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Continued fractions in p-adic numbers

By

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Abstract

The theory of continued fractions of irrational numbers is developed in the field $\mathbb{R}$ of real numbers, and there are many well known results on this subject. In this paper, we discuss some of the corresponding results for continued fractions of irrational numbers in the p-adic number field $\mathbb{Q}_p$.

§ 1. Introduction

Let α be an irrational real number. Then the regular continued fraction expansion $\mathrm{CF}(\alpha)$ of it is the following expression

$$\begin{aligned}\mathrm{CF}(\alpha) &= q_1 + \frac{1|}{|q_2} + \frac{1|}{|q_3} + \frac{1|}{|q_4} + \cdots \\ &:= q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{q_4 + \cdots}}},\end{aligned} \tag{1.1}$$

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where $q_n's$ $(n \geq 1)$ are integers determined from α inductively by

(1.2)

$$\begin{cases} q_1 := [\alpha_1], & (\text{Here } \alpha_1 = \alpha \text{ and } [\alpha_1] \text{ is the largest integer which does not exceed } \alpha_1.), \\ q_2 := [\alpha_2], & \left(\alpha_2 := \dfrac{1}{\alpha_1 - q_1} > 1\right), \\ q_3 := [\alpha_3], & \left(\alpha_3 := \dfrac{1}{\alpha_2 - q_2} > 1\right), \\ \quad \cdots & \\ q_n := [\alpha_n], & \left(\alpha_n := \dfrac{1}{\alpha_{n-1} - q_{n-1}} > 1\right). \end{cases}$$

We note, in particular, that the sequence $\{q_n\}_{n\geq 1}$ satisfies

$$q_n \in \mathbb{Z} \quad \text{for } n \geq 1 \quad \text{and} \quad q_n > 0 \quad \text{for } n \geq 2. \tag{1.3}$$

From this follows the fundamental equality

$$\mathrm{CF}(\alpha) = \alpha \tag{1.4}$$

Namely, if we put for each $n \geq 1$

$$\begin{aligned} a_n &= q_1 + \frac{1|}{|q_2} + \frac{1|}{|q_3} + \frac{1|}{|q_4} + \cdots + \frac{1|}{|q_n} \\ &:= q_1 + \cfrac{1}{q_2 + \cfrac{1}{\ddots + \cfrac{1}{q_{n-1} + \cfrac{1}{q_n}}}}, \end{aligned} \tag{1.5}$$

then the sequence $\{a_n\}_{n\geq 1}$ converges to α in $\mathbb{R}$.

Conversely, if the sequence $\{q_n\}_{n\geq 1}$ satisfy the condition (1.3) then the continued fraction (1.1) converges in $\mathbb{R}$ to an irrational number. Thus we see that the map CF gives one-to-one correspondence

$$\mathbb{R}\backslash\mathbb{Q} \quad \stackrel{\mathrm{CF}}{\longleftrightarrow} \quad \Big\{\text{the set of sequences } \{q_n\}_{n\geq 1} \text{ satisfying (1.3)}\Big\}. \tag{1.6}$$

We refer to [5], [2], [3] for details of these facts.

Now an interesting question is to ask whether a similar correspondence as above holds in the field $\mathbb{Q}_p$ of p-adic numbers for some prime number p. It turns out, however, that the continued fraction (1.1) does *not* converge in $\mathbb{Q}_p$ for *any* sequence $\{q_n\}_{n\geq 1}$ satisfying the condition (1.3). We shall show this in §3.

So we need to modify the expression (1.1) of "regular" continued fractions, if we wish to establish a meaningful theory of continued fractions in $\mathbb{Q}_p$.

Browkin [1] studied a construction of certain type of continued fractions which converge in p-adic numbers, where the general terms q_n are allowed to be non-integral rational numbers.

We shall consider the continued fractions $\mathrm{CF}_p^*(\{q_n\})$ with integral general terms q_n having slightly different shapes from (1.1). The simplest such continued fractions are of the following type, which will be called a "p-regular" continued fraction.

$$\begin{aligned} \mathrm{CF}_p^*(\{q_n\}) &= q_1 + \frac{p|}{|q_2} + \frac{p|}{|q_3} + \frac{p|}{|q_4} + \cdots \\ &:= q_1 + \cfrac{p}{q_2 + \cfrac{p}{q_3 + \cfrac{p}{q_4 + \cdots}}}, \end{aligned} \tag{1.7}$$

One of the main results of this paper is the following

Theorem 1.1. *Let q_1 be an integer and $q_2, q_3, q_4, \ldots$ be natural numbers. If q_n is not divisible by p for any $n \geq 2$, then* (1.7) *converges in $\mathbb{R}$ and in $\mathbb{Q}_p$ simultaneously.*

Moreover, we shall show that our construction $\mathrm{CF}_p^*(\{q_n\})$ gives a surjective map onto $\mathbb{Z}_p$.

Theorem 1.2. *Let $\{q_n\}_{n\geq 1}$ be a sequence of integers such that for $n \geq 2$, q_n is positive and is not divisible by p. Then the continued fraction* (1.7) *converges to a p-adic integer in $\mathbb{Q}_p$. Conversely any p-adic integer α is obtained as the limit of a continued fraction* (1.7) *satisfying the above conditions.*

§ 2. Convergence of "p-regular" continued fractions

Let p be a prime number, and $\mathbb{Q}_p$ the p-adic number field. In this section, we consider the continued fraction of the following type, which will be called a "p-regular" continued fraction:

$$q_1 + \cfrac{p}{q_2 + \cfrac{p}{q_3 + \cfrac{p}{q_4 + \cdots}}}. \tag{2.1}$$

As in the case of "regular" continued fractions, we shall denote this by

$$q_1 + \frac{p}{\lfloor q_2} + \frac{p}{\lfloor q_3} + \frac{p}{\lfloor q_4} + \cdots \, . \tag{2.2}$$

Our first task is to study the condition for the convergence of (1.7) in the real number field $\mathbb{R}$ or the p-adic number field $\mathbb{Q}_p$. For this purpose we introduce, for each positive integer n, independent indeterminates $x_1, ..., x_n$, and put

$$[x_1, ..., x_n] = x_1 + \frac{p}{\lfloor x_2} + \cdots + \frac{p}{\lfloor x_n}.$$

For $n = 1, 2, 3$ they are computed explicitly as

$$\begin{aligned} [x_1] &= x_1 = \frac{x_1}{1}, \\ [x_1, x_2] &= x_1 + \frac{p}{x_2} = \frac{p + x_1 x_2}{x_2}, \\ [x_1, x_2, x_3] &= x_1 + \frac{p}{x_2 + \frac{p}{x_3}} = \frac{p x_1 + p x_3 + x_1 x_2 x_3}{p + x_2 x_3}. \end{aligned}$$

We shall find the sequences of pairs of coprime polynomials $P_n(x_1, ..., x_n), Q_n(x_1, ..., x_n) \in \mathbb{Z}[x_1, ..., x_n]$ such that

$$[x_1, ..., x_n] = \frac{P_n(x_1, ..., x_n)}{Q_n(x_1, ..., x_n)}. \tag{2.3}$$

From the above computation of $[x_1, ..., x_n]$ for $n = 1, 2, 3$, we have

$$\begin{cases} P_1 = x_1, & Q_1 = 1, \\ P_2 = p + x_1 x_2, & Q_2 = x_2, \\ P_3 = p x_1 + p x_3 + x_1 x_2 x_3, & Q_3 = p + x_2 x_3. \end{cases}$$

If we set $P_0 = 1, Q_0 = 0$, then we observe that the following equalities hold for $n = 2, 3$.

$$\begin{cases} P_n = x_n P_{n-1} + p P_{n-2}. \\ Q_n = x_n Q_{n-1} + p Q_{n-2} \end{cases} \tag{2.4}$$

We shall show that (2.4) holds for all $n \geq 1$. It suffices to show that the polynomials defined by the recurring formulas (2.4) satisfies (2.3). This will be proved by induction on n. Let P_n, Q_n $(n \geq 1)$ be defined by (2.4), and assume that (2.3) holds for all natural numbers less than n. Then we can express $[x_1, ..., x_n]$ as follows:

$$[x_1, ..., x_n] = \left[x_1, ..., x_{n-2}, x_{n-1} + \frac{p}{x_n} \right]$$

$$
\begin{aligned}
&= \frac{P_{n-1}(x_1, ..., x_{n-2}, x_{n-1} + \frac{p}{x_n})}{Q_{n-1}(x_1, ..., x_{n-2}, x_{n-1} + \frac{p}{x_n})} \\
&= \frac{\left(x_{n-1} + \frac{p}{x_n}\right) P_{n-2} + pP_{n-3}}{\left(x_{n-1} + \frac{p}{x_n}\right) Q_{n-2} + pQ_{n-3}} \\
&= \frac{(x_n x_{n-1} + p)P_{n-2} + px_n P_{n-3}}{(x_n x_{n-1} + p)Q_{n-2} + px_n Q_{n-3}} \\
&= \frac{x_n(x_{n-1}P_{n-2} + pP_{n-3}) + pP_{n-2}}{x_n(x_{n-1}Q_{n-2} + pQ_{n-3}) + pQ_{n-2}} \\
&= \frac{x_n P_{n-1} + pP_{n-2}}{x_n Q_{n-1} + pQ_{n-2}} \\
&= \frac{P_n}{Q_n},
\end{aligned}
$$

which proves our assertion.

Next we show that the polynomials $P_n(x_1, ..., x_n), Q_n(x_1, ..., x_n)$ determined as above satisfy the equalities

$$
\begin{cases} P_n Q_{n-1} - P_{n-1} Q_n = & (-1)^n p^{n-1} \qquad (n \geq 1), \\ P_n Q_{n-2} - P_{n-2} Q_n = & (-1)^{n-1} p^{n-2} x_n \quad (n \geq 2). \end{cases} \tag{2.5}
$$

Let us prove the first equality of (2.5) by induction. Since

$$
P_1 Q_0 - P_0 Q_1 = x_1 \cdot 0 - 1 \cdot 1 = -1 = (-1)^1 p^0,
$$

the equality holds for $n = 1$. Now let $n \geq 2$ and assume that for $n - 1$ the equality

$$
P_{n-1} Q_{n-2} - P_{n-2} Q_{n-1} = (-1)^{n-1} p^{n-2}
$$

holds. Then by (2.4) we have

$$
\begin{aligned}
P_n Q_{n-1} - P_{n-1} Q_n &= (x_n P_{n-1} + pP_{n-2})Q_{n-1} - P_{n-1}(x_n Q_{n-1} + pQ_{n-2}) \\
&= -p(P_{n-1} Q_{n-2} - P_{n-2} Q_{n-1}) \\
&= -p(-1)^{n-1} p^{n-2} \\
&= (-1)^n p^{n-1}.
\end{aligned}
$$

The second equality of (2.5) is proved similarly. Namely from the definition of P_n and the equation (2.4) with induction hypothesis for $n - 1$, we have

$$
\begin{aligned}
P_n Q_{n-2} - P_{n-2} Q_n &= (x_n P_{n-1} + pP_{n-2})Q_{n-2} - P_{n-2}(x_n Q_{n-1} + pQ_{n-2}) \\
&= x_n(P_{n-1} Q_{n-2} - P_{n-2} Q_{n-1}) \\
&= x_n(-1)^{n-1} p^{n-2} \\
&= (-1)^{n-1} p^{n-2} x_n.
\end{aligned}
$$

Theorem 1.1. *Let q_1 be an integer and $q_2, q_3, q_4, \dots$ be natural numbers. If q_n is not divisible by p for any $n \geq 2$, then (1.7) converges in $\mathbb{R}$ and in $\mathbb{Q}_p$ simultaneously.*

Proof. Suppose that $\{q_n\}$ satisfies the same condition as Theorem 1.1 and let

$$a_n = [q_1, q_2, \dots, q_n] = q_1 + \frac{p|}{|q_2} + \frac{p|}{|q_3} + \cdots + \frac{p|}{|q_n} \qquad (n \geq 1).$$

Then for $n \geq 2$,

$$\begin{aligned} a_n - a_{n-2} &= \frac{P_n(q_1, \dots, q_n)}{Q_n(q_1, \dots, q_n)} - \frac{P_{n-2}(q_1, \dots, q_{n-2})}{Q_{n-2}(q_1, \dots, q_{n-2})} \\ &= \frac{P_n Q_{n-2} - P_{n-2} Q_n}{Q_n Q_{n-2}} \\ &= \frac{(-1)^{n-1} p^{n-2} q_n}{Q_n Q_{n-2}}. \end{aligned} \tag{2.6}$$

From the definition of Q_n we have $Q_n > 0$ for $n \geq 1$. Therefore from (2.6),

$$\begin{aligned} &a_1 \leq a_3 \leq a_5 \leq a_7 \leq \cdots, \\ &a_2 \geq a_4 \geq a_6 \geq a_8 \geq \cdots. \end{aligned} \tag{2.7}$$

On the other hand, for $n \geq 1$,

$$\begin{aligned} a_n - a_{n-1} &= \frac{P_n(q_1, \dots, q_n)}{Q_n(q_1, \dots, q_n)} - \frac{P_{n-1}(q_1, \dots, q_{n-1})}{Q_{n-1}(q_1, \dots, q_{n-1})} \\ &= \frac{P_n Q_{n-1} - P_{n-1} Q_n}{Q_n Q_{n-1}} \\ &= \frac{(-1)^n p^{n-1}}{Q_n Q_{n-1}}. \end{aligned} \tag{2.8}$$

This gives $a_{2m} \geq a_{2m-1}$ for $m \geq 1$. From (2.7) and (2.8), the sequences $\{a_{2m-1}\}$ (resp. $\{a_{2m}\}$) are monotonically increasing (resp. decreasing) and are bounded. So they converge in $\mathbb{R}$. We next prove that

$$\lim_{m \to \infty} a_{2m-1} = \lim_{m \to \infty} a_{2m}.$$

To show this assertion we need a lower bound of $Q_n(q_1, \dots, q_n)$.

Lemma 2.1. *We have the following inequality*

$$Q_n(q_1, \dots, q_n) \geq \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (n \geq 1),$$

where $\alpha = (1 + \sqrt{1 + 4p})/2$, $\beta = (1 - \sqrt{1 + 4p})/2$.

Proof of Lemma 2.1. From $Q_0 = 0, Q_1 = 1$ and $Q_n = x_n Q_{n-1} + pQ_{n-2}$ for $n \geq 2$, we can evaluate $Q_n(1, ..., 1)$ as

$$Q_n(1, ..., 1) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad (n \geq 1).$$

Since Q_n is a polynomial in $x_1, ..., x_n$ with non-negative integral coefficients, we have

$$Q_n(q_1, ..., q_n) \geq Q_n(1, ..., 1) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad (n \geq 1)$$

for $n \geq 1$. This proves the lemma. □

Substituting this for (2.8) and replacing n by $2m$ gives

$$\begin{aligned} |a_{2m} - a_{2m-1}| &\leq \left| \frac{p^{2m-1}(\alpha - \beta)^2}{(\alpha^{2m} - \beta^{2m})(\alpha^{2m-1} - \beta^{2m-1})} \right| \\ &= \left| \left(\frac{p}{\alpha^2}\right)^{2m-1} \times \frac{(\alpha - \beta)^2}{\alpha \left(1 - \left(\frac{\beta}{\alpha}\right)^{2m}\right)\left(1 - \left(\frac{\beta}{\alpha}\right)^{2m-1}\right)} \right| \to 0 \qquad (\text{as } m \to \infty), \end{aligned}$$

since $\alpha > |\beta|, \alpha > \sqrt{p}$. This completes the proof of convergence for a_n in $\mathbb{R}$. Next we prove that a_n converges in $\mathbb{Q}_p$. Let $|\cdot|_p$ be the normalized valuation on $\mathbb{Q}_p$ and show that $|Q_n|_p = 1$ for all $n \geq 1$. First we have $|Q_1|_p = |1|_p = 1, |Q_2|_p = |q_2|_p = 1$ which shows the assertion in the case $n = 1, 2$. Assume that $n \geq 3$ and the assertion holds for $n-1, n-2$. Then we have

$$\begin{aligned} |Q_n|_p &= |q_n Q_{n-1} + pQ_{n-2}|_p \\ &= \max\left(1, \frac{1}{p}\right) \qquad \text{by the induction hypothesis} \\ &= 1. \end{aligned} \tag{2.9}$$

From (2.8) and (2.9) we conclude that

$$\begin{aligned} |a_n - a_{n-1}|_p &= \left| \frac{(-1)^n p^{n-1}}{Q_n Q_{n-1}} \right|_p \\ &= p^{-(n-1)} \to 0 \qquad (n \to \infty). \end{aligned}$$

This completes the proof of Theorem 1.1. □

Remark 1. If we modify the condition in Theorem 1.1 and assume that q_n are not divisible by p for all but finitely many n, then Theorem 1.1 becomes false.
For example, let $q_1 = 0, q_2 = p$ and $q_n = p - 1$ for $n \geq 3$. Then we have

$$Q_0 = 0,$$

$$\begin{aligned}
Q_1 &= 1,\\
Q_2 &= q_2Q_1 + pQ_0 = p\cdot 1 + 0 = p,\\
Q_3 &= q_3Q_2 + pQ_1 = (p-1)p + p\cdot 1 = p^2,\\
Q_4 &= q_4Q_3 + pQ_2 = (p-1)p^2 + p\cdot p = p^3,\\
Q_5 &= q_5Q_4 + pQ_3 = (p-1)p^3 + p\cdot p^2 = p^4,\\
&\vdots\\
Q_n &= p^{n-1} \qquad (n \geq 1).
\end{aligned}$$

So we have

$$\begin{aligned}
|a_n - a_{n-1}|_p &= \left|\frac{(-1)^n p^{n-1}}{Q_nQ_{n-1}}\right|_p\\
&= \left|\frac{(-1)^n p^{n-1}}{p^{n-1}p^{n-2}}\right|_p\\
&= \left|\frac{(-1)^n}{p^{n-2}}\right|_p\\
&= p^{n-2}.
\end{aligned}$$

This shows that a_n does not converge in $\mathbb{Q}_p$.

§ 3. Some examples

In this section, we give some examples of continued fractions which converge in either $\mathbb{R}$ or $\mathbb{Q}_p$ and do not converge in another.

§ 3.1. CF which converges in $\mathbb{R}$ but not in $\mathbb{Q}_p$

Proposition 3.1. *Let q_1 be an integer and $q_2, q_3, q_4, \ldots$ natural numbers. Then the continued fraction*

$$q_1 + \frac{1|}{|q_2} + \frac{1|}{|q_3} + \frac{1|}{|q_4} + \cdots$$

(which is called a regular continued fraction in the theory of classic continued fractions) converges in $\mathbb{R}$ and do not converge in $\mathbb{Q}_p$.

Proof. Define the sequences of polynomials $P_n, Q_n \in \mathbb{Z}[x_1, \ldots, x_n]$ by

$$\begin{cases}
P_0 = 1, & Q_0 = 0\\
P_1 = x_1, & Q_1 = 1\\
P_n = x_nP_{n-1} + P_{n-2} & \text{for } n \geq 2,\\
Q_n = x_nQ_{n-1} + Q_{n-2} & \text{for } n \geq 2.
\end{cases}$$

Then we have

$$\begin{aligned}
&x_1 + \frac{1}{\lfloor x_2} + \frac{1}{\lfloor q_3} \cdots + \frac{1}{\lfloor x_n} = \frac{P_n}{Q_n} \ \ (n \geq 1),\\
&P_n Q_{n-1} - P_{n-1} Q_n = (-1)^n \ \ (n \geq 1),\\
&P_n Q_{n-2} - P_{n-2} Q_n = (-1)^{n-1} x_n \ \ (n \geq 2).
\end{aligned}$$

The proofs of these facts are similar to those in the proof of Theorem 1.1. Now, let $q_1, q_2, q_3, \ldots$ be as in the proposition and let

$$a_n = q_1 + \frac{1}{\lfloor q_2} + \frac{1}{\lfloor q_3} + \cdots + \frac{1}{\lfloor q_n}.$$

Then we have

$$\begin{aligned}
|a_n - a_{n-1}|_p &= \left| \frac{P_n(q_1, \ldots, q_n)}{Q_n(q_1, \ldots, q_n)} - \frac{P_{n-1}(q_1, \ldots, q_{n-1})}{Q_{n-1}(q_1, \ldots, q_{n-1})} \right|_p \\
&= \left| \frac{P_n Q_{n-1} - P_{n-1} Q_n}{Q_n Q_{n-1}} \right|_p \\
&= \left| \frac{(-1)^n}{Q_n Q_{n-1}} \right|_p \\
&= \frac{1}{|Q_n Q_{n-1}|_p} \\
&\geq 1 \qquad \text{since } Q_n, Q_{n-1} \text{ are integers.}
\end{aligned}$$

This shows that a_n does not converge in $\mathbb{Q}_p$. □

§ 3.2. CF which converges in $\mathbb{Q}_p$ but not in $\mathbb{R}$

Proposition 3.2. *Let a, b be integers with $0 < |a|, |b| < p$ and $-4p < ab < 0$. Then the continued fraction*

$$a + \frac{p}{\lfloor b} + \frac{p}{\lfloor a} + \frac{p}{\lfloor b} + \frac{p}{\lfloor a} + \frac{p}{\lfloor b} + \cdots$$

converges in $\mathbb{Q}_p$ and does not converge in $\mathbb{R}$.

Proof. The convergence of the continued fraction in $\mathbb{Q}_p$ is immediately from Theorem 1.1. (Note that in the proof of Theorem 1.1, the convergence of the continued fraction in $\mathbb{Q}_p$ does not require that $q_n > 0$ for $n \geq 2$.) Now suppose that the continued fraction converges to α in $\mathbb{R}$. Then we have

$$\alpha = a + \frac{p}{\lfloor b} + \frac{p}{\lfloor a} + \frac{p}{\lfloor b} + \frac{p}{\lfloor a} + \frac{p}{\lfloor b} + \cdots$$

$$= a + \frac{p|}{|b} + \frac{p|}{|\alpha}$$

$$= a + \cfrac{p}{b + \cfrac{p}{\alpha}}$$

$$= \frac{pa + (ab+p)\alpha}{b\alpha + p}.$$

So we have

$$b\alpha^2 - ab\alpha - pa = 0.$$

But this quadratic equation has discriminant

$$D = (-ab)^2 - 4b(-pa) = ab(ab+4p) < 0$$

from the assumption of a, b. This contradicts that α is a real number, and we conclude that the continued fraction does not converge in $\mathbb{R}$. □

Example 3.3. Let p be an odd prime such that $p \equiv 1(\mod 4)$, so that $\sqrt{-1} \in \mathbb{Q}_p$. Then from Theorem 1.1 there exists a p-regular continued fraction which converges to $k\sqrt{-1}$ in $\mathbb{Q}_p$ for some $k \in \mathbb{N}$. One can illustrate it explicitly for $p = 5, 13$ as

$$\sqrt{-1} = 2 + \frac{5|}{|-4} + \frac{5|}{|4} + \frac{5|}{|-4} + \frac{5|}{|4} + \frac{5|}{|-4} + \cdots \quad (\text{in } \mathbb{Q}_5),$$

$$3\sqrt{-1} = 2 + \frac{13|}{|-4} + \frac{13|}{|4} + \frac{13|}{|-4} + \frac{13|}{|4} + \frac{13|}{|-4} + \cdots \quad (\text{in } \mathbb{Q}_{13}).$$

§ 4. Image of the p-regular continued fraction map

Next we determine the set of p-adic numbers which do occur as the limiting value of continued fraction (1.7).

Theorem 1.2. *Let q_1 be an integer and $q_2, q_3, q_4, \ldots$ natural numbers which are not divisible by p. Then the continued fraction*

$$q_1 + \frac{p|}{|q_2} + \frac{p|}{|q_3} + \frac{p|}{|q_4} + \cdots \tag{4.1}$$

converges to a p-adic integer in $\mathbb{Q}_p$. Conversely any p-adic integer α is obtained as the limit of a continued fraction (1.7) *satisfying the above conditions.*

Proof. As before we set the following notations:

$$P_n = P_n(q_1, q_2, \ldots, q_n),$$

$$
\begin{aligned}
Q_n &= Q_n(q_1, q_2, ..., q_n),\\
a_n &= \frac{P_n}{Q_n} = q_1 + \frac{p|}{|q_2} + \cdots + \frac{p|}{|q_n},\\
|\cdot|_p &: \text{ the normalized valuation on } \mathbb{Q}_p.
\end{aligned}
$$

Then, as in the proof of Theorem 1.1, we have $|Q_n|_p = 1$ for $n \geq 1$. Further, from the definition of P_n, we have $|P_n|_p \leq 1$ for $n \geq 1$. These imply

$$|a_n|_p = |P_n/Q_n|_p = |P_n|_p/|Q_n|_p \leq 1.$$

So $a_n \in \mathbb{Z}_p$ for $n \geq 1$. Since $\mathbb{Z}_p$ is closed (see [4] p.17), the limiting value of a_n is also a p-adic integer.

Next, for given $\alpha \in \mathbb{Z}_p$, we construct $q_1, q_2, q_3, ...$ such that the continued fraction (4.1) converges to α in $\mathbb{Q}_p$. First, we set $\alpha_1 = \alpha$ and take an integer q_1 such that $p||(\alpha_1 - q_1)$. (Note that $\alpha_1 \in \mathbb{Z}_p$, so we can choose an integer q_1 with $p||(\alpha_1 - q_1)$.) Next, we put $\alpha_1 - q_1 = p\alpha_2^{-1}$. Then from the condition of q_1, we see that α_2 is a p-adic unit. Now we take a natural number q_2 such that $p||(\alpha_2 - q_2)$. Since α_2 is a p-adic unit (i.e., $\alpha_2 \in \mathbb{Z}_p - p\mathbb{Z}_p$), the integer q_2 is not divisible by p as required in Theorem 1.2. We continue in these fashion,

$$
\begin{aligned}
&\alpha_1 = \alpha,\\
&p||(\alpha_1 - q_1),\ \ \alpha_1 - q_1 = p\alpha_2^{-1},\\
&p||(\alpha_2 - q_2),\ \ \alpha_2 - q_2 = p\alpha_3^{-1},\\
&p||(\alpha_3 - q_3),\ \ \alpha_3 - q_3 = p\alpha_4^{-1},\\
&\qquad \vdots \qquad\qquad\qquad \vdots\\
&p||(\alpha_n - q_n), \alpha_n - q_n = p\alpha_{n+1}^{-1}.
\end{aligned}
$$

Then we see that α_n are p-adic units for $n \geq 2$ and q_n are not divisible by p for $n \geq 2$. Now we show that the continued fraction

$$q_1 + \frac{p|}{|q_2} + \frac{p|}{|q_3} + \frac{p|}{|q_4} + \cdots$$

converges to α. From the definition of α_n, we can express it by successive transformation as

$$
\begin{aligned}
\alpha &= \alpha_1\\
&= q_1 + p/\alpha_2\\
&= q_1 + \frac{p|}{|\alpha_2}\\
&= q_1 + \frac{p|}{|q_2 + p/\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
&= q_1 + \frac{p|}{|q_2} + \frac{p|}{|\alpha_3} \\
&= \cdots \\
&= q_1 + \frac{p|}{|q_2} + \frac{p|}{|q_3} + \cdots + \frac{p|}{|\alpha_n}.
\end{aligned}
$$

Hence, it follows that

$$
\begin{aligned}
\alpha - a_{n-1} &= \frac{P_n(q_1, ..., q_{n-1}, \alpha_n)}{Q_n(q_1, ..., q_{n-1}, \alpha_n)} - \frac{P_{n-1}(q_1, ..., q_{n-1})}{Q_{n-1}(q_1, ..., q_{n-1})} \\
&= \frac{P_n(q_1, ..., q_{n-1}, \alpha_n) Q_{n-1}(q_1, ..., q_{n-1})}{Q_{n-1}(q_1, ..., q_{n-1}) Q_n(q_1, ..., q_{n-1}, \alpha_n)} \\
&\quad - \frac{P_{n-1}(q_1, ..., q_{n-1}) Q_n(q_1, ..., q_{n-1}, \alpha_n)}{Q_{n-1}(q_1, ..., q_{n-1}) Q_n(q_1, ..., q_{n-1}, \alpha_n)} \\
&= \frac{(-1)^n p^{n-1}}{Q_{n-1}(q_1, ..., q_{n-1}) Q_n(q_1, ..., q_{n-1}, \alpha_n)}.
\end{aligned} \tag{4.2}
$$

On the other hand, as in the proof of Theorem 1.1, we see that $|Q_n|_p = 1$ for $n \geq 1$. So, from (4.2), we conclude that

$$
|\alpha - a_{n-1}|_p = \left| \frac{(-1)^n p^{n-1}}{Q_{n-1}(q_1, ..., q_{n-1}) Q_n(q_1, ..., q_{n-1}, \alpha_n)} \right|_p = \frac{1}{p^{n-1}} \to 0 \quad (\text{as } n \to \infty).
$$

□

Remark 2. Let $\alpha \in \mathbb{Z}_p$. From the proof of Theorem 1.2, for any sequence $q_1, q_2, q_3, ...$ which satisfy the condition in the proof, the continued fraction (4.1) converges to α. So there exist infinitely many sequences $\{q_n\}$ such that the continued fraction (4.1) converges to α. In the next section, we consider the continued fraction of another type and prove the Theorem 5.1 which establishes the uniqueness for certain type of p-adic continued fraction expansion.

§ 5. “p-semiregula” continued fractions in $\mathbb{Q}_p$

Let p be a prime number, $\mathbb{Q}_p$ the p-adic number field and $S = \{1, 2, ..., p-1\}$. In this section we consider the continued fraction of following type:

$$
p^{b_1} \left(q_1 + \frac{p^{b_2}|}{|q_2} + \frac{p^{b_3}|}{|q_3} + \frac{p^{b_4}|}{|q_4} + \cdots \right), \tag{5.1}
$$

where $q_1, q_2, q_3, ... \in S$, b_1 is an integer, and $b_2, b_3, b_4, ...$ are natural numbers.

We shall call it a “p-semiregular” continued fraction.

Theorem 5.1. *Let $q_1, q_2, q_3, \ldots \in S$, b_1 an integer, $b_2, b_3, b_4, \ldots$ natural numbers. Suppose that there exist infinitely many n such that $(q_n, b_n) \neq (p-1, 1)$. Then the continued fraction* (5.1) *converges to an irrational p-adic number. Conversely, for any irrational p-adic number α, there exist unique sequences $\{q_n\}$ and $\{b_n\}$ with $q_n \in S$ for $n \geq 1$, $b_1 \in \mathbb{Z}$, $b_n \in \mathbb{N}$ for $n \geq 2$ and $(q_n, b_n) \neq (p-1, 1)$ for infinitely many n, such that the continued fraction* (5.1) *converges to α.*

For the proof of Theorem 5.1, we start with several propositions.

Proposition 5.2. *If $q_n \in S$ for $n \geq 1$, $b_1 \in \mathbb{Z}$ and $b_n \in \mathbb{N}$ for $n \geq 2$, then the continued fraction (5.1) converges to a p-adic number.*

Proof. Let

$$\begin{aligned}
P_0 &= p^{b_1}, P_1 = p^{b_1} x_1, \\
Q_0 &= 0, Q_1 = 1, \\
P_n &= x_n P_{n-1} + p^{b_n} P_{n-2}, \\
Q_n &= x_n Q_{n-1} + p^{b_n} Q_{n-2}
\end{aligned}$$

for $n \geq 2$. Then we have

$$\begin{aligned}
&p^{b_1} \left(x_1 + \frac{p^{b_2}|}{|x_2} + \cdots + \frac{p^{b_n}|}{|x_n} \right) = \frac{P_n}{Q_n} \ (n \geq 1), \\
&P_n Q_{n-1} - P_{n-1} Q_n = (-1)^n p^{b_1 + b_2 + \cdots + b_n} \ (n \geq 1), \\
&P_n Q_{n-2} - P_{n-2} Q_n = (-1)^{n-1} p^{b_1 + b_2 + \cdots + b_{n-1}} x_n \ (n \geq 2).
\end{aligned}$$

The proofs of these facts are similar to those in the proof of Theorem 1.1. Now the proof in Theorem 1.1 can be repeated to prove the proposition. □

Proposition 5.3. *For a given $\alpha \in \mathbb{Q}_p \setminus \mathbb{Q}$ let*

$$q_n \in S, \quad b_n \in \mathbb{Z}, \quad \alpha_n \in \mathbb{Q}_p$$

be defined inductively by

$$\begin{array}{lll}
b_1 = \mathrm{ord}_p(\alpha), & \alpha_1 = p^{-b_1}\alpha, & \\
p|(\alpha_1 - q_1), & b_2 = \mathrm{ord}_p(\alpha_1 - q_1), & \alpha_1 - q_1 = p^{b_2}\alpha_2^{-1}, \\
p|(\alpha_2 - q_2), & b_3 = \mathrm{ord}_p(\alpha_2 - q_2), & \alpha_2 - q_2 = p^{b_3}\alpha_3^{-1}, \\
\vdots & \vdots & \vdots \\
p|(\alpha_n - q_n), & b_{n+1} = \mathrm{ord}_p(\alpha_n - q_n), & \alpha_n - q_n = p^{b_{n+1}}\alpha_{n+1}^{-1} \ (n \geq 1).
\end{array}$$

Then, the continued fraction (5.1) converges to α.

Proof. Similar to the proof of Theorem 1.2. □

Proposition 5.4. *Suppose that the continued fraction* (5.1) *converges to* $\alpha \in \mathbb{Q}_p$ *(rational numbers are allowed). Then the sequences* $\{q_n\}, \{b_n\}$ *are recovered from* α *as in Proposition 5.3.*

Proof. Let

$$\alpha_n = q_n + \frac{p^{b_{n+1}}|}{|q_{n+1}} + \frac{p^{b_{n+2}}|}{|q_{n+2}} + \cdots \quad (n \geq 1).$$

Then we can easily see that $|\alpha_n|_p = 1$ for $n \geq 1$ and

$$\begin{aligned} \alpha &= p^{b_1}\alpha_1, \\ \alpha_n &= q_n + \frac{p^{b_{n+1}}}{\alpha_{n+1}} \qquad (n \geq 1). \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{ord}_p(\alpha) &= \operatorname{ord}_p(p^{b_1}\alpha_1) = b_1, \\ \operatorname{ord}_p(\alpha_1 - q_1) &= \operatorname{ord}_p\left(\frac{p^{b_2}}{\alpha_2}\right) = b_2, \\ \operatorname{ord}_p(\alpha_2 - q_2) &= \operatorname{ord}_p\left(\frac{p^{b_3}}{\alpha_3}\right) = b_3, \\ &\vdots \\ \operatorname{ord}_p(\alpha_n - q_n) &= \operatorname{ord}_p\left(\frac{p^{b_n}}{\alpha_{n+1}}\right) = b_{n+1}. \end{aligned}$$

So, the sequences $\{q_n\}$ and $\{b_n\}$ satisfy the conditions in Proposition 5.3. □

Proposition 5.5. *For a given* $\alpha \in \mathbb{Q}$ *let*

$$q_n \in S, \quad b_n \in \mathbb{Z}, \quad \alpha_n \in \mathbb{Q}$$

be defined inductively by

$$\begin{array}{lll} b_1 = \operatorname{ord}_p(\alpha), & \alpha_1 = p^{-b_1}\alpha, & \\ p|(\alpha_1 - q_1), & b_2 = \operatorname{ord}_p(\alpha_1 - q_1), & \alpha_1 - q_1 = p^{b_2}\alpha_2^{-1}, \\ p|(\alpha_2 - q_2), & b_3 = \operatorname{ord}_p(\alpha_2 - q_2), & \alpha_2 - q_2 = p^{b_3}\alpha_3^{-1}, \\ \vdots & \vdots & \vdots \\ p|(\alpha_n - q_n), & b_{n+1} = \operatorname{ord}_p(\alpha_n - q_n), & \alpha_n - q_n = p^{b_{n+1}}\alpha_{n+1}^{-1}\ (n \geq 1). \end{array}$$

Then, either $\alpha_n - q_n = 0$ *for some* n, *or* $q_n = p - 1, b_n = 1$ *for all* $n \gg 1$.

Proof. Suppose that $\alpha_n - q_n \neq 0$ for all n, and write

$$\alpha_n = \frac{r_n}{s_n} \text{ with } (r_n, s_n) = 1.$$

Now, consider the sequence of natural numbers $\{|r_n| + p|s_n|\}$. From the definition, we have

$$\begin{aligned} &|r_{n+1}| + p|s_{n+1}| \\ &\quad \leq |s_n| + \frac{p}{p^{b_{n+1}}} (|r_n| + (p-1)|s_n|) \\ &\quad \leq |s_n| + (|r_n| + (p-1)|s_n|) \\ &\quad = |r_n| + p|s_n|. \end{aligned}$$

The equality holds if and only if $q_n = p-1, b_{n+1} = 1$. Since the numbers $|r_n| + p|s_n|$ become a constant for sufficiently large n, we have

$$q_n = p-1, b_n = 1 \ \ (n \gg 1).$$

□

Proposition 5.6.

$$(p-1) + \frac{p}{\lfloor p-1} + \frac{p}{\lfloor p-1} + \frac{p}{\lfloor p-1} + \cdots = -1 \quad (\ \textit{in } \mathbb{Q}_p).$$

Proof. We prove the following equality which is obviously equivalent to the assertion.

$$p + \frac{p}{\lfloor p-1} + \frac{p}{\lfloor p-1} + \frac{p}{\lfloor p-1} + \cdots = 0 \qquad (\text{ in } \mathbb{Q}_p\ ).$$

In the proof of Proposition 5.2, we set

$$q_1 = p, \qquad q_n = p-1 \quad (n \geq 2), \qquad b_1 = 0, \qquad b_n = 1 \quad (n \geq 2).$$

Then we have

$$\begin{aligned} P_0 &= 1, \\ P_1 &= p, \\ P_2 &= (p-1)P_1 + pP_0 = (p-1)p + p \cdot 1 = p^2, \\ P_3 &= (p-1)P_2 + pP_1 = (p-1)p^2 + p \cdot p = p^3, \\ &\vdots \\ P_n &= p^n \qquad (n \geq 1). \end{aligned}$$

On the other hand, as in the proof of Theorem 1.1, we have $|Q_n|_p = 1$ for $n \geq 1$. So we conclude

$$\left| p + \frac{p}{|p-1} + \frac{p}{|p-1} + \cdots + \frac{p}{|p-1} \right|_p = \left| \frac{P_n}{Q_n} \right|_p = \frac{1}{p^n} \to 0 \qquad (\text{as } n \to \infty).$$

□

We now have most of the ingredients to prove Theorem 5.1.

Proof of Theorem 5.1. Let $\alpha \in \mathbb{Q}_p \setminus \mathbb{Q}$. First, Proposition 5.3 asserts that there exist $q_n \in S$, for $n \geq 1$, $b_1 \in \mathbb{Z}$, and $b_n \in \mathbb{N}$ for $n \geq 2$ such that

$$p^{b_1} \left(q_1 + \frac{p^{b_2}}{|q_2} + \frac{p^{b_3}}{|q_3} + \frac{p^{b_4}}{|q_4} + \cdots \right) = \alpha.$$

Now we show that $(q_n, b_n) \neq (p-1, 1)$ for infinitely many n. If this is not the case, there exists $N \in \mathbb{N}$ such that $(q_n, b_n) = (p-1, 1)$ for all $n \geq N$. Then from Proposition 5.6, we have

$$q_N + \frac{p^{b_{N+1}}}{|q_{N+1}} + \frac{p^{b_{N+2}}}{|q_{N+2}} + \frac{p^{b_{N+3}}}{|q_{N+3}} + \cdots = -1.$$

So it follows that

$$\alpha = p^{b_1} \left(q_1 + \frac{p^{b_2}}{|q_2} + \frac{p^{b_3}}{|q_3} + \frac{p^{b_4}}{|q_4} + \cdots + \frac{p^{b_{N-1}}}{|-1} \right).$$

This contradict that α is an irrational number. Hence the sequences $\{q_n\}$, $\{b_n\}$ satisfy all the condition in Theorem 5.1. Further, Proposition 5.4 guarantees the uniqueness of $\{q_n\}$ and $\{b_n\}$. Next, let $q_n \in S$ for $n \geq 1$, b_1 an integer, b_n natural numbers for $n \geq 2$, and suppose that there exist infinitely many n such that $(q_n, b_n) \neq (p-1, 1)$. Then Proposition 5.2 says that the continued fraction

$$p^{b_1} \left(q_1 + \frac{p^{b_2}}{|q_2} + \frac{p^{b_3}}{|q_3} + \frac{p^{b_4}}{|q_4} + \cdots \right)$$

converges to a p-adic number. If this limiting value is a rational number, then from Proposition 5.4 and Proposition 5.5, (q_n, b_n) must be $(p-1, 1)$ for sufficiently large n. This contradict the assumption of $\{q_n\}$ and $\{b_n\}$. This completes the proof of Theorem 5.1. □

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