offs increase the speeds and that the increases are proportional to fractional powers of ϵ.

In this article, we turn our attention to a third class of fronts, which we have not previously considered, namely pushed fronts propagating into unstable states. A front is said to be pushed [16] when the reaction term is strictly positive along the front, and when its propagation speed is larger than the linear spreading speed. As was the case for the pulled fronts and bistable fronts, the forward asymptotic state is degenerate in the presence of a cut-off, and the wave no longer decays exponentially in space. Another consequence of a cut-off is that the geometric selection criterion for pushed fronts —that the wave speed is selected by requiring the heteroclinic orbit to approach along a strong stable manifold— is absent. Hence, there is a need to determine how cut-offs affect pushed fronts.

Specifically, we will analyze the wave speeds for pushed fronts in the Nagumo equation. This equation is a prototype model in mathematical biology and arises in many applications, see for example [11]. As is typical of equations with pushed fronts, there is a critical wave speed for traveling fronts. We will demonstrate that this traveling front persists when a cut-off is added; and, at the same time, we will derive the rigorous asymptotics for the wave speed. In contrast to the situation for bistable fronts, cut-offs decrease the speed of pushed fronts. Moreover, the decrease is proportional to a fractional power of ϵ. Our results complement those for pushed fronts in the cut-off Nagumo equation that were presented in [3], following a variational approach.

The approach based on the geometric desingularization method is developed here in such a way that it can be applied to pushed fronts in general scalar R-D equations with cut-offs.

§ 1.1. Nagumo equation with cut-off: propagation of a pushed front into an unstable state

The main equation that we study in this article is

$$\phi_t = \phi_{xx} + \phi(1-\phi)(\phi-\gamma)H(\phi-\varepsilon). \tag{1.3}$$

where H is as above. We consider the parameter regime $-\frac{1}{2} < \gamma < 0$, and we work with the traveling wave variable $\xi = x - ct$ to study fronts.

In the absence of a cut-off, i.e., when $\varepsilon = 0$, traveling fronts connect $\phi^- = 1$ to $\phi^+ = 0$. The fronts that travel with the critical wave speed $c_0 = \frac{1}{\sqrt{2}} - \sqrt{2}\gamma$ decay at the strong rate, $e^{-\frac{1}{\sqrt{2}}\xi}$, as $\xi \to \infty$, see for example [3]. These are the fronts we will study. They are labeled as pushed fronts [16], because $f(\phi) = \phi(1-\phi)(\phi-\gamma)$ is strictly positive along them for $\gamma < 0$ and because the wave speed c_0 is larger than the linear speed $2\sqrt{-\gamma}$ for $\gamma \in (-\frac{1}{2}, 0)$. For completeness, we observe that there are also fronts that travel with speed $c > c_0$ and these decay at a weaker rate; however, we will not study these.

§ 1.2. Statement of the main result

The following theorem is the main result of this article:

Theorem 1.1. For any $\gamma \in (-\frac{1}{2}, 0)$ there exists a $\varepsilon_0 > 0$ and sufficiently small such that for every $\varepsilon \in (0, \varepsilon_0]$ there exists a unique critical wave speed $c(\varepsilon)$ (dependent on γ) for which the traveling front solution, which propagates between $\phi^- = 1$ and $\phi^+ = 0$ and which decays at the strong exponential rate when $\varepsilon = 0$, persists. Moreover, $c(\varepsilon) = c(0) + \Delta c(\varepsilon)$, where $c(0) = \frac{1}{\sqrt{2}} - \sqrt{2}\gamma$ and Δc is a negative, $\mathcal{C}^1$-smooth function in γ and ε (including at $\varepsilon = 0$) that satisfies

$$\Delta c(\varepsilon) = K_\gamma \varepsilon^{1+2\gamma} + o(\varepsilon^{1+2\gamma}) \tag{1.4}$$

with

$$K_\gamma = \frac{\Gamma(4)}{\Gamma(1+2\gamma)\Gamma(3-2\gamma)} \frac{\sqrt{2}\gamma}{(1+2\gamma)^{2\gamma}}. \tag{1.5}$$

Here and in the following, $\Gamma(\cdot)$ denotes the standard Gamma function, see Section 6.1 of [1]. Moreover, the dependence of $c(\varepsilon)$ on the parameter γ is suppressed for convenience of notation.

Theorem 1.1 also yields the critical wave speed $c_{GL}(\varepsilon)$ for pushed fronts in the real Ginzburg-Landau equation with cut-off, $\phi_t = \phi_{xx} + \phi(1-\phi)(1+a\phi)H(\phi - \varepsilon)$, where $a > 2$, see Section IV.B of [13]. Indeed, if we identify the parameters $a = -\frac{1}{\gamma}$, where $\gamma \in (-\frac{1}{2}, 0)$ and also transform the independent variables, time $t = -\gamma\hat{t}$ and space $x = \sqrt{-\gamma}\hat{x}$, then the Ginzburg-Landau equation becomes: $\phi_{\hat{t}} = \phi_{\hat{x}\hat{x}} + \phi(1-\phi)(\phi - \gamma)$, which is precisely the Nagumo equation. Moreover, in light of these transformations, the wave speed for the Nagumo equation gets multiplied by a factor of $\sqrt{a} = \sqrt{-\frac{1}{\gamma}}$ to yield $c_{GL}(\varepsilon)$. This result confirms the scaling of $\Delta c(\varepsilon)$ with ε found for the Ginzburg-Landau equation in Section IV.B of [13].

Remark 1. One may compare the formula for K_γ with that obtained in [3], where a variational approach is used. They differ by a factor of $(1+2\gamma)^{2\gamma}$.

Remark 2. The formulas for $\Delta c(\varepsilon)$ and K_γ turn out to be the same here for pushed fronts with $\gamma \in (-\frac{1}{2}, 0)$ as they are for bistable fronts with $\gamma \in (0, \frac{1}{2})$, see [8]. However, there are several important structural differences in the proof arising from the sign of γ and from the features of bistable and pushed fronts.

§ 1.3. Outline of the method of proof

The proof of Theorem 1.1 will be carried out in the context of the traveling front ODE system,

$$u' = v, \qquad v' = -cv - u(1-u)(u-\gamma)H(u-\varepsilon), \qquad \varepsilon' = 0, \tag{1.6}$$

where the prime denotes the derivative with respect to $\xi = x - ct$.

In the absence of a cut-off, *i.e.,* with $\varepsilon = 0$ and $H \equiv 1$, the front solution is a heteroclinic connection of (1.6) between the fixed points $Q_0^- = (1,0,0)$ and $Q_0^+ = (0,0,0)$. In the (u,v) plane, Q_0^- is a saddle fixed point, and Q_0^+ is a stable node for $\gamma \in (-\frac{1}{2}, 0)$ and c near $c(0)$; the associated eigenvalues are $\lambda_\pm^- = -\frac{c}{2} \pm \sqrt{c^2 + 4(1-\gamma)}$ and $\lambda_\pm^+ = -\frac{c}{2} \pm \sqrt{c^2 + 4\gamma}$, respectively. The heteroclinics lie in the coincidence of one branch of $W^u(Q_0^-)$ and one branch of the strong stable manifold $W^{ss}(Q_0^+)$ precisely for $c = c(0)$, and that is the geometric criterion that determines the unique, critical wave speed when $\varepsilon = 0$. (This contrasts with the fact that Q_0^+ is a saddle for $\gamma \in (0, \frac{1}{2})$, and hence uniqueness of the critical wave speed there stems from the codimension of a saddle-saddle connection, see Section 2 of [8].)

For $\varepsilon \neq 0$, system (1.6) has fixed points at $Q_\varepsilon^- = (1,0,\varepsilon)$ and $Q_\varepsilon^+ = (0,0,\varepsilon)$, and Theorem 1.1 will establish that the heteroclinic orbit —present when $\varepsilon = 0$ and $c = c(0)$— persists for $\varepsilon > 0$ and sufficiently small. However, demonstrating this persistence is technically challenging due to the facts that Q_ε^+ is a degenerate equilibrium of (1.6) for $\varepsilon \neq 0$ and the traveling wave no longer decays exponentially as $\xi \to \infty$.

In order to prove this theorem, we will first desingularize the degenerate equilibrium Q_ε^+ in (1.6) using the blow-up method, [6]. See Section 2 below. The phase space of the blown-up vector field naturally decomposes into three regions, an outer region in which $u = \mathcal{O}(1)$, an inner region near $u = 0$, and an intermediate region that lies between them. In the phase space of this blown-up vector field, we will construct a singular ($\varepsilon = 0$) heteroclinic connection Γ between Q_0^- and Q_0^+ that traverses across the three regions. See Section 3. Finally, in Section 4, we will prove that, for each $\varepsilon > 0$ sufficiently small, the singular heteroclinic orbit Γ persists as a heteroclinic connection between Q_ε^- and Q_ε^+ for a unique value $c(\varepsilon)$ of c in (1.6). That connection will correspond precisely to the sought-after front solution of (1.3) propagating with speed $c(\varepsilon)$. The corresponding persistence proof will also yield the leading-order asymptotics of $c(\varepsilon)$, thus establishing formulas (1.4) and (1.5) in Theorem 1.1.

Remark 4. Examination of the local dynamics near the stable node Q_0^+ reveals another reason for the parameter restriction $\gamma > -\frac{1}{2}$. In the (u,v) plane with $c = c(0)$, the eigenvectors corresponding to $\lambda_\pm^+$ are $\mathbf{e}_\pm^+ = (1, \lambda_\pm^+)^T$. Hence, for $\gamma \in (-\frac{1}{2}, 0)$, the strong stable eigendirection is given by $\mathbf{e}_-^+$ and the weak stable eigendirection by $\mathbf{e}_+^+$, when $c = c_0$. At $\gamma = -\frac{1}{2}$, the two eigenvalues are equal, and hence there is no unique strong stable eigendirection. Then, for $\gamma < -\frac{1}{2}$, the roles of the eigendirections are switched.

§ 2. Desingularization of the degenerate equilibrium point Q_ε^+

The degeneracy of the state $Q_\varepsilon^+ = (0, 0, \varepsilon)$ of the system (1.6) can be removed by desingularizing ('blowing-up') this point to an invariant two-sphere, [6]. In the context of (1.6), the desired blow-up transformation takes the form

$$u = \bar{r}\bar{u}, \qquad v = \bar{r}\bar{v}, \qquad \varepsilon = \bar{r}\bar{\varepsilon}. \tag{2.1}$$

Here, $(\bar{u}, \bar{v}, \bar{\varepsilon}) \in \mathcal{S}^2_+ = \{(\bar{u}, \bar{v}, \bar{\varepsilon}) | \bar{u}^2 + \bar{v}^2 + \bar{\varepsilon}^2 = 1\}$, with $\bar{\varepsilon} \geq 0$ and $\bar{r} \in [0, r_0]$ for $r_0 > 0$ sufficiently small. In other words, the transformation maps Q_ε^+ to the two-sphere $\mathcal{S}^2$ in $\mathbf{R}^3$. The powers of r were chosen so that as much structure is retained in the vector field as possible in the limit $r \to 0$.

The blow-up regularizes the dynamics in a neighborhood of Q_ε^+. The dynamics on and near the resulting two-sphere can then be studied using standard techniques from dynamical systems theory.

To study the dynamics on (and near) $\mathcal{S}^2_+$ of the flow induced by (1.6) under the blow-up, we introduce local coordinate charts: we define a phase-directional chart K_1 by setting $\bar{u} = 1$ in (2.1), and a rescaling chart K_2 by setting $\bar{\varepsilon} = 1$ in (2.1). Moreover, the manifold $\mathcal{S}^2_+$ will be invariant under the induced dynamics in both of these charts, and hence we will be able to regularize the singular limit $\varepsilon \to 0$ in (1.6).

Remark 5. Given any object $\Box$ in the original (u, v, ε) variables, we denote the corresponding blown-up object by $\bar{\Box}$. Also, in charts K_i $(i = 1, 2)$, the corresponding object will be denoted by $\Box_i$, as required.

§ 3. Construction of the singular heteroclinic orbit Γ

In this section, we construct the singular heteroclinic connection Γ between Q_0^- and Q_0^+, working separately in each of the outer, inner, and intermediate regions.

§ 3.1. Outer region

In the outer region where $u = \mathcal{O}(1)$, the ODE system (1.6) corresponds precisely to the traveling front equations without cut-off,

$$u' = v, \qquad v' = -cv - u(1-u)(u-\gamma), \tag{3.1}$$

as $H = 1$ there. There is a traveling front solution that connects the rest states at $(1, 0)$ and $(0, 0)$ and that travels with the critical wave speed $c_0 = \frac{1}{\sqrt{2}} - \sqrt{2}\gamma$. It is known explicitly:

$$u(\xi) = \frac{1}{1 + e^{\frac{1}{\sqrt{2}}(\xi - \xi^-)}}, \qquad v(u, c_0) = \frac{1}{\sqrt{2}} u(u-1), \tag{3.2}$$

where ξ^- denotes an arbitrary phase.

The portion of the singular heteroclinic that lies in the outer region is now constructed as follows. For ρ small, with $\rho \geq \varepsilon_0$, we let Σ^- denote the hyperplane $\{u = \rho\}$ in (u, v, ε) space, and we write $P_0^- : (\rho, v^-, 0)$ for the point of intersection of $W^u(Q_0^-)$ with Σ^-. (Here and in the following, we suppress the ρ-dependence of Σ^- and P_0^-, for the sake of brevity.) The section Σ^- defines a local section for the flow of (1.6), and the segment of $W^u(Q_0^-)$ lying between Q_0^- and P_0^-, which we label Γ^-, is precisely the portion of the singular heteroclinic connection Γ that lies in this outer region, i.e., in $\{u \geq \rho\}$ with $\rho \geq \varepsilon_0$.

This completes the analysis of the outer region for $\varepsilon = 0$. However, before going on to the other regions, it is useful to state one result concerning the equation of variations about this solution. We let $c = c_0 + \Delta c$, with $\Delta c = c - c_0 = o(1)$. Then, noting that the manifold $W^u(Q_\varepsilon^-)$ is analytic in the state variables u and v (at least as long as $u \geq \varepsilon$), as well as in the parameter c, we may assume an expansion for $W^u(Q_\varepsilon^-)$ of the form

$$v(u, c) = \Sigma_{j=0}^{\infty} \frac{1}{j!} \frac{\partial^j v}{\partial c^j}(u, c_0)(\Delta c)^j. \tag{3.3}$$

Only the first two terms in this expansion will be needed below. The leading-order term is given by (3.2). The first-order term in Δc is found from the variational equation associated to (3.1) taken along $v(u, c_0)$:

$$\frac{\partial}{\partial u}\left(\frac{\partial v}{\partial c}(u, c_0)\right) = -1 + 2\frac{u - \gamma}{u(1-u)}\frac{\partial v}{\partial c}(u, c_0).$$

For $u \in (0, 1]$, the unique solution $\frac{\partial v}{\partial c}(u, c_0)$ that satisfies $\frac{\partial v}{\partial c}(1, c_0) = 0$ is given by

$$\frac{\partial v}{\partial c}(u, c_0) = \frac{1}{3 - 2\gamma} u^{-2\gamma}(1 - u)F(3 - 2\gamma, -2\gamma; 4 - 2\gamma; 1 - u), \tag{3.4}$$

where $F(\cdot, \cdot; \cdot; \cdot)$ denotes the hypergeometric function, see Section 15 of [1]. Moreover, $\frac{\partial v}{\partial c}$ is strictly positive for all $u \in (0, 1)$. This solution is given in Lemma 2.1 of [8]. We refer the reader to Appendix A of [8] for the proof. We also note that this derivative is regular in the limit $u \to 0^+$, since $-1/2 < \gamma < 0$ here. This result will be useful in Section 4.5, below.

§ 3.2. Inner region

In the inner region where $u < \varepsilon$, we have $H = 0$. Hence, the dynamics of (1.6) is governed by the corresponding cut-off equations,

$$u' = v, \qquad v' = -cv, \qquad \varepsilon' = 0, \tag{3.5}$$

where we have appended the trivial equation for ϵ. We study these equations in the rescaling chart K_2, $\{\varepsilon_2 = 1\}$, where the blow-up transformation (2.1) is given by

$$u = r_2 u_2, \quad v = r_2 v_2, \quad \varepsilon = r_2.$$

Substituting this transformation into (3.5), we find the equivalent system

$$u_2' = v_2, \qquad v_2' = -cv_2, \qquad r_2' = 0. \tag{3.6}$$

This is the system with which we work in this section.

The dynamics of (3.6) are straightforward. All points on the u_2-axis are equilibria of (3.6) for $r_2 (= \varepsilon)$ fixed. However, only points on the line $\ell_2^+ = \{(0,0,r_2) | r_2 \in [0, r_0]\}$ can correspond to Q_ε^+ for $\varepsilon > 0$ after blow-down (i.e., after transformation to the original (u, v, ε) coordinates). Therefore, we will only consider those equilibria of system (3.6) here.

Now, in the singular limit ($r_2 \to 0$), the $u_2 - v_2$ subsystem of (3.6) may be written equivalently as the scalar equation $\frac{dv_2}{du_2} = -c_0$, where $c_0 = \frac{1}{\sqrt{2}} - \sqrt{2}\gamma$, recall Section 3.1. We are interested in the solution that satisfies $v_2(0) = 0$. Therefore, the unique solution is given by $v_2(u_2) = -c_0 u_2$. We label the corresponding orbit in phase space as Γ_2^+, and we observe that it is precisely the stable manifold $W_2^s(Q_{0_2}^+)$.

Finally, we define the section Σ_2^+ for the flow of (3.6) by

$$\Sigma_2^+ = \{(1, v_2, \varepsilon) | (v_2, \varepsilon) \in [-v_0, 0] \times [0, \varepsilon_0]\},$$

for $v_0 > 0$ sufficiently small and fixed, and we note that Σ_2^+ represents a natural boundary for the inner region. Since $u = \varepsilon$ is equivalent to $u_2 = 1$, Σ_2^+ marks the transition between the cut-off and unmodified regimes in the original first-order system (1.6) after blow-up and transformation to chart K_2. The orbit Γ_2^+ intersects Σ_2^+ in the point $P_{0_2}^+ = (1, -c_0, 0)$, as $v_2(1) = -c_0$. Therefore, Γ_2^+ is the portion of the singular orbit Γ that lies in this inner region. The geometry in chart K_2 is illustrated in Figure 1.

§ 3.3. Intermediate region

The intermediate region, where $\varepsilon < u < \mathcal{O}(1)$ and hence $H = 1$, provides the connection between the outer and inner regions and is most conveniently studied in chart K_1. Here, the blow-up transformation (2.1) is given by

$$u = r_1, \quad v = r_1 v_1, \quad \text{and} \quad \varepsilon = r_1 \varepsilon_1. \tag{3.7}$$

In the new $(r_1, v_1, \varepsilon_1)$ coordinates, system (1.6) becomes

$$r_1' = r_1 v_1, \qquad v_1' = -cv_1 - v_1^2 + \gamma - (1+\gamma) r_1 + r_1^2 \qquad \varepsilon_1' = -\varepsilon_1 v_1. \tag{3.8}$$

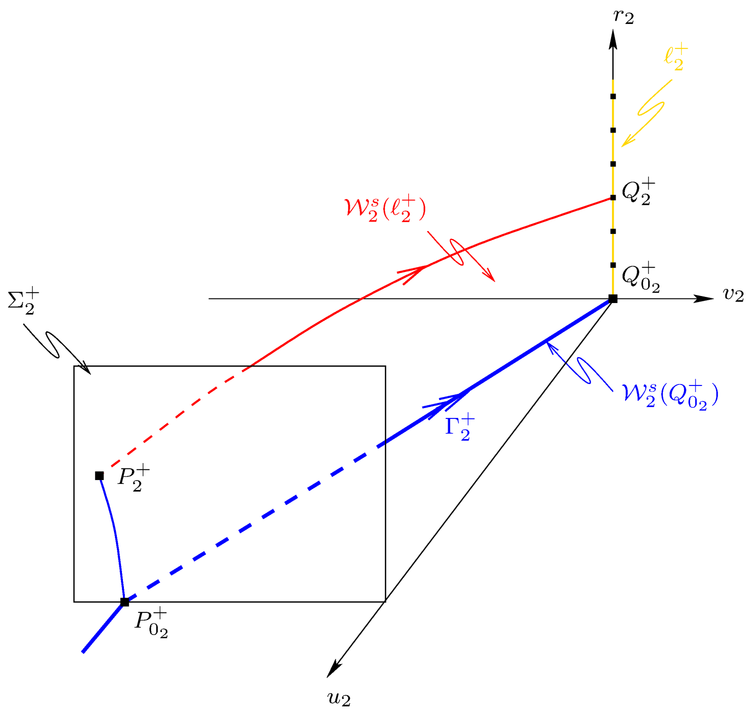

Figure 1. The geometry in chart K_2.

Since c reduces to $c_0 = \frac{1}{\sqrt{2}} - \sqrt{2}\gamma$ for $\varepsilon(= r_1\varepsilon_1) = 0$, it follows that the equilibria of (3.8) are located at $P_1^s = (0, -\frac{1}{\sqrt{2}}, 0)$ and $P_1^u = (0, \sqrt{2}\gamma, 0)$, respectively. These equilibria correspond to the strong stable and weak stable eigendirections, respectively, of the linearization at Q_0^+ of the Nagumo equation without cut-off for $\gamma \in (-\frac{1}{2}, 0)$. (In other words, the blow-up transformation teases apart the asymptotics of solutions in a neighborhood of Q_ε^+ and, hence, desingularizes the dynamics of (1.6) down to $\varepsilon = 0$.) Both P_1^s and P_1^u are hyperbolic saddle equilibria of (3.8), with eigenvalues $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}(1+2\gamma)$, and $\frac{1}{\sqrt{2}}$, respectively, and $\sqrt{2}\gamma, -\frac{1}{\sqrt{2}}(1+2\gamma)$, and $-\sqrt{2}\gamma$.

The relevant equilibrium for us is P_1^s, since $v_1 = \frac{v}{u} \to -\frac{1}{\sqrt{2}}$ as $u \to 0^+$, recall (3.2). The linear analysis of P_1^s is similar to that carried out in [8]. Hence, the rest of the construction of the singular heteroclinic in the intermediate region proceeds along the same lines as that used in [8], with some important modifications to account for the sign of γ.

In particular, we observe that the hyperplanes $\{r_1 = 0\}$ and $\{\varepsilon_1 = 0\}$ are invariant for (3.8), as well as that both hyperplanes correspond to the singular limit as $\varepsilon \to 0^+$ in (1.6). The resulting reduced dynamics determines the location of the singular heteroclinic orbit Γ in this intermediate region. Specifically, in $\{\varepsilon_1 = 0\}$, the orbit passing through $P_{0_1}^-$ (which is the image of the point P_0^- under the blow-up transformation) is asymptotic to P_1^s as $\xi \to \infty$. We denote this orbit by Γ_1^-, and we observe that Γ_1^-

corresponds to the unstable manifold $W^u(Q_0^-)$, after blow-up and transformation to K_1. (Alternatively, Γ_1^- can be interpreted as the equivalent, in K_1, of the tail of the traveling front solution to (1.3) in the absence of a cut-off.)

Next, in the invariant hyperplane $\{r_1 = 0\}$, the orbit through $P_{0_1}^+$ (which is the image of the point $P_{0_2}^+$ in Σ_2^+ under the coordinate transformation between charts K_2 and K_1) asymptotes to P_1^s in backward time, i.e., as $\xi \to -\infty$. We denote that orbit by Γ_1^+, and it is given implicitly by

$$\ln|\varepsilon_1| - \frac{1}{2}\ln|(v_1 + \frac{1}{\sqrt{2}})(v_1 - \sqrt{2}\gamma)| - \frac{c_0}{\sqrt{c_0^2 + 4\gamma}}\operatorname{arccoth}\left(\frac{2v_1 + c_0}{\sqrt{c_0^2 + 4\gamma}}\right) = 0, \tag{3.9}$$

as one obtains from integrating (3.8) on $\{r_1 = 0\}$. For $\gamma \in (-\frac{1}{2}, 0)$, v_1 goes from $-\frac{1}{\sqrt{2}}$ at P_1^s down to $v_{0_1}^+ = -\frac{1}{\sqrt{2}} + \sqrt{2}\gamma < -\frac{1}{\sqrt{2}}$ at $P_{0_1}^+$, along the orbit Γ_1^+. Also, the quantity $\frac{2v_1+c_0}{\sqrt{c_0^2+4\gamma}}$ goes from -1 down to $-1 + \frac{4\gamma}{1+2\gamma}$.

In summary, we find that the union of the two segments Γ_1^- and Γ_1^+ constitutes that portion of Γ that lies in the intermediate region. (We refer the reader to Figure 2 in [8] for an illustration of a similar though slightly different situation, with the difference that here the v coordinate of $P_{0_1}^+$ lies to the left of $-1/\sqrt{2}$ for $\gamma \in (-1/2, 0)$.)

§ 3.4. Summary of the construction of Γ

In summary, the singular heteroclinic connection Γ (more specifically, the corresponding orbit $\bar{\Gamma}$ in the phase space of the blown-up vector field) has been defined as the union of the orbits $\bar{\Gamma}^-$ and $\bar{\Gamma}^+$ with the singularities $\bar{Q}_0^-$, $\bar{P}^s$, and $\bar{Q}_0^+$. It is illustrated in Figure 2.

§ 4. Existence and asymptotics of $c(\varepsilon)$

In this section, we establish the persistence of the singular heteroclinic orbit Γ for $\varepsilon > 0$ sufficiently small. To that end, we consider successively the dynamics obtained in the three regions above, the outer, inner, and intermediate.

§ 4.1. Persistence of the invariant manifolds in the outer and inner regions

In the outer region, the unstable manifold $W^u(Q_0^-)$ persists in an analytic fashion to the unstable manifold $W^u(Q_\varepsilon^-)$ (at least as long as $u \geq \varepsilon$ for $\varepsilon > 0$ sufficiently small). For each fixed $\varepsilon > 0$, $W^u(Q_\varepsilon^-)$ corresponds precisely to that segment of the sought-after persistent heteroclinic in the outer region. In addition, we are interested in the unstable manifold $W^u(\ell^-) = \bigcup_{\varepsilon\in[0,\varepsilon_0]} W^u(Q_\varepsilon^-)$, which is simply a foliation in $\varepsilon \in [0, \varepsilon_0]$ with fibers $W^u(Q_\varepsilon^-)$.

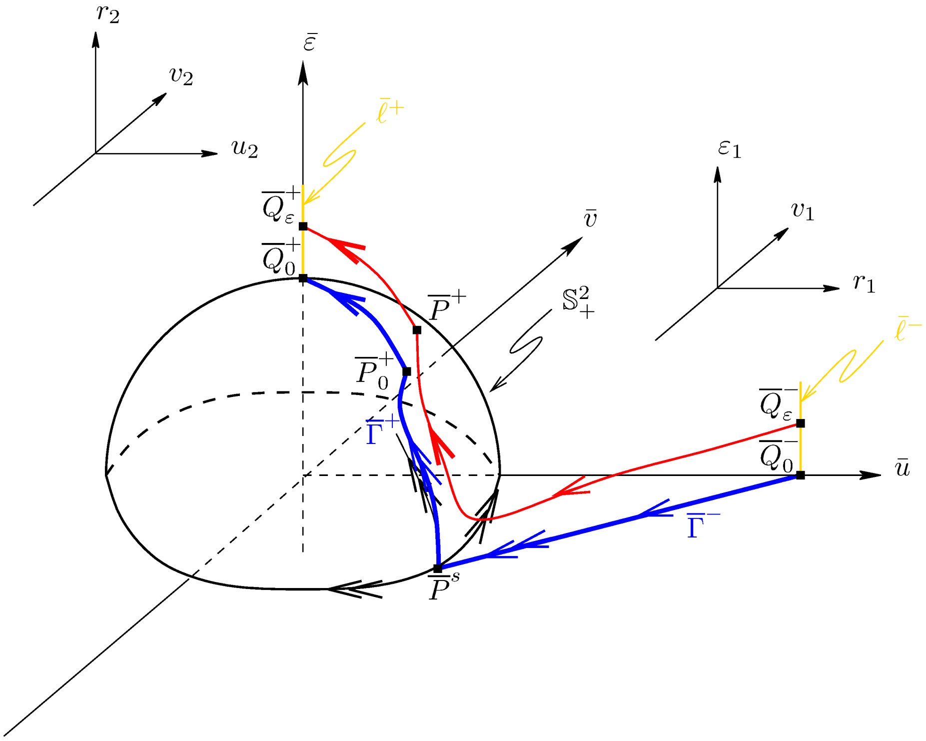

Figure 2. The global geometry of the blown-up vector field.

In the inner region, the stable manifold $W_2^s(Q_{0_2}^+)$, which is given explicitly by $v_2(u_2) = -c_0 u_2$ for $\varepsilon = 0$, perturbs analytically for $r_2(= \varepsilon) > 0$ small and $u_2 \leq 1$ to the manifold $W_2^s(Q_\varepsilon^+)$. In fact, the persistent manifold is also known explicitly in this chart; it is given by the graph of $v_2 = -cu_2$, for $c = c_0[1 + o(1)]$. For ε fixed, $W_2^s(Q_\varepsilon^+)$ corresponds to the segment of the sought-after persistent heteroclinic that is located in the inner region (after blow-down). In addition, we are also interested in the corresponding stable manifold $W_2^s(\ell_2^+)$ of the line of equilibria ℓ_2^+; this is the union of the persistent manifolds over $\varepsilon \in [0, \varepsilon_0]$.

§ 4.2. Outline of proof strategy in the intermediate region

To prove Theorem 1.1, we will show that for each $\varepsilon > 0$ sufficiently small, the two manifolds $W^u(\ell^-)$ and $W_2^s(\ell^+)$ connect in the intermediate region for a unique value of c in (1.6). The existence of that connection is equivalent to the persistence of the singular heteroclinic orbit Γ. The strategy of the proof is as follows.

We will henceforth denote the corresponding c-value by $c(\varepsilon)$. Also, we will show that $c(\varepsilon)$ reduces to c_0 in the singular limit as $\varepsilon \to 0^+$, and hence we will identify $c(0)$ and c_0 once the existence of $c(\varepsilon)$ has been proven below in Proposition 4.1. That proof will be carried out entirely in the intermediate region, i.e., in chart K_1. In a first step, we define the two sections Σ_1^- and Σ_1^+ for the flow of (3.8), as follows:

$$\begin{aligned}\Sigma_1^- &= \{(\rho, v_1, \varepsilon\rho^{-1}) | (v_1, \varepsilon) \in [-v_0, 0] \times [0, \varepsilon_0]\} \quad \text{and} \\ \Sigma_1^+ &= \{(\varepsilon, v_1, 1) | (v_1, \varepsilon) \in [-v_0, 0] \times [0, \varepsilon_0]\},\end{aligned} \tag{4.1}$$

where v_0 is as defined above. (The restriction to the negative v_1-axis is possible due to the fact that we are only interested in the dynamics of (3.8) in a neighborhood of P_1^s; recall the discussion in Section 3.3.) We note that Σ_1^- corresponds to the section Σ^- introduced in Section 3.1, after blow-up and transformation to chart K_1; moreover, we again suppress the ρ-dependence of that section, for convenience of notation. Similarly, Σ_1^+ is equivalent to Σ_2^+ under the change of coordinates between the charts. Clearly, Σ_1^- separates the outer region from the intermediate region, while Σ_1^+ defines the boundary between the intermediate and inner regions.

Now, the crucial step in showing existence and uniqueness of $c(\varepsilon)$ consists in describing the transition map $\Pi_1 : \Sigma_1^- \to \Sigma_1^+$ sufficiently accurately. In other words, we will require that, for $\varepsilon > 0$ small enough, the point of intersection of $W^u(Q_\varepsilon^-)$ with the section Σ^-, which we denote by P^-, is mapped to the point of intersection P_2^+ of $W_2^s(Q_\varepsilon^+)$ with Σ_2^+ in the transition through the intermediate region. (Here, we note that the corresponding orbit constitutes that portion of the persistent heteroclinic that lies in this region; moreover, we omit the parameter dependence of the points P^- and P_2^+, for brevity.) Also, the required persistence proof will reveal that $\Delta c(\varepsilon) = c(\varepsilon) - c_0$ must be negative and must satisfy (1.4) and (1.5).

§ 4.3. Normal form in the intermediate region

It is useful to recast (3.8) in a form that is more convenient. We begin with the coordinate transformation $c = c_0 + (c - c_0) = \frac{1}{\sqrt{2}} - \sqrt{2}\gamma + \Delta c$ and the new variable $z = v_1 + \frac{1}{2}c_0 = v_1 + \frac{1}{2\sqrt{2}}(1 - 2\gamma)$. System (3.8) becomes

$$\begin{aligned}r_1' &= -\left[\tfrac{1}{2\sqrt{2}}(1-2\gamma) - z\right] r_1, \\ z' &= \left[\tfrac{1}{2\sqrt{2}}(1-2\gamma) - z\right] \Delta c - z^2 + \tfrac{1}{8}(1+2\gamma)^2 - (1+\gamma)r_1 + r_1^2, \\ \varepsilon_1' &= \left[\tfrac{1}{2\sqrt{2}}(1-2\gamma) - z\right] \varepsilon_1.\end{aligned} \tag{4.2}$$

(Here, we observe that the terms that are linear in z in (3.8)(b) cancel due to our choice of constant in the definition of z.) Next, from the right-hand sides of the vector field, we divide out the factor of $\left[\frac{1}{2\sqrt{2}}(1-2\gamma) - z\right]$, which is positive for the range of z values considered here,

$$\begin{aligned}r_1' &= -r_1 \\ z' &= \Delta c - \frac{z^2 - \frac{1}{8}(1+2\gamma)^2}{\frac{1}{2\sqrt{2}}(1-2\gamma) - z} + \frac{-(1+\gamma)r_1 + r_1^2}{\frac{1}{2\sqrt{2}}(1-2\gamma) - z}, \\ \varepsilon_1' &= \varepsilon_1.\end{aligned} \tag{4.3}$$

This transformation corresponds to a rescaling of ξ that leaves the phase portrait of the system unchanged; correspondingly, the prime now denotes differentiation with respect to a new independent variable ζ. Moreover, since the equations in (4.3) are autonomous, we may assume without loss of generality that $\zeta^- = 0$ in Σ_1^- independent of the choice of ξ^- in (3.2).

Now, we derive the normal form for (4.3), as follows:

Proposition 4.1. Let $\mathcal{V} = \{(r_1, z, \varepsilon_1) | (r_1, z, \varepsilon_1) \in [0, \rho] \times [-z_0, 0] \times [0, 1]\}$, where $z_0 = v_0 + \frac{1}{2\sqrt{2}}(1 - 2\gamma)$ with v_0 as in (4.1). Then, there exists a C^∞-smooth coordinate transformation

$$\psi : \mathcal{V} \to \psi(\mathcal{V}) \qquad \text{with} \quad (r_1, z, \varepsilon_1) \to (r_1, \hat{z}, \varepsilon_1)$$

where $\hat{z}(z, r_1) = z + \mathcal{O}(r_1)$, such that (4.3) can be written as

$$\begin{aligned} r_1' &= -r_1 \\ \hat{z}' &= \Delta c - \frac{\hat{z}^2 - \frac{1}{8}(1+2\gamma)^2}{\frac{1}{2\sqrt{2}}(1-2\gamma) - \hat{z}} \\ \varepsilon_1' &= \varepsilon_1. \end{aligned} \tag{4.4}$$

Proof. The result follows from standard normal form theory; see for example [5] and the references therein. In particular, we note that the r_1-dependent terms in (4.3)(b) are non-resonant and that hence they can be removed completely via a near-identity coordinate change ψ. (Here, we append the equation $\Delta c' = 0$, and make a linear shift in the variable $\hat{z}$ so that the fixed point is at the origin. In addition, if one uses standard normal form theory with the vector field in polynomial form, one can binomially expand.)

Moreover, ψ can only depend on the variables r_1 and z, since (4.3)(b) is independent of ε_1. Therefore, $\hat{z} = z + \mathcal{O}(r_1)$, as claimed. □

This normal form is the system of equations we will work with in the remainder of this section.

§ 4.4. Uniqueness of Δc

Let P_1^- and P_1^+ denote the points that correspond to P^- and P_2^+, respectively, after transformation to chart K_1, and let $\hat{P}_1^-$ and $\hat{P}_1^+$ be the respective corresponding points after application of the normal form transformation ψ defined in Proposition 4.1. Finally, let $\hat{z}^-$ and $\hat{z}^+$ denote the associated $\hat{z}$-values that are obtained from z^- and z^+, respectively. We find

Lemma 4.1. For any $\rho \in (\varepsilon, 1)$ with $\varepsilon \in (0, \varepsilon_0]$, and for each Δc sufficiently small, the points $\hat{P}_1^- : (\rho, \hat{z}^-, \varepsilon\rho^{-1})$ and $\hat{P}_1^+ : (\varepsilon, \hat{z}^+, 1)$ are such that

(4.5)
$$\hat{z}^- = \hat{z}^-(\rho, \Delta c) = -\frac{1}{2\sqrt{2}}(1+2\gamma)+\nu(\rho, \Delta c)\Delta c, \quad \text{with} \quad \nu(\rho, 0) = \frac{1}{\rho}\frac{\partial v}{\partial c}(\rho, c_0)[1+\nu_1(\rho)] > 0$$

and

$$\hat{z}^+ = \hat{z}^+(\Delta c, \varepsilon) = -\left[\frac{1}{2\sqrt{2}}(1-2\gamma) + \Delta c\right] + \omega(\Delta c, \varepsilon)\varepsilon. \tag{4.6}$$

Here, $\nu(\rho, \Delta c)$ is a $\mathcal{C}^\infty$-smooth function in ρ and Δc, while ν_1 is $\mathcal{C}^\infty$ smooth down to $\rho = 0$, with $\nu_1(0) = 0$. Finally, $\omega(\Delta c, \varepsilon)$ is $\mathcal{C}^\infty$ smooth in Δc and ε, including in a neighborhood of $(0, 0)$.

This lemma is precisely the same as Lemma 2.2 in [8], where the analysis was carried out for $\gamma \in (0, \frac{1}{2})$. We note that the proof given there also applies directly to the case under consideration here of $\gamma \in (-\frac{1}{2}, 0)$, and hence we refer there for the proof.

For given Δc small, $\varepsilon \in (0, \varepsilon_0]$, and $\rho \in (\varepsilon, 1)$, we now consider the solution to (4.4) with initial $\hat{z}$ value $\hat{z}(0) = \hat{z}^-(\rho, \Delta c)$, where $\hat{z}^-$ is as in (4.5). Let $\hat{z}_-^+$ denote the corresponding value of that solution at time ζ^+, $\hat{z}(\zeta^+)$. Then, we have:

Lemma 4.2. For $\hat{z}_-^+ = \hat{z}(\zeta^+)$, there holds $\frac{\partial \hat{z}_-^+}{\partial c}(\rho, \Delta c) > 0$. Moreover, there can exist at most one value of Δc such that $\hat{z}_-^+(\rho, \Delta c) = \hat{z}^+$, where $\hat{z}^+$ is as in (4.6).

This lemma is precisely the same as Lemma 2.3 in [8], where the analysis was carried out for $\gamma \in (0, \frac{1}{2})$. We note that the proof given there also applies directly to the case under consideration here of $\gamma \in (-\frac{1}{2}, 0)$, and hence we refer there for the proof.

Lemma 4.2 implies, in particular, that, if it exists, a connection between the points $\hat{P}_1^-$ and $\hat{P}_1^+$ under the flow of (4.4) can exist for at most one value of Δc in (4.4). As a consequence, persistence of the singular heteroclinic orbit Γ constructed in Section 3 is also only possible for at most one value of Δc.

§ 4.5. Existence and asymptotics of Δc

We are now in a position to complete the proof of Theorem 1.1. We will prove that there exists a function $\Delta c = \Delta c(\varepsilon)$ so that the singular heteroclinic orbit Γ persists, for $\varepsilon > 0$ positive and sufficiently small and $c = c_0 + \Delta(\varepsilon)$ in (1.6). In particular, we integrate the normal form system obtained in (4.4)(b), taking into account the estimates for $\hat{z}^-$ and $\hat{z}^+$ found in Lemma 4.1 above, and noting that this proof differs in several important respects from that of Proposition 4.2 in [8] due to the difference in the range of the values of the parameter γ under consideration here. Moreover, this function Δc will be unique by the results of the previous section.

Proposition 4.2. Let $\gamma \in (-\frac{1}{2}, 0)$. Then, for every $\varepsilon \in (0, \varepsilon_0]$, with $\varepsilon_0 > 0$ sufficiently small, there exists a function $c(\varepsilon) = c_0 + \Delta c(\varepsilon)$, with $\Delta c(0) = 0$, such that the singular orbit Γ persists if and only if $c = c(\varepsilon)$ in (1.6). Moreover, Δc is negative, as well as $\mathcal{C}^1$ smooth in γ and ε, including at $\varepsilon = 0$.

Proof. Given the normal form (4.4), we need to determine Δc so that $\hat{P}_1^-$ is mapped to $\hat{P}_1^+$ under Π_1. We first integrate the second component of (4.4) using separation of variables to obtain

(4.7)

$$\begin{aligned}\zeta^+ - \zeta^- - \frac{1}{2}\ln|2\hat{z}^2 + 2\Delta c\hat{z} - \frac{1}{\sqrt{2}}(1-2\gamma)\Delta c - \frac{1}{4}(1+2\gamma)^2|\Big|_{\hat{z}^-}^{\hat{z}^+} \\ - \frac{\frac{1}{\sqrt{2}}(1-2\gamma) + \Delta c}{\sqrt{\frac{1}{2}(1+2\gamma)^2 + \sqrt{2}(1-2\gamma)\Delta c + (\Delta c)^2}} \\ \times \operatorname{arccoth}\left(\frac{2\hat{z} + \Delta c}{\sqrt{\frac{1}{2}(1+2\gamma)^2 + \sqrt{2}(1-2\gamma)\Delta c + (\Delta c)^2}}\right)\Bigg|_{\hat{z}^-}^{\hat{z}^+} = 0.\end{aligned}$$

Next, we recall that $\zeta^+ = -\ln\frac{\varepsilon}{\rho}$ and $\zeta^- = 0$; moreover, we make use of $\hat{z}^+ = -[\frac{1}{2\sqrt{2}}(1-2\gamma) + \Delta c] + \omega(\Delta c, \varepsilon)\varepsilon$ and $\hat{z}^- = -\frac{1}{2\sqrt{2}}(1+2\gamma) + \nu(\rho, \Delta c)\Delta c$, as given in (4.6) and (4.5), respectively. Substituting into (4.7), using the identity

$$\operatorname{arccoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right),$$

and expanding the result in terms of Δc and ε, we find that

(4.8)

$$\begin{aligned}-\ln\frac{\varepsilon}{\rho} - \frac{1}{2}\ln\left[-2\gamma - \sqrt{2}(1-2\gamma)\omega(\Delta c, \varepsilon)\varepsilon + \mathcal{O}(2)\right] \\ + \frac{1}{2}\ln\left[-\sqrt{2}[1 + (1+2\gamma)\nu(\rho, 0) + \mathcal{O}(1)]\Delta c\right] - \frac{1}{2}\left\{\frac{1-2\gamma}{1+2\gamma} + \frac{8\sqrt{2}\gamma}{(1+2\gamma)^3}\Delta c + \mathcal{O}((\Delta c)^2)\right\} \\ \times\left\{\ln\left[-2\gamma + \frac{4\sqrt{2}\gamma}{1+2\gamma}\Delta c - \sqrt{2}(1+2\gamma)\omega(\Delta c, \varepsilon)\varepsilon + \mathcal{O}(2)\right]\right. \\ \left. - \ln\left[-\sqrt{2}\frac{1 + (1+2\gamma)\nu(\rho, 0) + \mathcal{O}(1)}{(1+2\gamma)^2}\Delta c\right]\right\} = 0.\end{aligned}$$

Here, $\mathcal{O}(1)$ denotes terms of at least order 1 in Δc, and $\mathcal{O}(2)$ stands for terms of at least order 2 in Δc and ε, both $\mathcal{O}(1)$ and $\mathcal{O}(2)$ are $\mathcal{C}^\infty$-smooth, and uniform in their respective arguments, if ρ is restricted to compact subsets of $(0,1)$.

Since (4.8) can have a solution for at most one value of Δc by Lemma 4.2, we will restrict ourselves to $\Delta c < 0$ and show that a solution exists in that case. That solution will then necessarily be unique.

Taking into account that $\nu(\rho,0) > 0$ by Lemma 4.1, we exponentiate (4.8) to obtain

$$\begin{aligned} \left(\frac{\varepsilon}{\rho}\right)^{2(1+2\gamma)} &= 2(-2\gamma)^{-2}\left\{[1+(1+2\gamma)\nu(\rho,0)]\Delta c\right\}^{1+2\gamma} \\ &\quad \cdot \left\{\frac{1+(1+2\gamma)\nu(\rho,0)}{(1+2\gamma)^2}\Delta c\right\}^{1-2\gamma}[1+\mathcal{O}(1)], \end{aligned} \tag{4.9}$$

where the $\mathcal{O}(1)$-terms are now C^∞-smooth in Δc, $\Delta c \ln(\Delta c)$, and ε. (Here, the occurrence of logarithmic terms in Δc is due to the $\Delta c \ln(\Delta c)$-terms in (4.8).) Clearly, the relation in (4.9) is satisfied at $(\Delta c, \varepsilon) = (0,0)$; moreover, it is C^1-smooth in ε, Δc, γ, and ρ in a uniform fashion, for ε and Δc sufficiently small (including at $(\Delta c,\varepsilon) = (0,0)$) and γ and ρ in a compact subset of $(-\frac{1}{2},0)$ and $(0,\rho_1)$, respectively. Finally, since $1+(1+2\gamma)\nu(\rho,0) > 0$, it follows from the Implicit Function Theorem that (4.9) has a solution $\Delta c(\varepsilon,\gamma,\rho)$ that is C^1-smooth in ε (down to $\varepsilon = 0$), γ and ρ, as claimed.

By definition, that solution yields precisely the value of Δc for which a heteroclinic connection exists between the points Q_ε^- and Q_ε^+ in (1.6). Hence, $\Delta c = \Delta c(\varepsilon,\gamma)$ must hold, i.e., Δc cannot depend on ρ. In other words, Δc must be independent of the definition of the section Σ^-, which was chosen arbitrarily.

Finally, to determine the asymptotics of $\Delta c(\varepsilon) \equiv \Delta c(\varepsilon,\gamma)$, we solve (4.9) to leading order to obtain

$$\Delta c(\varepsilon) = K_\gamma \varepsilon^{1+2\gamma} + o(\varepsilon^{1+2\gamma}), \tag{4.10}$$

where the constant K_γ is defined by

$$K_\gamma = \frac{\sqrt{2}\gamma(1+2\gamma)^{1-2\gamma}}{1+(1+2\gamma)\nu(\rho,0)}\frac{1}{\rho^{1+2\gamma}} \equiv \frac{\sqrt{2}\gamma(1+2\gamma)^{1-2\gamma}}{(1+2\gamma)\delta(\gamma)} < 0, \tag{4.11}$$

and we have suppressed the ρ dependence in δ. Here,

$$\delta(\gamma) = \left[\frac{1}{1+2\gamma} + \nu(\rho,0)\right]\rho^{1+2\gamma} \tag{4.12}$$

denotes a strictly positive function that is C^∞ smooth in $\gamma \in (-\frac{1}{2},0)$, for any $\rho \in (0,1)$ fixed and sufficiently small. This completes the proof of Proposition 4.2. □

We emphasize that the definition of K_γ has to be independent of ρ, since $\Delta c(\varepsilon)$ is defined by the global condition that the singular heteroclinic orbit Γ persists for ε

sufficiently small: while the Implicit Function Theorem is applied for ρ fixed in the proof of Proposition 4.2, our argument is valid for arbitrary ρ. (To state it differently, although the function $\nu(\rho, 0)$, as defined in (2.22), may depend on the definition of Σ_1^- and, hence, on ρ, that dependence must cancel, as a matter of principle, once the dynamics of (2.4) in the outer region has been taken into account.) Therefore, the function $\delta(\gamma)$ also cannot depend on ρ, and we may obtain the value of δ by evaluating (4.12) for any $\rho \in (0, 1)$; in particular, we may pass to the limit in which $\rho = 0$. Recalling the definition of $\nu(\rho, 0)$ from (18), we obtain

$$\delta(\gamma) = \lim_{\rho \to 0^+} \{\rho^{1+2\gamma}\nu(\rho, 0)\} = \lim_{\rho \to 0^+} \{\rho^{2\gamma} \frac{\partial v}{\partial c}(\rho, c_0)\}. \tag{4.13}$$

Therefore, it remains to evaluate the above limit. From the explicit solution $\frac{\partial v}{\partial c}$ of the variational equation, recall (3.4), we find:

Lemma 4.3. The function defined in (4.12) satisfies

$$\delta(\gamma) = \lim_{\rho \to 0^+} \{\rho^{2\gamma} \frac{\partial v}{\partial c}(\rho, c_0)\} = \frac{\Gamma(3 - 2\gamma)\Gamma(1 + 2\gamma)}{\Gamma(4)}, \tag{4.14}$$

where $\frac{\partial v}{\partial c}(u, c_0)$ is as given in (3.4).

Proof. Evaluating (3.4) at $u = \rho$, for $\rho > 0$ and small, and noting that the hypergeometric function F converges absolutely at $\rho = 0$ since $\mathrm{Re}(1 + 2\gamma) > 0$, see Section 15.1 of [1], we find

$$\rho^{2\gamma} \frac{\partial v}{\partial c}(\rho, c_0) = \frac{1}{3 - 2\gamma}(1 - \rho)F(3 - 2\gamma, -2\gamma; 4 - 2\gamma; 1 - \rho).$$

Then, making use of identity

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \qquad \text{for } c \notin \mathbf{Z}_- \text{ and } \mathbf{Re}(c - a - b) > 0, \tag{4.15}$$

see (15.1.20) in [1], and taking the limit $\rho \to 0^+$ in the resulting equation, we find

$$\delta(\gamma) = \frac{1}{3 - 2\gamma} \frac{\Gamma(4 - 2\gamma)\Gamma(1 + 2\gamma)}{\Gamma(4)}.$$

Then, since $\Gamma(4 - 2\gamma) = (3 - 2\gamma)\Gamma(3 - 2\gamma)$, we find that (4.14) follows, which completes the proof. $\square$

We are now in a position to complete the proof of Theorem 1.1. In fact, by combining (4.11) and (4.14), we obtain

$$K_\gamma = \frac{\Gamma(4)}{\Gamma(1 + 2\gamma)\Gamma(3 - 2\gamma)} \frac{\sqrt{2}\gamma}{(1 + 2\gamma)^{2\gamma}},$$

which is precisely (1.5). This completes the proof of Theorem 1.1.

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