

Dynamics of pulses on a thin strip-like domain in \mathbf{R}^2

Dedication to Prof. Nishiura on his sixtieth birthday

By

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Abstract

The movement of pulse solutions on a sufficiently narrow strip-like domain in \mathbf{R}^2 , say Ω is considered. Suppose the width of Ω is constant and it is given by $\Omega := \{C(s) + \delta z\nu(s); -\infty < s < +\infty, |z| < z_0\}$ for sufficiently small $\delta > 0$ and a sufficiently smooth curve $C = \{C(s)\}$ in \mathbf{R}^2 , where s is arcwise length parameter of the center curve C and $\nu(s)$ is a unit normal vector at $C(s)$. Then it is shown that a pulse solution moves according to the gradient of $\kappa^2(s)$, where $\kappa(s)$ is the curvature of the center curve C .

§ 1. Introduction

In this paper, we consider a sufficiently narrow strip-like domain in \mathbf{R}^2 and investigate how solutions behave on it. For the researches on thin domains, there have been so many works in various fields such as fluid dynamics, nonlinear waves, dissipative systems though we omit to refer them.

From the biological point of view, problems on thin domains imply many situations such as problems near cell membranes and problems in thin tubular domains like nerve axons (refer to the book [14]). We focus on the problems in thin tubular domains in this paper. One of the typical problems in thin tubular domains is the motion of nerve impulses along nerve axons, which have been described by reaction-diffusion model equations such as the FitzHugh-Nagumo model and the Hodgkin-Huxley model (see e.g. [14]). In those models, a nerve impulse is represented as a pulse-like localized solution on one dimensional axis. However, real axons are not one-dimensional but sufficiently

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thin tubular and curved domains. Then it is natural to consider how the geometrical properties of the domains influence the motion of the pulse solutions. Intuitively, the problems in sufficiently thin tubular domains are expected to be reduced to one dimensional problems in some sense. Thus, our interest is how geometrical properties of the original thin domains is reflected in the reduced one dimensional problems.

There have been several results for such problems. In [15], a thin tubular domain $\Omega_\delta \subset \mathbf{R}^{m+1}$ defined by $\Omega_\delta := \{C(s) \times \delta D(s); 0 \leq s \leq s_0\}$ was treated, where $C := \{C(s)\}$ is a smooth curve in \mathbf{R}^{m+1} with arclength parameter s and $D(s)$ is a subset of m dimensional hyperplane intersecting normally C at $C(s)$ (Fig.1). We call the curve C "Center curve" of Ω_δ in this paper. It was shown that the dynamics of the scalar

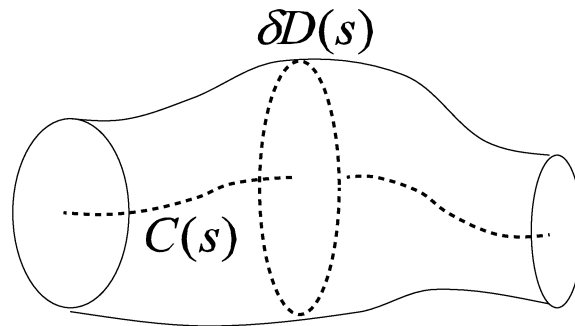


Figure 1. Domain Ω_δ .

equation

$$(1.1) \quad u_t = \Delta u + f(u)$$

in Ω_δ with the Neumann boundary condition is reduced to the dynamics of one-dimensional scalar equation

$$(1.2) \quad v_t = \frac{1}{a(s)} \{a(s)v_s\}_s + f(v), \quad 0 < s < s_0$$

with the Neumann boundary condition at $s = 0, s_0$ as $\delta \rightarrow 0$, where $a(s) := |D(s)|$, the area of $D(s)$ as m dimensional surface. By using (1.2), the existence of stationary front solutions of (1.1) was shown in [15], which are some extended results of [11] and [9]. In (1.2), there do not appear the geometrical properties of the center curve C because such geometrical properties are expected as the higher terms compared with the modulations of $a(s)$ in some sense. Moreover, the comparison principle was the main tool for the proof, which is not applicable to general reaction-diffusion systems including FitzHugh-Nagumo systems and other important model systems.

For the results treating reaction-diffusion systems, we can refer to [10] together with results for large diffusivity problems (e.g. [1], [5], [12], [8]). But all of the results

were on the first approximate motions which do not include the geometrical properties of domains such as curvatures.

In order to investigate effects of the geometrical properties of the center curve C , we consider the case that $a(s)$ is constant. The typical example is a domain with constant section. As the first step, we only consider the domain in \mathbf{R}^2 , that is, a domain with equal width as in Fig2 but we treat general reaction-diffusion systems.

For a domain with equal width in \mathbf{R}^2 , the Allen-Cahn equation

$$(1.3) \quad u_t = \varepsilon^2 \Delta u + u(1 - u^2)$$

was considered in [7] for the domain $\Omega(1)$ defined below as in Fig2 and it was shown that the motion of a front solution is essentially given by

$$\dot{h} = -\varepsilon^4 A(s) \{\kappa^2(s)\}_s|_{s=h},$$

where h denotes the location of 0-level point of a front solution, $A(s)$ is a positive function and $\kappa(s)$ is the curvature of $C(s)$. That is, the front solution moves depending on the gradient of $\kappa^2(s)$. The existence of stable stationary front solutions at points with minimal κ^2 was also shown.

In this paper, we consider an infinitely long thin strip-like domain with equal width, that is, $\mathbf{R}^2 \supset \Omega(\delta) := \{C(s) + \delta z \nu(s); -\infty < s < +\infty, |z| < z_0\}$ for sufficiently small $\delta > 0$, where $C = \{C(s)\}$ is a smooth curve in \mathbf{R}^2 and $\nu(s)$ is a unit normal vector at $C(s)$ (Fig2). We suppose the domain $\Omega(\delta)$ does not intersect itself for simplicity.

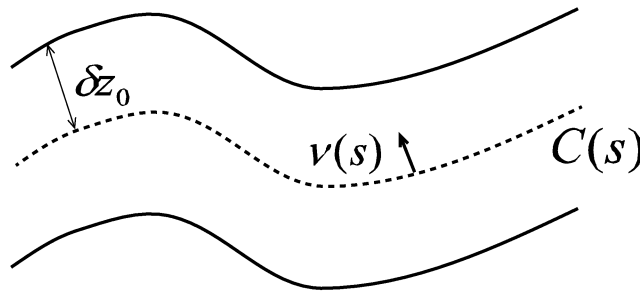


Figure 2. Domain $\Omega(\delta) \subset \mathbf{R}^2$ with equal width.

Equations which we consider here are general types of reaction-diffusion systems:

$$(1.4) \quad \mathbf{u}_t = D \Delta \mathbf{u} + F(\mathbf{u}), \quad t > 0, \mathbf{x} := (x, y) \in \Omega(\delta), \mathbf{u} \in \mathbf{R}^N$$

with the Neumann boundary condition and $D := \text{diag}\{d_1, d_2, \dots, d_N\}$ with positive components. To extract the geometrical properties of the center curve $C(s)$, we assume:

H1) There exists a linearly stable stationary symmetric pulse solution $P(s) \in \mathbf{R}^N$

with $P(s) \rightarrow 0$ as $|s| \rightarrow \infty$ for the one dimensional equation

$$(1.5) \quad \mathbf{u}_t = D\mathbf{u}_{ss} + F(\mathbf{u}), \quad -\infty < s < +\infty.$$

Let L_1 be the linearized operator $L_1 := D\partial_s^2 + F'(P(s))$ and $\varphi^*(s)$ be the adjoint eigenfunction satisfying $L_1^*\varphi^* = 0$ and $\langle P_s, \varphi^* \rangle_s = 1$, where $L_1^* := D\partial_s^2 + {}^tF'(P(s))$ and $\langle \cdot, \cdot \rangle_s$ denotes the L^2 inner product with respect to $s \in \mathbf{R}$.

Here, we rewrite $\mathbf{u}(t, x, y)$ of the arguments (x, y) to $\mathbf{u} = \mathbf{u}(t, z, s)$ of the arguments (z, s) by

$$\begin{pmatrix} z \\ s \end{pmatrix} = C(s) + \delta z\nu(s).$$

Theorem 1.1. *If $\mathbf{u}(0, z, s)$ is sufficiently close to $P(s - h_0)$ for $h_0 \in \mathbf{R}$, then the solution $\mathbf{u}(t, z, s)$ keeps close to $P(s - h(t))$ and $h(t)$ satisfies*

$$(1.6) \quad \dot{h} = -\frac{1}{3}\delta^2 z_0^2 \langle D(\kappa^2(s+h)P_s)_s, \varphi^* \rangle_s + O(\delta^3),$$

where $\kappa(s)$ is the curvature of $C(s)$.

Corollary 1.2. *If $D = \varepsilon^2 D'$ for sufficiently small $\varepsilon > 0$, then the solution $\mathbf{u}(t, z, s)$ keeps close to $P((s - h(t))/\varepsilon)$ and $h(t)$ satisfies*

$$\dot{h} = -\delta^2 \varepsilon^2 M_1 (\kappa^2(s))_s|_{s=h} + O(\varepsilon^3 \delta^2 + \delta^3),$$

where $M_1 := \frac{1}{3}z_0^2 \langle D'(s\tilde{P}_s)_s, \varphi^* \rangle_s$ and $\tilde{P}(s)$ is a linearly stable stationary symmetric pulse solution $\tilde{P}(s) \in \mathbf{R}^N$ satisfying

$$(1.7) \quad 0 = D'\tilde{P}_{ss} + F(\tilde{P}), \quad -\infty < s < +\infty$$

with $\tilde{P}(s) \rightarrow 0$ as $|s| \rightarrow \infty$.

Thus, the pulse solution essentially moves along the center curve C depending on the gradient of $\kappa^2(s)$. But the direction of the movement crucially depends on each model equation. As one example which we can calculate the equation (1.6) explicitly, a pulse solution for the Gray-Scott model will be treated.

The organization of this paper is as follows: Precise assumptions and the main results are stated in Section 2. The formal derivation of (1.6) and the application to the Gray-Scott model are in Sections 3 and 4, respectively. Proofs will be given in Section 5.

§ 2. Main results

In this section, we mention the precise assumptions and statements. Let us consider the reaction-diffusion systems (1.4) in $\Omega(\delta) := \{C(s) + \delta z\nu(s); -\infty < s < +\infty, |z| < z_0\}$ and assume (H1). We give assumptions for the linearized operator L_1 .

(H2) The spectrum of L_1 consists of $\sigma_0 \cup \sigma_1$, where $\sigma_0 := \{0\}$ and $\sigma_1 \subset \{\lambda \in \mathbf{C}; \text{Re}(\lambda) < -\gamma_0\}$ for $\gamma_0 > 0$. Moreover, 0 is a simple eigenvalue with the associated eigenfunction P_s .

Let $\Omega_0 := (-z_0, +z_0) \times \mathbf{R}$. Then we have the following theorem which is the precise statement of Theorem 1.1:

Theorem 2.1. *If $\|\mathbf{u}(0, z, s) - P(s - h_0)\|_{L^\infty(\Omega_0)} \leq O(\delta)$, then the solution $\mathbf{u}(t, z, s)$ satisfies $\|\mathbf{u}(t, z, s) - P(s - h(t))\|_{L^\infty(\Omega_0)} \leq O(\delta)$ uniformly for $t > 0$ and*

$$(2.1) \quad \dot{h} = -\frac{1}{3}\delta^2 z_0^2 \langle D(\kappa^2(s + h)P_s)_s, \varphi^* \rangle_s + O(\delta^3)$$

holds, where $\kappa(s)$ is the curvature of $C(s)$.

Let $H_1(h) := -\frac{1}{3}z_0^2 \langle D(\kappa^2(s + h)P_s)_s, \varphi^* \rangle_s$ and consider the ODE

$$(2.2) \quad \frac{dh}{dT} = H_1(h),$$

where $T := \delta^2 t$.

Theorem 2.2. *Let $h = h^*$ be an equilibrium of (2.2). If $H'_1(h^*) < 0$ (or $H'_1(h^*) > 0$) holds, then there exists a stable stationary pulse solution (or an unstable stationary one, respectively), say $\mathbf{u}^*(z, s)$ satisfying*

$$\|\mathbf{u}^*(z, s) - P(s - h^*)\|_{L^\infty(\Omega_0)} \leq O(\delta).$$

As mentioned in Corollary 1.2, $H_1(h)$ can be calculated explicitly in a special case as follows:

Corollary 2.3. *If $D = \varepsilon D'$ for sufficiently small $\varepsilon > 0$, then*

$$H_1(h) = -\varepsilon^2 M_1(\kappa^2(s))_s|_{s=h} + O(\varepsilon^3)$$

holds, where $M_1 := \frac{1}{3}z_0^2 \langle D'(s\tilde{P}_s)_s, \varphi^* \rangle_s$ and $\tilde{P}(s) \in \mathbf{R}^N$ is a linearly stable stationary symmetric pulse solution with $\tilde{P}(s) \rightarrow 0$ as $|s| \rightarrow \infty$ for the equation

$$(2.3) \quad \mathbf{u}_t = D' \mathbf{u}_{ss} + F(\mathbf{u}), \quad -\infty < s < +\infty.$$

§ 3. Derivation of (2.1)

By the coordinate transformation from (x, y) to (z, s) , we write $\mathbf{u}(t, x, y) = U(\tau, z, s)$ and (1.4) becomes

$$\begin{cases} \delta U_\tau = D \left\{ \frac{1}{\delta^2} U_{zz} - \frac{1}{\delta} \frac{\kappa(s)}{1 - \delta z \kappa(s)} U_z + \frac{1}{1 - \delta z \kappa(s)} \left(\frac{1}{1 - \delta z \kappa(s)} U_s \right)_s \right\} + F(U), \\ U_z = 0 \quad (z = \pm z_0), \end{cases}$$

where $\tau := \delta t$. Let $\mathcal{L}_\delta(U)$ be the right hand side of the above equation. We also transform the equation by $\zeta := s - h(\tau)$ and then $U = U(\tau, z, \zeta)$ satisfies

$$(3.1) \quad \delta U_\tau - \delta h_\tau U_\zeta = \mathcal{L}_\delta(U).$$

Expanding $U(\tau, z, \zeta) = P(\zeta) + \delta U_1(\tau, z, \zeta) + \dots$ and $h_\tau = H_0 + \delta H_1 + \dots$, we first have the following proposition:

Proposition 3.1. *U_1 and U_2 are independent of z argument.*

This proposition is easily checked by considering terms with orders δ^{-1} and δ^0 in (3.1).

By Proposition 3.1, we can assume U_1 and U_2 are in $E_\zeta^\perp := \{U \in L^2(\mathbf{R}); \langle U, \varphi^* \rangle_\zeta = 0\}$.

Next, we consider terms of orders δ^1 and δ^2 in (3.1). Let $\frac{1}{1 - \delta z \kappa} \left(\frac{1}{1 - \delta z \kappa} U_\zeta \right)_\zeta = U_{\zeta\zeta} + \delta z K_1 U + \delta^2 z^2 K_2 U + \dots$, where $K_1 U := \{\kappa U_{\zeta\zeta} + (\kappa U_\zeta)_\zeta\}$ and $K_2 U := \{\kappa^2 U_{\zeta\zeta} + \kappa(\kappa U_\zeta)_\zeta + (\kappa^2 U_\zeta)_\zeta\}$ and so on. Equating terms of δ^1 in (3.1), we have

$$(3.2) \quad -H_0 P_\zeta = D \{ \partial_z^2 U_3 + \partial_\zeta^2 U_1 + z K_1 P \} + F'(P) U_1.$$

Integrating both side of (3.2) with respect to z by using the Neumann boundary conditions at $z = \pm z_0$ and the oddness of $z K_1 P$ with respect to z , we see

$$(3.3) \quad -H_0 P_\zeta = D \partial_\zeta^2 U_1 + F'(P) U_1 = L_1 U.$$

Hence taking the L^2 inner product with $\varphi^*(\zeta)$, we have

$$-H_0 \langle P_\zeta, \varphi^* \rangle_\zeta = \langle L_1 U_1, \varphi^* \rangle_\zeta = 0$$

and $H_0 = 0$ is obtained. Then (3.3) leads to $L_1 U_1 = 0$ and $U_1 = 0$ holds by $U_1 \in E_\zeta^\perp$.

Next, equating terms of δ^2 in (3.1), we have

$$(3.4) \quad -H_1 P_\zeta = D \{ \partial_z^2 U_4 - \kappa \partial_z U_3 + \partial_\zeta^2 U_2 + z^2 K_2 P \} + F'(P) U_2.$$

Integrating both side of (3.4) with respect to z by using the Neumann boundary conditions at $z = \pm z_0$, we find (3.4) becomes

$$(3.5) \quad -2z_0 H_1 P_\zeta = -\kappa [DU_3]_{-z_0}^{z_0} + 2z_0 L_1 U_2 + \frac{2}{3} z_0^3 DK_2 P.$$

Here U_3 satisfies

$$0 = D\partial_z^2 U_3 + zDK_1 P$$

from (3.2) and $U_1 = 0$. Integrating the equation, we have

$$0 = D\partial_z U_3 + \frac{1}{2}(z^2 - z_0^2)DK_1 P$$

and therefore

$$(3.6) \quad [DU_3]_{-z_0}^{z_0} = \frac{2}{3} z_0^3 DK_1 P$$

holds. Substituting (3.6) into (3.5), we obtain

$$(3.7) \quad H_1 = -\frac{z_0^2}{3} \langle D(\kappa^2 P_\zeta)_\zeta, \varphi^* \rangle_\zeta,$$

where $\kappa = \kappa(\zeta + h)$.

§ 4. Application to the Gray-Scott model

In this section, we consider the application to the Gray-Scott model

$$(4.1) \quad \begin{cases} u_t = \Delta u - uv^2 + \varepsilon^2 a(1 - u), \\ v_t = \varepsilon^2 \Delta v - \varepsilon^{\frac{1}{2}} bv + uv^2 \end{cases}$$

in $\Omega(\delta)$ with positive constants a, b and a sufficiently small $\varepsilon > 0$. For the one dimensional equation of (4.1)

$$(4.2) \quad \begin{cases} u_t = u_{xx} - uv^2 + \varepsilon^2 a(1 - u), \\ v_t = \varepsilon^2 v_{xx} - \varepsilon^{\frac{1}{2}} bv + uv^2, \end{cases}$$

the existence of a stable stationary symmetric pulse solution $P(x) \in \mathbf{R}^2$ was shown in [2]. And also the adjoint eigenfunction φ^* was explicitly calculated for the Gray-Scott model in [3] as follows: Let the stationary pulse solution $P(x) = {}^t(\Phi(x), \Psi(x))$. Since $P(x)$ is even and the adjoint eigenfunction $\varphi^*(x)$ is odd, it suffices to consider only for $x > 0$. In [2] $P(x)$ is explicitly given by

$$\Phi(x) = \begin{cases} \varepsilon^{3/4} \{p_0 + o(1)\} & (0 < x \ll 1), \\ \Phi_0(x) & (x > 0), \end{cases} \quad \Psi(x) = \varepsilon^{-1/4} \{q_0(\xi) + O(\varepsilon^{1/2})\} \quad (x > 0),$$

where $p_0 := 3b\sqrt{b/a}$, $q_0(\xi) := \frac{1}{2}\text{sech}^2(\frac{\sqrt{b}}{2}\xi)$, $\xi := \varepsilon^{-3/4}x$ and $\Phi_0(x)$ is a function satisfying

$$0 = \Phi_0'' + \varepsilon^2 a(1 - \Phi_0), \quad \Phi_0(0) = \varepsilon^{3/4}p_0, \quad \Phi_0(\infty) = 1.$$

In [3] the adjoint eigenfunction $\varphi^*(x) = {}^t(\Phi^*(x), \Psi^*(x))$ normalized by $\langle P_x, \varphi^* \rangle_x = 1$ is uniquely given by

$$\Phi^*(x) = \begin{cases} -\varepsilon^{5/4}r_0\{p_1(\xi) + o(1)\} & (0 < x \ll 1), \\ -\varepsilon^{5/4}r_0\{r_1e^{-\varepsilon\sqrt{a}x} + o(1)\} & (x > 0), \end{cases} \quad \Psi^*(x) = -\varepsilon^{1/4}r_0\{q_1(\xi) + o(1)\} \quad (x > 0),$$

where $r_0 > 0$, $q_1(\xi) := \text{sech}^2(\frac{\sqrt{b}}{2}\xi) \tanh(\frac{\sqrt{b}}{2}\xi)$,

$$p_1(\xi) := \int_0^\xi \left\{ - \int_0^{\xi'} q_0^2(\xi'')q_1(\xi'')d\xi'' + p_2 \right\} d\xi',$$

$p_2 := \int_0^\infty q_0^2(\xi)q_1(\xi)d\xi$ and $r_1 := p_1(\infty) > 0$. Substituting the above into $H_1(h)$, we can calculate directly the explicit form of $H_1(h)$ as follows:

Theorem 4.1. For (4.1), $H_1(h)$ in (2.2) is given by

$$H_1(h) = -\varepsilon^2 M_2(\kappa^2(s))_s|_{s=h} + O(\varepsilon^3),$$

where

$$M_2 := -\frac{2}{3}r_0z_0^2 \int_0^\infty (s(\text{sech}^2s)_s)_s \text{sech}^2s \tan s ds > 0.$$

Theorem 4.1 implies the pulse solution for (4.1) moves toward less curved point of the center curve $C(s)$ of $\Omega(\delta)$.

§ 5. Proofs

§ 5.1. Proof of Theorem 2.1

Proofs are mainly based on [3] and [7].

In Section 3, we constructed approximate functions upto $O(\delta^3)$. Quite similarly we can construct them upto $O(\delta^4)$, that is, let

$$U(h^\delta; \delta)(z, \zeta) := P(\zeta) + \delta^2 U_2(h^\delta)(z, \zeta) + \delta^3 U_3(h^\delta)(z, \zeta) + \delta^4 U_4(h^\delta)(z, \zeta) + \delta^5 U_5(h^\delta)(z, \zeta)$$

where $\zeta := s - h^\delta$ and $h^\delta = h^\delta(T)$ satisfies $\frac{dh^\delta}{dT} = H_1(h) + \delta H_2(h)$ for $T := \delta^2 t$. Then $\mathbf{u}^\delta(t, x, y) := U(h^\delta(\delta^2 t); \delta)(z, \zeta)$ satisfies

$$\mathbf{u}_t^\delta = D\Delta \mathbf{u}^\delta + F(\mathbf{u}^\delta) + O(\delta^4)$$

in $\Omega(\delta)$ because $U^\delta(h^\delta)(z, \zeta) := U(h^\delta; \delta)(z, \zeta)$ satisfies $\mathcal{L}_\delta(U^\delta) + \delta^2(H_1(h) + \delta H_2(h))(U_\zeta^\delta - U_h^\delta) = O(\delta^4)$ in Ω_0 , where $\mathcal{L}_\delta(U)$ is defined in Section 3. Expand

$$\mathcal{L}_\delta(U) = D\left\{\frac{1}{\delta^2}U_{zz} + U_{ss}\right\} + F(U) + D\left\{\frac{1}{\delta}\Xi(\delta)U_z + \delta z K_1 U + \delta^2 z^2 K_2 U + \dots + \delta^n z^n K_n U + \dots\right\},$$

where $\Xi(\delta) := -\frac{\kappa(s)}{1 - \delta z \kappa(s)}$ and K_j are defined in Section 3. Since (1.4) in $\Omega(\delta)$ and

$$(5.1) \quad U_t = \mathcal{L}_\delta(U)$$

in Ω_0 are equivalent by $\mathbf{u}(t, x, y) = U(t, z, s)$, we consider (5.1) in Ω_0 hereafter. Let $X := C_{unif}(\Omega_0)$ with L^∞ norm and

$$L_2(h)U := D\left\{\frac{1}{\delta^2}U_{zz} + U_{ss}\right\} + F'(P(s - h))U.$$

$\langle \cdot, \cdot \rangle_2$ denotes L^2 inner product in $L^2(\Omega_0)$.

Proposition 5.1. *Suppose $\delta > 0$ is sufficiently small. Then the spectrum of $L_2(h)$ consists of $\sigma_0 \cup \sigma_2$, where $\sigma_0 := \{0\}$ and $\sigma_2 \subset \{\lambda \in \mathbf{C}; \operatorname{Re}(\lambda) < -\gamma_0\}$. 0 is a simple eigenvalue with the associated eigenfunction P_ζ .*

Proof. By using $\zeta := s - h$, we see the operator $L_2(h)$ becomes $L_2 := D\left\{\frac{1}{\delta^2}\partial_z^2 + \partial_\zeta^2\right\} + F'(P(\zeta))$, which means it is enough to consider L_2 . Expanding $U \in X$ as

$$U(z, \zeta) = \sum_{n=0}^{\infty} \cos \frac{n\pi(z + z_0)}{2z_0} \mathbf{a}_n(\zeta)$$

and substituting into the eigenvalue problem $(L_2 - \lambda)U = \mathbf{g}$ for $\mathbf{g} \in X$, we have

$$(5.2) \quad -\tau_n D \mathbf{a}_n + (L_1 - \lambda) \mathbf{a}_n = \mathbf{g}_n,$$

where $\mathbf{g} = \sum_{n=0}^{\infty} \cos \frac{n\pi(z+z_0)}{2z_0} \mathbf{g}_n(\zeta)$ and $\tau_n := (\frac{n\pi}{2z_0\delta})^2$. Since $L_1 = D\partial_\zeta^2 + F'(P(\zeta))$, (5.2) is written as

$$(D\partial_\zeta^2 - \tau_n D - \lambda) \mathbf{a}_n + F'(P(\zeta)) \mathbf{a}_n = \mathbf{g}_n,$$

or equivalently

$$(5.3) \quad \{Id + (D\partial_\zeta^2 - \tau_n D - \lambda)^{-1} F'(P(\zeta))\} \mathbf{a}_n = (D\partial_\zeta^2 - \tau_n D - \lambda)^{-1} \mathbf{g}_n$$

for $\lambda \in \rho(D\partial_\zeta^2 - \tau_n D)$, where Id is the identity. Let $d_{min} := \min\{d_1, d_2, \dots, d_N\}$ for $D = \operatorname{diag}\{d_1, d_2, \dots, d_N\}$. Then the spectral set of $(D\partial_\zeta^2 - \tau_n D)$ satisfies $\sigma(D\partial_\zeta^2 - \tau_n D) \subset (-\infty, -\tau_n d_{min}]$ and $\|(D\partial_\zeta^2 - \tau_n D - \lambda)^{-1}\| \leq \frac{C_0}{|\lambda + \tau_n d_{min}|}$ holds for λ outside of a sector and $C_0 > 0$. Hence if $|\lambda + \tau_n d_{min}|$ is sufficiently large, say $|\lambda + \tau_n d_{min}| \geq C_1$ for $C_1 > 0$,

the operator $\{Id + (D\partial_\zeta^2 - \tau_n D - \lambda)^{-1} F'(P(\zeta))\}$ is invertible and $\|\mathbf{a}_n\| \leq \frac{C_2}{|\lambda + \tau_n d_{min}|} \|\mathbf{g}_n\|$ holds for $C_2 > 0$ in (5.3).

Since $\tau_0 = 0$ for $n = 0$, (5.2) is solvable for $\lambda \in \rho(L_1)$, the resolvent set of L_1 .

For $n \geq 1$, $\tau_n \geq \frac{C_3}{\delta^2}$ holds for $C_3 > 0$. Hence $|\lambda + \tau_n d_{min}| \geq C_1$ holds for any $n \geq 1$ if $Re(\lambda) > -\frac{C_4}{\delta^2}$ for $C_4 > 0$. Thus, $\rho(L_1) \cap \{\lambda \in \mathbf{C}; Re(\lambda) > -\frac{C_4}{\delta^2}\}$ is included in the resolvent set $\rho(L_2)$. \blacksquare

Let the sector $S_{\theta, \gamma} := \{\lambda \in \mathbf{C}; |arg(\lambda + \gamma)| < \theta\}$. Then we may assume by Proposition 5.1 that the resolvent set $\rho(L_2(h))$ includes $S_{\theta_0, \gamma_0} \setminus \sigma_0$ for $\pi/2 < \theta_0 < \pi$ and that $\|(L_2(h) - \lambda)^{-1}\| \leq \frac{C_5}{|\lambda|}$ holds, specially

$$(5.4) \quad \|(L_2(h) - \lambda)^{-1} \mathbf{g}\| \leq \frac{C_5}{|\lambda + \tau_1 d_{min}|} \|\mathbf{g}\| \leq \frac{C_5}{\tau_1 d_{min} \sin \theta_0} \|\mathbf{g}\| \leq C_6 \delta^2 \|\mathbf{g}\|$$

holds for $\lambda \in S_{\theta_0, \gamma_0}$ and $C_5 > 0$, $C_6 > 0$ if $\bar{\mathbf{g}} = 0$, where $\bar{U} := \frac{1}{2z_0} \int_{-z_0}^{z_0} U dz$ for $U \in X$. We also define the operator $SU := U - \bar{U}$ and the notation $\langle U \rangle := \bar{U}$. Then (5.4) implies

$$(5.5) \quad \|(L_2(h) - \lambda)^{-1} \mathbf{g}\| \leq C_6 \left\{ \frac{1}{|\lambda|} \|\bar{\mathbf{g}}\| + \delta^2 \|S\mathbf{g}\| \right\}$$

for $\mathbf{g} \in X$ and $\lambda \in S_{\theta_0, \gamma_0} \setminus \sigma_0$.

The adjoint operator $L_2^*(h)$ of $L_2(h)$ has also the same properties as $L_2(h)$. Specially, 0 is a simple eigenvalue of $L_2^*(h)$ with the associated eigenfunction $\varphi^*(s - h) = \varphi^*(\zeta)$. Let $E_2(h) := span\{P_\zeta(\cdot - h)\}$, $E_2^\perp(h) := \{U \in X; \langle U, \varphi^*(\cdot - h) \rangle_2 = 0\}$ and projections $Q_2(h) : X \rightarrow E_2(h)$, $R_2(h) : X \rightarrow E_2^\perp(h)$. We note

$$(5.6) \quad \|(L_2(h) - \lambda)^{-1} \mathbf{g}\| \leq C_6 \left\{ \frac{1}{|\lambda| + \gamma_0} \|\bar{\mathbf{g}}\| + \delta^2 \|S\mathbf{g}\| \right\}$$

holds for $\mathbf{g} \in E_2^\perp(h)$ and $\lambda \in S_{\theta_0, \gamma_0} \setminus \sigma_0$.

Let $X_1 := C_{unif}(\mathbf{R}^1)$, $E_1(h) := span\{P_\zeta(\cdot - h)\}$ in X_1 and $E_1^\perp(h) := \{U \in X_1; \langle U, \varphi^*(\cdot - h) \rangle_\zeta = 0\}$. We also define the projections in X_1 , $Q_1(h) : X_1 \rightarrow E_1(h)$, $R_1(h) : X_1 \rightarrow E_1^\perp(h)$. Then the followings are easily checked by the facts $Q_2(h)V = \frac{1}{2z_0} \langle V, \varphi^*(\cdot - h) \rangle_2 P_\zeta(\cdot - h)$, $R_2(h)V = V - \frac{1}{2z_0} \langle V, \varphi^*(\cdot - h) \rangle_2 P_\zeta(\cdot - h)$ and so on.

Proposition 5.2. For $V \in X$, $Q_2(h)S = SQ_2(h)$, $R_2(h)S = SR_2(h)$, $Q_1(h)\bar{V} = \overline{Q_2(h)V}$ and $R_1(h)\bar{V} = \overline{R_2(h)V}$ hold.

Let $\mathcal{M}_0 := \{P(\cdot - h); h \in \mathbf{R}\}$. Since any $U \in X$ in the neighborhood of \mathcal{M} is uniquely expressed as $U(z, s) = P(s - h) + V(z, s)$ with $V \in E_2^\perp(h)$, we express the equation $U(t, z, \zeta)$ of (5.1) by

$$U(t, z, s) = U^\delta(h(t))(z, s - h) + V(t, z, s)$$

with $V \in E_2^\perp(h)$. Substituting the representation into (5.1), we have

$$V_t = \mathcal{L}'_\delta(U^\delta(h))V + \{\dot{h} - \delta^2(H_1 + \delta H_2)\}\{U_\zeta^\delta(h) - U_h^\delta(h)\} + G(V) + O(\delta^4)$$

with $|G(V)| \leq O(|V|^2)$ for sufficiently small $\delta > 0$ and $V \in E_2^\perp(h)$.

Fix $h_0 \in \mathbf{R}$ arbitrarily and define the map $\Pi(h)$ from $E_2^\perp(h_0)$ to $E_2^\perp(h)$ as the solution $V = \Pi(h)W$ of

$$\begin{cases} \frac{dV}{dh} = \frac{1}{2z_0} \langle V, \phi_\zeta^*(s-h) \rangle_2 P_\zeta(s-h), \\ V(h_0) = W \in E_2^\perp(h_0). \end{cases}$$

Operating $R_2(h)$ and taking inner product with $\varphi^*(\cdot - h)$ in X by transforming $V = \Pi(h)W$, we see

$$(5.7) \quad \begin{cases} W_t = \mathcal{A}(h)W + J(h, W), \\ h_t = H(h, W), \end{cases}$$

where

$$\mathcal{A}(h)W := \Pi(-h)R_2(h)\mathcal{L}'_\delta(U^\delta)\Pi(h)W,$$

$$J(h, W) := -H(h, V)\Pi(-h)R_2(h)\Pi_h W$$

$$\begin{aligned} & + \{H(h, V) - \delta^2(H_1 + \delta H_2)\}\Pi(-h)R_2(h)\{U_\zeta^\delta - U_h^\delta\} + \Pi(-h)R_2(h)G + O(\delta^4) \\ & = \{H(h, V) - \delta^2(H_1 + \delta H_2)\}\Pi(-h)R_2(h)\{U_\zeta^\delta - U_h^\delta\} + \Pi(-h)R_2(h)G + O(\delta^4), \end{aligned}$$

$$H(h, W) := \frac{-\langle \mathcal{L}'_\delta(U^\delta)V + G(V) + O(\delta^4), \varphi^* \rangle_2 + \delta^2(H_1(h) + \delta H_2(h)) \langle U_\zeta^\delta, \varphi^* \rangle_2}{\langle U_\zeta^\delta - \Pi_h W, \varphi^* \rangle_2}$$

and $V = \Pi(h)W$. Here we note that $R_2(h)\Pi_h W = 0$ holds. Let $A(h) := \Pi(-h)L_2(h)\Pi(h)$. Since

$$\begin{aligned} \mathcal{L}'_\delta(U^\delta)V &= L_2(h)V + \delta^2 F''(P)(U_2 + \delta U_3 + O(\delta^2))V + O(\delta^4)V \\ &+ D\left\{\frac{1}{\delta}\Xi(\delta)V_z + \delta z K_1 V + \delta^2 z^2 K_2 V + O(\delta^3)V\right\}, \end{aligned}$$

we write

$$\mathcal{A}(h) = A(h) + \delta^2 B(\delta) + K(\delta),$$

where $\delta^2 B(\delta) := R_2(h)\{F'(U^\delta) - F'(P)\}R_2(h)$ and

$$\begin{aligned} K(\delta)V &:= R_2(h)D\left\{\frac{1}{\delta}\Xi(\delta)\partial_z + \delta z K_1 + \delta^2 z^2 K_2 + O(\delta^3)\right\}V \\ &= R_2(h)D\left\{\frac{1}{\delta}\Xi(\delta)V_z + \frac{1}{1 - \delta z \kappa} \left(\frac{1}{1 - \delta z \kappa} V_\zeta\right)_\zeta - V_{\zeta\zeta}\right\} \end{aligned}$$

for $V \in E_2^\perp(h)$.

Lemma 5.3. *The set $S_{\theta_0, \gamma_0} \setminus D(C_7\delta)$ is in the resolvent set of $\mathcal{A}(h)$ for a sufficiently large constant $C_7 > 0$, where $D(r) := \{\lambda \in \mathbf{C}; |\lambda| \leq r\}$. For $\lambda \in S_{\theta_0, \gamma_0} \setminus D(C_7\delta)$, $\|(\mathcal{A}(h) - \lambda)^{-1}\| \leq \frac{C_8}{|\lambda| + \gamma_0}$ holds in $E_2^\perp(h)$ for $C_8 > 0$*

Proof. In this proof, we fix h arbitrarily and omit to write h explicitly. That is, R_2 denotes $R_2(h)$ and so on. For $\lambda \in S_{\theta_0, \gamma_0} \setminus D(C_7\delta)$, we consider $(\mathcal{A} - \lambda)V = \mathbf{f}$ for V and $\mathbf{f} \in E_2^\perp$. Let $V = \bar{V} + W = R_2\bar{V} + R_2W$ and $\mathbf{f} = \bar{\mathbf{f}} + \mathbf{g} = R_2\bar{\mathbf{f}} + R_2\mathbf{g}$ with $\bar{W} = \bar{\mathbf{g}} = 0$. That is, we may assume $\bar{V}, W, \bar{\mathbf{f}}, \mathbf{g} \in E_2^\perp$. Then it follows that

$$(L_1 - \lambda)\bar{V} + (L_2 - \lambda)W + (\delta^2 B(\delta) + K(\delta))(\bar{V} + W) = \bar{\mathbf{f}} + \mathbf{g}.$$

On the other hand, integrating with respect to z , we also have

$$(L_1 - \lambda)\bar{V} + \langle (\delta^2 B(\delta) + K(\delta))(\bar{V} + W) \rangle = \bar{\mathbf{f}}$$

because $R_2 S = S R_2$ holds. Subtracting each other, we see

$$(5.8) \quad \begin{cases} (L_2 - \lambda)W + S(\delta^2 B(\delta) + K(\delta))(\bar{V} + W) = \mathbf{g}, \\ (L_1 - \lambda)\bar{V} + \langle (\delta^2 B(\delta) + K(\delta))(\bar{V} + W) \rangle = \bar{\mathbf{f}}. \end{cases}$$

Proposition 5.4. $\|(L_2 - \lambda)^{-1} S K(\delta)\| \leq O(\delta)$ holds for $\lambda \in S_{\theta_0, \gamma_0} \setminus \sigma_0$.

Proof. First, we shall show $\|(L_2 - \lambda)^{-1} S R_2 D\Xi(\delta) \partial_z\| \leq O(\delta^2)$. Put $v := (L_2 - \lambda)^{-1} S R_2 D\Xi(\delta) w_z \in E_2^\perp$ for $w \in E_2^\perp$. Then $(L_2 - \lambda)v = S R_2 D\Xi(\delta) w_z$ holds. Since

$$S R_2 D\Xi(\delta) w_z = S\{D\Xi(\delta) w_z - Q_2 D\Xi(\delta) w_z\} = S D\Xi(\delta) w_z = D\Xi(\delta) w_z - \langle D\Xi(\delta) w_z \rangle$$

hold, we define v_1 by

$$(L_2 - \lambda)v_1 = S\{D\Xi(\delta)w - \int^z D\Xi_z(\delta)w dz\} - z \langle D\Xi(\delta)w_z \rangle.$$

Then (5.6) implies $\|v_1\|, \|\partial_z v_1\|, \|\partial_z^2 v_1\| \leq O(\delta^2)\|w\|$ because

$$\langle S\{D\Xi(\delta)w - \int^z D\Xi_z(\delta)w dz\} - z \langle D\Xi(\delta)w_z \rangle \rangle = 0.$$

Since $(R_2 w)_z = (S w)_z = w_z$ holds, $\partial_z v_1$ satisfies

$$(L_2 - \lambda)\partial_z v_1 = D\Xi(\delta)w_z - \langle D\Xi(\delta)w_z \rangle.$$

Defining $v_2 := v - \partial_z v_1$, we see

$$(5.9) \quad \begin{cases} (L_2 - \lambda)v_2 = 0, \\ \partial_z v_2 = -\partial_z^2 v_1 \quad (z = \pm z_0), \end{cases}$$

which means $\|v_2\| \leq O(\|v_1\| + \|\partial_z v_1\| + \|\partial_z^2 v_1\|) \leq O(\delta^2)\|w\|$. Hence $\|v\| \leq \|v_2\| + \|\partial_z v_1\| \leq O(\delta^2)\|w\|$. Thus

$$(5.10) \quad \|(L_2 - \lambda)^{-1} S R_2 D \Xi(\delta) \partial_z\| \leq O(\delta^2)$$

is shown.

Other terms such as $\delta z K_1$ and $\delta^2 z^2 K_2$ are estimated as

$$\|(L_2 - \lambda)^{-1} S R_2 (z^n D K_j)\| \leq C_9 \delta^2$$

for $C_9 > 0$, which is shown by quite a similar manner to Proposition 4.1 in [6] and we omit the details. \blacksquare

The first equation (5.8) is written as

$$(5.11) \quad W + (L_2 - \lambda)^{-1} S \{\delta^2 B(\delta) + K(\delta)\} W = (L_2 - \lambda)^{-1} \mathbf{g} - (L_2 - \lambda)^{-1} S \{\delta^2 B(\delta) + K(\delta)\} \bar{V}.$$

Since $\|(L_2 - \lambda)^{-1} S \{\delta^2 B(\delta) + K(\delta)\}\| \leq O(\delta)$ by Proposition 5.4, $Id + (L_2 - \lambda)^{-1} S \{\delta^2 B(\delta) + K(\delta)\}$ is invertible for $\lambda \in S_{\theta_0, \gamma_0} \setminus D(C_7 \delta)$, where $C_7 > 0$ is sufficiently large constant independent of δ and W is solvable in (5.11) for given \bar{V} .

On the other hand, the second equation of (5.8) is written as

$$(L_1 - \lambda) \bar{V} + \langle (\delta^2 B(\delta) + K(\delta)) \bar{V} \rangle = \bar{\mathbf{f}} - \langle (\delta^2 B(\delta) + K(\delta)) W \rangle$$

and hence

$$\bar{V} + (L_1 - \lambda)^{-1} \langle (\delta^2 B(\delta) + K(\delta)) \bar{V} \rangle = (L_1 - \lambda)^{-1} \{ \bar{\mathbf{f}} - \langle (\delta^2 B(\delta) + K(\delta)) W \rangle \}.$$

The left hand side of the equation is invertible with respect to \bar{V} because $K(\delta) \bar{V}$ does not include the term $\frac{1}{3} \Xi(\delta) \partial_z$ and other terms of $K(\delta)$ are estimated by quite a similar manner to Proposition 4.1 in [6]. Then \bar{V} is given by

$$\begin{aligned} \bar{V} &= \{ Id + (L_1 - \lambda)^{-1} \langle (\delta^2 B(\delta) + K(\delta)) \cdot \rangle \}^{-1} (L_1 - \lambda)^{-1} \{ \bar{\mathbf{f}} - \langle (\delta^2 B(\delta) + K(\delta)) W \rangle \} \\ &=: V^{**} \bar{\mathbf{f}} + V^* W \end{aligned}$$

and therefore

$$\|V^{**}\| \leq \frac{C_{10}}{|\lambda| + \gamma_0}, \quad \|V^*\| \leq \frac{C_{10}}{(|\lambda| + \gamma_0) \delta}$$

hold. Substituting \bar{V} into (5.11), we see that the right hand side of (5.11) is estimated as

$$\|(L_2 - \lambda)^{-1} \mathbf{g} - (L_2 - \lambda)^{-1} S \{\delta^2 B(\delta) + K(\delta)\} \bar{V}\| \leq \frac{C_{11}}{|\lambda| + \gamma_0} \{ \|\mathbf{g}\| + \delta^2 \|\bar{\mathbf{f}}\| + \delta \|W\| \}$$

because $\frac{1}{\delta}\Xi\partial_z\bar{V} = 0$ holds and therefore $K(\delta)\bar{V}$ does not include terms of $O(1/\delta)$, which implies

$$\|(L_2 - \lambda)^{-1}S\{\delta^2 B(\delta) + K(\delta)\}\bar{V}\| \leq O(\delta^2)\|\bar{V}\|.$$

Thus W is solvable as

(5.12)

$$W = [Id + (L_2 - \lambda)^{-1}S\{\delta^2 B(\delta) + K(\delta)\}(Id + V^*)]^{-1}(L_2 - \lambda)^{-1}\{\mathbf{g} - S(\delta^2 B(\delta) + K(\delta))V^{**}\bar{\mathbf{f}}\},$$

which means $\|W\| \leq \frac{C_{12}}{|\lambda| + \gamma_0}\{\|\mathbf{g}\| + \delta^2\|\bar{\mathbf{f}}\|\}$. Hence $\|\bar{V}\| \leq \frac{C_{12}}{|\lambda| + \gamma_0}\{\|\mathbf{g}\| + \|\bar{\mathbf{f}}\|\}$ also holds.

Thus this lemma is proved. \blacksquare

Defining $W(D_1) := \{W \in E_2^\perp(h_0); \|W\| \leq D_1\delta^4\}$ and $W(D_1, D_2) := \{V \in C(\mathbf{R}; E_2^\perp(h_0)); \|W\| \leq D_1\delta^4, \|W(\zeta) - W(\zeta')\| \leq D_2\delta^4|\zeta - \zeta'|\}$, we see $H(h, W) = \delta^2 H_1(h) + O(\delta^3)$ and $J(h, W) = O(\delta^4)$ hold together with

$$|H(h, W) - H(h', W')| \leq C_{13}\{\delta^2|h - h'| + \frac{1}{\delta}\|W - W'\|\},$$

$$\|J(h, W) - J(h', W')\| \leq C_{13}\{\delta^4|h - h'| + \delta\|W - W'\|\}$$

for $W, W' \in W(D_1)$. Here we used the facts $R_2(h)U_\zeta^\delta = O(\delta^2)$ and so on. Then, quite a similar way to [3], we can take appropriate D_1 and D_2 such that there exists an exponentially attractive invariant manifold $\mathcal{M} := \{(\Sigma(h), h) \in E^\perp(h_0) \times \mathbf{R}; h \in \mathbf{R}\}$ of (5.7) with $\Sigma \in W(D_1, D_2)$. This means the solution $U(t, z, s)$ of (5.1) is given by

$$U(t, z, s) = U^\delta(h(t))(z, s - h(t)) + \Pi(h(t))\Sigma(h(t))$$

in the neighborhood of \mathcal{M}_0 and $h(t)$ is the solution of $\dot{h} = H(h, \Sigma(h)) = \delta^2\{H_1(h) + O(\delta)\}$. Thus the proof of the theorem is completed. \blacksquare

§ 5.2. Proof of Theorem 2.2

The invariant manifold \mathcal{M} is Lipschitz continuous with $O(\delta^4)$ small Lipschitz constant. Hence $H(h, \Sigma(h))$ can be written as $H(h, \Sigma(h)) = \delta^2\{H_1(h) + \delta H^*(h; \delta)\}$ satisfying $|H^*(h; \delta) - H^*(h'; \delta)| \leq C_{14}|h - h'|$.

Let h^* be an equilibrium of (2.2) satisfying $H'_1(h^*) < 0$ and let $I(h) := H_1(h) + \delta H^*(h; \delta)$. We shall show that $I(h)$ has an equilibrium $h = h(\delta)$ satisfying $h(\delta) = h^* + O(\delta)$ and $I(h)$ is monotone decreasing in the neighborhood of $h = h(\delta)$, which means $h(\delta)$ is a stable equilibrium of the ODE

$$(5.13) \quad \dot{h} = H(h, \Sigma(h)) = \delta^2 I(h).$$

Defining $\gamma_1 := -H'_1(h^*) > 0$ and substituting $h = h^* + l$ into $I(h^* + l) = 0$, we have $0 = -\gamma_1 l + O(l^2) + \delta H^*(h^* + l)$ and

$$l = \delta \frac{1}{\gamma} H^*(h^* + l) + O(l^2).$$

Then it is easy to show the right hand side of the above equation is a contraction in the set $D_\delta := \{|l| \leq C_{15}\delta\}$ for an appropriate $C_{15} > 0$. Thus the fixed value, say $l = l^*(\delta) = O(\delta)$ gives the equilibrium $h(\delta) = h^* + l^*(\delta)$ of $I(h)$. In the neighborhood of $h(\delta)$, $I(h)$ is written as

$$I(h(\delta) + h) = -\gamma_1 h + O(h^2 + \delta^2) + \delta H^*(h^* + l^* + h) - \gamma_1 l^*(\delta).$$

Hence

$$I(h(\delta) + h) - I(h(\delta) + h') = \{-\gamma_1 + O(|h| + |h'| + \delta)\}(h - h'),$$

which means $I(h(\delta) + h)$ is monotone decreasing for sufficiently small h and h' .

Other cases are similarly shown. ■

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