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Comparison of Front Speed Asymptotics of G-Equation and its Variants

To Professor Yasumasa Nishiura on his 60th birthday with friendship and appreciation

By

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Abstract

G-equations are well-known front propagation models in turbulent combustion and describe the front motion law in the form of local normal velocity equal to a constant (laminar speed) plus the normal projection of fluid velocity. In level set formulation, G-equations are Hamilton-Jacobi equations with convex (L^1 type) but non-coercive Hamiltonians. The large time front speed is also known as the turbulent burning velocity s_T, a fundamental object in turbulent combustion research. We review recent progress on the sensitivity of the s_T asymptotics in strong spatially periodic flows. We illustrate how s_T asymptotics may alter under the variations of compressibility, viscosity, strain, and nonlinearities (L^1 vs. L^2 type). These variations of G-equations arise either from physical modeling, numerical approximation, or asymptotics of reaction-diffusion-advection equations. Viscosity in G-equation works against the flow. It arrests front stagnation in compressible flows and reduces front speed-up in incompressible flows. Flow induced strain in G-equations decreases front speeds in compressible and shear flows. Modifying the L^1 nonlinearity to L^2 type in the viscous G-equation makes a dramatic difference in s_T asymptotics for cellular flows. However, such nonlinearity modification does not change the s_T asymptotics of the inviscid G-equation for cellular flows. Future work remains on how s_T may vary under the variations of G-equations in more complex flows.

§ 1. Introduction

Turbulent combustion is a challenging, far from equilibrium, nonlinear and multiscale dynamic phenomenon [26, 32]. A first principle physical-chemical modeling

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requires at least a system of reaction-diffusion-advection equations coupled with the Navier-Stokes equations. However, theoretical understanding and efficient modeling of the turbulent flame propagation often rely on simplified models such as the passive scalar reaction-diffusion-advection equations (RDA) and Hamilton-Jacobi equations (HJ), as documented in books [31, 26, 33] and research papers [1, 3, 8, 9, 11, 18, 21, 24, 27, 29, 30, 32, 36].

The passive scalar reaction-diffusion-advection equation is:

$$T_t + V(x,t) \cdot DT = d\,\Delta T + \frac{1}{\tau_r} f(T), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where T represents the reactant temperature, D is the spatial gradient operator, $V(x,t)$ is a prescribed fluid velocity, f is a nonlinear reaction function; d is the molecular diffusion constant, $\tau_r > 0$ is reaction time scale. The flow field V is known or statistically known. For an isothermal reaction, the scalar is a reactant concentration though we shall still denote it by T. The common form of the reaction function is $f(T) = T(1-T)$, so called Kolmogorov-Petrovsky-Piskunov-Fisher (KPP-Fisher); $f(T) = T^m(1-T)$ ($m \geq 2$, higher order KPP-Fisher); $f(T) = e^{-E/T}(1-T)$ ($E > 0$), Arrhenius combustion nonlinearity; $F(T) = 0$, $T \in [0,\theta] \cup \{1\}$, $f(T) > 0$, $T \in (\theta, 1)$, ignition combustion nonlinearity. KPP or generalized KPP comes from isothermal autocatalytic reaction-diffusion system with equal diffusion constants (or unit Lewis number), [5, 33]. Equation (1.1) is well-known to admit propagating front solutions [2] if the advection is absent ($V = 0$). Turbulent combustion concerns with the setting of flame propagation when the reactant (a fluid) is stirred on a broad range of scales. Though the flame front will be wrinkled by the fluid velocity, its average location eventually moves at a steady speed s_T in each specified direction, the so called "turbulent burning velocity". The prediction of the turbulent flame speed is a fundamental problem in turbulent combustion theory [31, 27, 26]. For KPP nonlinearity, it is known [12, 32, 4, 21, 33] that s_T is given by a variational principle on the large time growth rate of a viscous quadratically nonlinear Hamilton-Jacobi equation (QHJ). More precisely, consider compactly supported non-negative initial data $T(x,0)$, then for each direction e and wave number $\lambda > 0$, let $H_e(\lambda)$ be the principal Lyapunov exponent of the linear advection-diffusion equation:

$$\phi_t = d\,\Delta\phi + (2\,d\,\lambda\,e - V(x,t)) \cdot D\phi + [d\,\lambda^2 - \lambda\,e \cdot V(x,t) + \tau_r^{-1} f'(0)]\,\phi,$$

with initial data $\phi(x,0) = 1$. Under suitable stationarity and ergodicity condition of the flow field [21], the following limit exists almost surely and is independent of x:

$$\bar{H}_e(\lambda) = \lim_{t \to +\infty} \frac{1}{t} \ln \phi(x,t).$$

The turbulent front speed along the e direction, a deterministic quantity, is:

$$s_T(e) = \inf_{\lambda > 0} \frac{\bar{H}_e(\lambda)}{\lambda}. \tag{1.2}$$

The function $u = \ln \phi$ satisfies the viscous QHJ:

$$u_t = d\,\Delta u + d\,|Du|^2 + (2\,d\,\lambda\,e - V(x,t)) \cdot \nabla u + d\,\lambda^2 - \lambda\,e \cdot V(x,t) + \tau_r^{-1}\,f'(0). \tag{1.3}$$

So $\bar{H}_e$ is also the linear growth rate of QHJ solution u. When V is a periodic flow field in (x,t), $\bar{H}_e$ reduces to a principal eigenvalue. When V is space-time periodic and in a scale-separation form $V = V(x,t,\epsilon^{-\alpha}x,\epsilon^{-\alpha}t)$, $d = \epsilon\,k$, $\tau_r = \epsilon^{-1}$, $\alpha \in (0,1]$, the limiting behavior of $T = T^\epsilon$ is [17]: $\lim_{\epsilon\to 0} T^\epsilon = 0$ locally uniformly in $\{(x,t) : Z < 0\}$ and $T^\epsilon \to 1$ locally uniformly in the interior of $\{(x,t) : Z = 0\}$, where $Z \in C(\mathbb{R}^n \times [0,+\infty))$ is the unique viscosity solution of the variational inequality

$$\max(Z_t - \bar{H}(D_x Z, x, t) - f'(0), Z) = 0, \quad (x,t) \times \mathbb{R}^n \times (0,+\infty), \tag{1.4}$$

with initial data $Z(x,0) = 0$ in the support of $T(x,0)$, and $Z(x,0) = -\infty$ otherwise. The set $\Gamma_t = \partial\{x \in \mathbb{R}^n : Z(x,t) < 0\}$ can be viewed as a front. The effective Hamiltonian $\bar{H} = \bar{H}(p,x,t)$ is defined as a solution of the following cell problem: for each $(p,x,t) \in \mathbb{R}^n \times \mathbb{R}^n \times (0,+\infty)$ there are a unique number $\bar{H}(p,x,t)$ and a function $w(y,\tau) \in C^{0,1}(\mathbb{R}^n \times (0,+\infty))$ periodic in both y and τ such that

$$w_\tau - a(\alpha)\,k\,\Delta_y w - \,k\,|p + D_y w|^2 + V(x,t,y,\tau) \cdot (p + D_y w) = -\bar{H}(p,x,t), \tag{1.5}$$

where $a(\alpha) = 0$ if $\alpha \in (0,1)$, $a(\alpha = 1) = 1$. One noticed that (1.5) is a periodic version of (1.3) if $\alpha = 1$ (the fast variables (y,τ) playing the role of (x,t) in (1.3)), and an inviscid periodic version if $\alpha \in (0,1)$. Both viscous and inviscid QHJs are simplified asymptotic models for characterizing s_T in the context of KPP. However, the KPP front speeds require an additional minimization (1.2) on top of QHJ (1.3) or variational inequality (1.4) on top of QHJ (1.5).

The RDAs (1.1) and in particular the KPP are first principle equations, though they are limited to the unit Lewis number (equal diffusion rates) regime. Another approach in turbulent flame modeling is the level set formulation [24] of interface motion laws with the interface width ignored. The simplest motion law is that the normal velocity of the interface (V_n) is equal to a constant s_l (the laminar speed) plus the projection of fluid velocity along the normal $\vec{n}$. The laminar speed is the flame speed when fluid is at rest. Let the flame front be the zero level set of a function $G(x,t)$, then the normal direction is $DG/|DG|$, the normal velocity is $-G_t/|DG|$. The motion law becomes the so called G-equation, a popular model in turbulent combustion [31, 26]:

$$G_t + V(x,t) \cdot DG + s_l |DG| = 0. \tag{1.6}$$

Chemical kinetics and Lewis number effects are all included in the laminar speed s_l which is provided by a user. Formally, under the G-equation model, for a specified unit direction p,

$$s_T(p) = -\lim_{T\to+\infty} \frac{G(x,T)}{T}.$$

Here $G(x,t)$ is the solution of equation (1.6) with initial data $G(x,0) = p \cdot x$. The existence of s_T has been rigourously established in [34] and [6] independently for incompressible periodic flows. When V is t independent, s_T is the effective Hamiltonian of the following cell problem

$$s_l|p+DG| + V(x)\cdot(p+DG) = \bar{H}(p) = s_T.$$

Here $\bar{H}(p)$ is the unique number such that the above equation admits periodic approximate solutions. The formal analysis of (1.6) and s_T is also performed in the framework of renormalization group methods [29, 30, 36]. See also [25] on a spectral closure approximation, and [13] for a numerical study of G-equation in comparison with combustion system modeling thermal-diffusive instabilities of free-propagating premixed lean hydrogen-air flames. Though G-equation is a phenomenological model, it is more flexible in that many factors influencing front motion can be incorporated into s_l. For example, the strain effect of a turbulent fluid flow is modeled by extending s_l to $s_l + \overrightarrow{n}\cdot DV \cdot \overrightarrow{n}$. The G-equation with flow induced strain is [26]:

$$G_t + s_l|DG| + V(x,t)\cdot DG + \frac{DG}{|DG|}\cdot DV \cdot DG = 0. \tag{1.7}$$

Then formally, $s_T(p) = -\lim_{T\to+\infty}\frac{G(x,T)}{T}$. Here $G(x,t)$ is the solution of equation (1.7) with initial data $G(x,0) = p\cdot x$. So far we are not able to prove the existence of s_T except for some simple situations like the one-dimensional (1d) compressible flow and the shear flow. It is conjectured by some experts [27] in combustion theory that the strain term will slow down flame propagation. Though theoretically it is hard to verify for general flow, we confirm this conjecture for 1d compressible flow and the unidirectional shear flow. Besides the strain effect of the flow, a flame front is affected by its own geometry or the curvature. One such model proposed in [26] is to replace s_l by $s_l + d\kappa$. Then the G-equation becomes

$$G_t - d\kappa|DG| + s_l|DG| + V(x,t)\cdot DG = 0. \tag{1.8}$$

Here d is the so called Markstein diffusivity and κ is the mean curvature of the flame front, i.e, $\kappa = \mathrm{div}(\frac{DG}{|DG|})$. The curvature G-equation (1.8) is very difficult to analyze. To obtain some ideas of the diffusion effect, a natural simplification is to change the mean curvature term κ to ΔG (a linearization of the curvature in some sense). This leads to the viscous G-equation

$$G_t - d\Delta G + s_l|DG| + V(x,t)\cdot DG = 0. \tag{1.9}$$

The above viscous G-equation also serves as a basic model to understand the diffusion effect introduced in the numerical computation of equation (1.6). For the viscous case,

$s_T(p,d) = -\lim_{T\to+\infty} \frac{G(x,T)}{T}$. Here $G(x,t)$ is the solution of equation (1.9) with initial data $G(x,0) = p \cdot x$. It is also the effective Hamiltonian of the following cell problem

$$-d\Delta G + s_l|p + DG| + V(x) \cdot (p + DG) = s_T(p,d) = \bar{H}(p,d).$$

Here $\bar{H}(p,d)$ is the unique number such that the above equation admits periodic solutions. The most general G-equation is to combine both the strain effect and the curvature effect.

Motivated by KPP asymptotics and G-equation, we are interested in the QHJ of the form:

$$F_t - d\Delta_x F + s_l\,|D_x F|^2 + V(x,t) \cdot D_x F = 0, \tag{1.10}$$

where $d \geq 0$, and s_l is a positive constant. Hereafter, we shall refer to (1.10) as F-equation if $d = 0$, and viscous F-equation if $d > 0$. Note if V is t independent and $F = F(x,t)$ is the solution of (1.10) with initial data $F(x,0) = p\cdot x$, the large time limit $\lim_{T\to+\infty} \frac{F(x,T)}{T} = -\bar{H}(p,d)$, where $\bar{H}(p,d)$ is the effective Hamiltonian of the following cell problem

$$-d\Delta F + s_l|p + DF|^2 + V(x) \cdot (p + DF) = s_T(p,d) = \bar{H}(p,d)$$

which is the same as (1.5) when V is t independent and we change V to $-V$.

Though various passive scalar models as shown above have been proposed to study s_T, their predictions may be potentially different or sometimes asymptotically equivalent. It requires delicate analysis to understand these subtleties. In this paper, we report on recent progress in analyzing these turbulent combustion models, and compare the properties of s_T in the G-equations (1.6)-(1.7), the F-equation (1.10) and RDAs for steady (time-independent) periodic flows ($V = V(x)$) with mean equal to zero. The s_T's are compared in terms of different nonlinearities and the flow induced strains, as well as presence or absence of viscosity. In [11], the comparison of KPP with the G-equation (1.6) for periodic shear flows of non-zero mean showed under-estimation of the G-equation in terms of the transverse magnitude of the mean flow. Our comparison results will be presented for 1d compressible flows, mean zero periodic shear flows as well as for cellular and cats' eye flows. The latter two flows appeared as canonical flow examples in dynamo and convection-enhanced diffusion problems [7, 10].

The paper is organized as follows. While comparing G(F)-equations, we shall take affine initial data $p \cdot x$, for a unit direction p, and normalize $s_l = 1$. We shall also write the flow field V as a function $v(x)$ in the one space dimensional case. In section 2, we show in G-equation that compressible flows slow down and even quench front speeds while viscosity helps to stop front quenching. On the other hand, incompressible flows (e.g. cellular flows) enhance front speeds yet viscosity reduces such speedup to sublinear growth (so called speed bending in combustion). For cellular flow, the s_T of any viscous G-equation is uniformly bounded [16] in the limit of large flow amplitude A, while it grows like $O(A/\log A)$ in the inviscid G-equation. In section 3, we compare the s_T's in G-equations (1.6) with and without the strain effect. We show that the strain term slows down the propagation speed s_T in one-dimensional compressible flows and in shear flows. In section 4, we compare the asymptotic behavior of s_T's from the inviscid G-equation (1.6) and the inviscid F-equation (1.10) when $V(x)$ is scaled to $A\,V(x)$, $A \gg 1$. The asymptotic growth rate $\lim_{A\to+\infty} \frac{s_T}{A}$ is the same for these two equations. The limit is characterized by the rotation number of the dynamical system $\dot{x} = V(x)$. In section 5, we discuss the front speed of the viscous F-equation which also shares the same growth rate. As an example, the cat's eye flow is shown to have only one direction for speed bending. For cellular flows, the $\frac{A}{\log(A)}$ growth pattern of s_T in G-equation (1.6) also holds for the F-equation (1.10). In section 6, we conclude with some open problems for future research.

§ 2. G-equations and Compressibility

For 1d inviscid G-equation without strain, if $V(x) = v(x)$ is a continuous one-periodic function, the turbulent flame speed s_T is the unique number such that the following equation ($p = 1$)

$$|1 + G'| + v(x)(1 + G') = s_T$$

admits approximate periodic solutions. For simplicity, let us assume that $\max v(x) > 0$. Then s_T has an explicit formula [15]:

$$s_T = \begin{cases} 0 & \text{if } \{x|\ v(x) = -1\} \neq \emptyset \\ \left(\int_0^1 \frac{1}{1+v(x)}\, dx\right)^{-1} > 0 & \text{otherwise.} \end{cases}$$

However for the viscous G-equation with viscosity $d > 0$,

$$s_T(d) = 0 \quad \text{iff} \quad \int_0^1 v(x)\, dx + 1 = 0. \tag{2.1}$$

In particular if $v(x)$ is mean zero, then $s_T(d) \neq 0$, quenching is arrested no matter how small d is and how large the maximum of v is! In fact by a continuity argument [15], $s_T(d) > 0$.

Now let us consider a front moving along direction $p = (1, 0)$ and in the two-dimensional (incompressible) cellular flow with amplitude A:

$$V(x_1, x_2) = A\left(\sin(2\pi x_1)\cos(2\pi x_2), -\sin(2\pi x_1)\cos(2\pi x_2)\right). \tag{2.2}$$

corresponding to Hamiltonian $\mathcal{H}(x_1, x_2) = (A/2\pi)\sin(2\pi x_1)\sin(2\pi x_2)$. The $s_T = s_T(A, d)$ is a function of two variables (A, d). It is known [23, 1, 20] that

$$s_T(A, 0) \sim O(A/\log A), \quad A \gg 1. \tag{2.3}$$

However, with any fixed small viscosity $d > 0$, the amazing result [16] is that for a constant $C(d)$

$$s_T(A, d) \leq C(d),\ \forall\, d > 0,\ A \geq 2. \tag{2.4}$$

Front speed is drastically reduced. The microscopic explanation is that the viscosity corresponds to the addition of Brownian noise in the generalized characteristics of the inviscid G-equation, making it difficult for a Lagrangian particle to travel through vortices (or hop from a saddle point to another without getting trapped in a vortex). Here in the incompressible flow, the viscosity term of the G-equation is biased towards slowing down the transport (or s_T), just the opposite of its effect in compressible flows. The analytical explanation is that solution G is smoother due to viscosity, hence its spatial gradient is smaller, and the front speed in periodic incompressible flow $s_T = s_l\langle p + \nabla G\rangle$ is reduced, $\langle\cdot\rangle$ is average on a periodic cell. The reaction-diffusion front speeds in cellular flows obey the asymptotics of $O(A^{1/4})$ at large A, [3, 19, 28, 37, 38].

§ 3. G-equations with and without Strain

§ 3.1. One-dimensional G-equation and Compressible Flow

Now let us consider the one space dimensional G-equation with the strain term. Using the same method as in [16], we can show that there exists a unique number $\hat{s}_T$ such that the following equation

$$(1 + v')|1 + G'| + v(x)(1 + G') = \hat{s}_T$$

admits approximate periodic solutions. The $\hat{s}_T$ is the flame speed of G-equation under strain, and is given by

$$\hat{s}_T = \begin{cases} 0 & \text{if } \{x|\ v'(x) + v(x) = -1\} \neq \emptyset \\ \left(\int_0^1 \frac{1}{1+v'(x)+v(x)}\,dx\right)^{-1} > 0 & \text{otherwise.} \end{cases}$$

In [35], we have:

Theorem 3.1. *The flame speed in the G-equation with strain is no faster than that of the G-equation without strain:*

$$s_T \geq \hat{s}_T.$$

If $s_T > 0$, then "=" holds if and only if $v \equiv 0$.

§ 3.2. Strain Effects in Shear Flows

Suppose that $V(x,y) = (v(y),0)$. For $(m,n) \in \mathbb{R}^2$, denote $\lambda(m,n)$ as the unique number such that the G-equation for the shear flow

$$\sqrt{m^2 + (n+u')^2} + mv(x) = \lambda = \lambda(m,n)$$

has a periodic viscosity solution. We also write $\hat{\lambda}(m,n)$ as the unique number such that the G-equation for the shear flow with strain term

$$\sqrt{m^2 + (n+u')^2} + \frac{m(n+u')v'}{\sqrt{m^2 + (n+u')^2}} + mv(x) = \hat{\lambda} = \hat{\lambda}(m,n).$$

It is clear that

$$\lambda,\ \hat{\lambda} \geq |m| + \max_{\mathbb{T}^1} mv \tag{3.1}$$

To ensure that both of them are non negative, we further assume that

$$\int_0^1 v(x)\,dx = 0.$$

Then we have [35]:

Theorem 3.2. *The flame speed under the strain of shear flows is no faster than that in shear flows without the strain.*

$$\lambda(m,n) \geq \hat{\lambda}(m,n).$$

If $m \neq 0$ and $\lambda(m,n) > |m| + \max_{\mathbb{T}^1} mv$, then "=" holds if and only if $v \equiv 0$.

§ 4. Inviscid G-equation and F-equation

Let α_A be the effective Hamiltonian (front speed) from the inviscid G-equation without strain term

$$|p + DG| + AV(x) \cdot (p + DG) = \alpha_A$$

and β_A the effective Hamiltonian (front speed) from the invicid F-equation

$$|p + DF|^2 + AV(x) \cdot (p + DF) = \beta_A.$$

Both α_A and β_A can be characterized by variational (inf-max) formulas. Precisely speaking,

$$\alpha_A = \inf_{\phi \in C^1(\mathbb{T}^n)} \{|p + D\phi| + AV(x) \cdot (p + D\phi)\}$$

and

$$\beta_A = \inf_{\phi \in C^1(\mathbb{T}^n)} \{|p + D\phi|^2 + AV(x) \cdot (p + D\phi)\}$$

Hence α_A, $\beta_A \le O(A)$. Since $t^2 \ge t - 1$, $\beta_A \ge \alpha_A - 1$. The following theorem says that α_A/A and β_A/A have the same asymptotic limit.

Theorem 4.1.

$$\lim_{A \to +\infty} \frac{\alpha_A}{A} = \lim_{A \to +\infty} \frac{\beta_A}{A} = \inf_{\phi \in C^1(\mathbb{T}^n)} \max_{\mathbb{T}^n} \{V(x) \cdot (p + \phi)\} \equiv c_0.$$

In particular, if the G-equation shows speed bending effect, so does the F-equation.

A dynamical system characterization of the growth rate c_0 in Theorem 4.1 is in terms of "rotation number".

Definition 4.1. *The function $\xi = \xi(t)$ is called an orbit if $\dot{\xi}(t) = V(\xi(t))$. Moreover, the function $\xi : [0, T] \to \mathbb{R}^n$ is called periodic orbit and T is called a period if ξ is an orbit satisfying $\xi(0) - \xi(T)$ is an integer vector.*

Theorem 4.2. *There exists an orbit ξ such that*

$$\lim_{T \to +\infty} \frac{p \cdot \xi(T)}{T} = \lim_{T \to +\infty} \frac{1}{T} \int_0^T p \cdot V(\xi(t))\, dt = c_0.$$

In particular, when $n = 2$, $c_0 > 0$ if and only if there exists a periodic orbit $\xi : [0, T] \to \mathbb{R}^2$ such that

$$\frac{p \cdot (\xi(T) - \xi(0))}{T} > 0.$$

where T is the period.

Example 4.1 (cat's eye flow). *For the cat's eye flow, the stream function is $H = \sin 2\pi x_1 \sin 2\pi x_2 + \delta \cos 2\pi x_1 \cos 2\pi x_2$ for $\delta \in [0, 1]$. When $\delta = 0$, it becomes the cellular flow. When $\delta > 0$, the zero level curve $\{H = 0\}$ is a periodic orbit and has a rotation vector parallel to (1,1). Also, since H is an even function, we have:*

$$\inf_{\phi \in C^1(\mathbb{T}^n)} \max_{\mathbb{T}^n} \{(p + D\phi) \cdot V(x)\} = \begin{cases} > 0 & \text{if } p \text{ does not parallel with } (-1, 1) \\ = 0 & \text{if } p \text{ parallels with } (-1, 1). \end{cases}$$

Hence for $\delta \neq 0$, the bending effect only occurs along the direction in parallel with (-1,1).

If V is the celluar flow, α_A grows like $O(\frac{A}{\log(A)})$. The following theorem [35] says that the same growth law holds for the F-equation.

Theorem 4.3. *If V is the cellular flow, then at large A*

$$\beta_A = O\left(\frac{A}{\log(A)}\right).$$

Hence the quadratic nonlinearity does not make a difference on the front speed asymptotics of the inviscid Hamilton-Jacobi equations in cellular flows.

§ 5. Viscous G-equation and F-equation

Fix $d > 0$, for $A > 0$, we denote χ_A as the effective Hamiltonian (front speed) of the cell problem associated with the viscous G-equation

$$-d\Delta G + |p + DG| + AV(x) \cdot (p + DG) = \chi_A,$$

and κ_A as the effective Hamiltonian (front speed) of the following cell problem

$$-d\Delta F + |p + DF|^2 + AV(x) \cdot (p + DF) = \kappa_A.$$

Similar to the inviscid case, both χ_A and κ_A can be given by inf-max formulas.

$$\chi_A = \inf_{\phi \in C^2(\mathbb{T}^n)} \max_{\mathbb{T}^n} \{-d\Delta\phi + |p + D\phi| + AV(x) \cdot (p + D\phi)\}.$$

and

$$\kappa_A = \inf_{\phi \in C^2(\mathbb{T}^n)} \max_{\mathbb{T}^n} \{-d\Delta\phi + |p + D\phi|^2 + AV(x) \cdot (p + D\phi)\}.$$

Clearly, χ_A, $\kappa_A \le O(A)$ and $\chi_A \le \kappa_A + 1$. The following theorem says that κ_A/A has the same asymptotic limit as α_A/A and β_A/A in two space dimensions [35].

Theorem 5.1. *For $n = 2$,*

$$\lim_{A \to +\infty} \frac{\kappa_A}{A} = \inf_{\phi \in C^1(\mathbb{T}^2)} \{\max_{\mathbb{T}^2} V(x) \cdot (p + D\phi)\} = c_0.$$

Theorem 5.1 implies that in cat's eye flows, κ_A has linear growth in in all but $(-1, 1)$ direction, and sublinear growth (bending) in $(-1, 1)$ direction. In cellular flows, the theorem implies that κ_A grows sublinearly. The κ_A satisfies a lower bound $O(A^{1/3})$ in cellular flows, see [19, 35]; while χ_A is uniformly bounded in A. The quadratic nonlinearity makes a difference on the front speed asymptotics of the viscous Hamilton-Jacobi equations. The exact growth asymptotics of κ_A in cellular flows remain to be found.

§ 6. Concluding Remarks

We compared the turbulent flame speeds (s_T) of G-equation and the analogous quadratically nonlinear Hamilton-Jacobi equation (the F-equation) in the presence of steady and periodic compressible flows, shear flows, and incompressible flows (cellular and cat's eye flows). The viscosity term is seen to have an opposing effect on flows. The strain effect is shown to decrease s_T for compressible and shear flows. The s_T of the viscous F-equation has the same growth rate as the inviscid F and G-equations. Nonlinearity is seen to play a larger role on front speeds in viscous G and F-equations than in their inviscid counterparts.

Comparisons of s_T in the viscous F and G-equations remain to be explored for more complex flows, including three dimensional steady flows. Likewise, the effects of strain, nonlinearity, and interface curvature remain to be studied in this more general setting. The curvature regularization is a degenerate diffusion, and presents additional mathematical difficulties. A result on curvature dependent G-equation in shear flows is given in [16]. The front speed grows at the same rate as that of the inviscid G-equation. Though some experts in combustion believe that the curvature effect on s_T is minor, mathematical analysis in case of non-shear flows remains to be developed.

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