On the parity of poly-Euler numbers

By

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Abstract

Poly-Euler numbers are introduced in [9] via special values of an $L$-function as a generalization of the Euler numbers. In this article, poly-Euler numbers with negative index are mainly treated, and the parity of them is shown as the main theorem. Furthermore the divisibility of poly-Euler numbers are also discussed.

§ 1. Introduction

For every integer $k$, we define poly-Euler numbers $E_{n}^{(k)}$ $(n = 0, 1, 2, \ldots)$, which is introduced as a generalization of the Euler number, by

\[
\frac{\text{Li}_k(1 - e^{-4t})}{4t\cosh t} = \sum_{n=0}^{\infty} \frac{E_{n}^{(k)}}{n!} t^n.
\]

Here,

\[
\text{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad (|x| < 1, \ k \in \mathbb{Z})
\]

is the $k$-th polylogarithm. When $k = 1$, $E_{n}^{(1)}$ is the Euler number defined by

\[
\frac{1}{\cosh t} = \sum_{n=0}^{\infty} \frac{E_{n}^{(1)}}{n!} t^n.
\]
The reason why we refer to $E_{n}^{(k)}$s as “poly-Euler numbers” will be mentioned in the next section from the point of view of the relation between the poly-Bernoulli number and Arakawa-Kaneko’s zeta-function. In this article, we treat some number theoretical properties of poly-Euler numbers with negative index ($k \leq 0$). Our main theorem is Theorem 3.1 described in Section 3, which mentions the parity of poly-Euler numbers can be determined definitely. In Section 4, we discuss the divisibility of poly-Euler numbers via congruence relations of them. Tables 1 and 2 cited at the end of this article are the lists of numerical values of poly-Euler numbers. General properties of poly-Euler numbers including the case of positive index are treated in [8].

§2. The poly-Bernoulli numbers and Arakawa-Kaneko’s zeta-function

For every integer $k$, the poly-Bernoulli numbers $B_{n}^{(k)}$ and the modified poly-Bernoulli numbers $C_{n}^{(k)}$ introduced by Kaneko [4] are defined by

$$\frac{\mathrm{Li}_{k}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} \frac{B_{n}^{(k)}}{n!} t^{n}; \quad \frac{\mathrm{Li}_{k}(1-e^{-t})}{e^{t}-1} = \sum_{n=0}^{\infty} \frac{C_{n}^{(k)}}{n!} t^{n},$$

respectively. When $k = 1$, the above generating functions become

$$\frac{te^{t}}{e^{t}-1} \quad \text{and} \quad \frac{t}{e^{t}-1},$$

respectively. Therefore $B_{n}^{(k)}$ and $C_{n}^{(k)}$ are generalizations of the classical Bernoulli numbers. Some number theoretic properties of the poly-Bernoulli number were given by Kaneko [4], Arakawa and Kaneko [2] and others. Furthermore the combinatorial interpretations of $B_{n}^{(-k)}$ were given by Brewbaker [3] and Launois [6]. Recently, Shikata [10] gives the alternative proof of the result of Brewbaker.

It is known that the poly-Bernoulli numbers are special values of Arakawa-Kaneko’s zeta-function. Arakawa and Kaneko [1] introduced a zeta-function:

$$\xi_{k}(s) := \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} \frac{\mathrm{Li}_{k}(1-e^{-t})}{e^{t}-1} dt \quad (k \geq 1).$$

We refer to the above function as Arakawa-Kaneko’s zeta-function. Arakawa-Kaneko’s zeta-function satisfies $\xi_{1}(s) = s\zeta(s + 1)$, and

$$\xi_{k}(-n) = \sum_{l=0}^{n} (-1)^{l} \left( \begin{array}{c} n \\ l \end{array} \right) B_{l}^{(k)} = (-1)^{n} C_{n}^{(k)}$$

for any non-positive integer $n$, where $\zeta(s)$ is the Riemann zeta-function. Hence Arakawa-Kaneko’s zeta-function is a kind of generalization of the Riemann zeta-function. Furthermore, we should mention that Arakawa-Kaneko’s zeta-function is applied to research
On the parity of poly-Euler numbers

In this section, we determine the parity of poly-Euler numbers \((n+1)E_{n}^{(-k)}\).

**Theorem 3.1.** For any non-negative integer \(k\), \((n+1)E_{n}^{(-k)}\) is even (odd, respectively) integer, when \(n\) is odd (even, respectively).

**Proof of Theorem 3.1.** We first review an explicit formula:

**Lemma 3.2** (Ohno-Sasaki [8]). For any non-negative integers \(k\) and \(n\), we have

\[
(n+1)E_{n}^{(-k)} = (-1)^{k} \sum_{l=0}^{k} (-1)^{l} \binom{k}{l} \sum_{m=1}^{n+1} \binom{n+1}{m} \left(4l+2\right)^{n+1-m},
\]
where the symbol $\left\{ \begin{array}{l} k \\ l \end{array} \right\}$ is the Stirling number of the second kind defined by the recurrence relation

$$\left\{ \begin{array}{l} k + 1 \\ l \end{array} \right\} = \left\{ \begin{array}{l} k \\ l - 1 \end{array} \right\} + l \left\{ \begin{array}{l} k \\ l \end{array} \right\}$$

with

$$\left\{ \begin{array}{l} 0 \\ 0 \end{array} \right\} = 1, \quad \left\{ \begin{array}{l} k \\ 0 \end{array} \right\} = \left\{ \begin{array}{l} 0 \\ l \end{array} \right\} = 0 \ (k, l \neq 0)$$

for any integers $k$ and $l$.

See [8] for the detailed proof of Lemma 3.2.

Remark. We easily see that the right-hand side of (3.1) is an integer, since the Stirling numbers are integers. Furthermore this fact indicates that the denominator of poly-Euler number $E_n^{(-k)}$ is at most $n + 1$.

We prove Theorem 3.1 by using the above lemma. Note that

$$\sum_{m=1, m\text{:odd}}^{n+1} \binom{n+1}{m} (4l + 2)^{n+1-m} \equiv \begin{cases} 1 \pmod{2} & n \text{ : even}, \\ 0 \pmod{2} & n \text{ : odd} \end{cases}$$

for any non-negative integer $l$. Hence, when $n$ is odd, we have Theorem 3.1 immediately from Lemma 3.2.

On the other hand, when $n$ is even, we have

$$(n + 1)E_n^{(-k)} \equiv \sum_{l=0}^{k} l! \left\{ \begin{array}{l} k \\ l \end{array} \right\} = \left\{ \begin{array}{l} k \\ 0 \end{array} \right\} + \left\{ \begin{array}{l} k \\ 1 \end{array} \right\} = \left\{ \begin{array}{l} k + 1 \\ 1 \end{array} \right\} = 1 \pmod{2}$$

for any non-negative integer $k$. Here, we have used the recurrence relation (3.2). Thus the proof of Theorem 3.1 is completed.

§ 4. Congruence relations of poly-Euler numbers

In the previous section, we definitely determined the parity of poly-Euler numbers. In this section, we treat the divisibility of poly-Euler numbers via congruence relations of them. In particular, we consider the case of

$$(n + 1)E_n^{(-k)} \pmod{n + 1},$$

which allows us to evaluate whether $E_n^{(-k)}$ is an integer. In [8], we treat the case when $n + 1$ is an odd prime. Hence we discuss the composite cases here.
Theorem 4.1. For any non-negative integer \(k\), we have

\[
6E_{5}^{(-k)} \equiv \begin{cases} 
4 \pmod{6} & \text{if } k \text{ is even}, \\
0 \pmod{6} & \text{if } k \text{ is odd}.
\end{cases}
\]

Proof of Theorem 4.1. From Lemma 3.2, we have

\[
(4.1) \quad 6E_{5}^{(-k)} = (-1)^{k} \sum_{l=0}^{k} \frac{(-1)^{l}l!}{l!} \sum_{j=0}^{2} \binom{6}{2j+1} (4l+2)^{5-2j}.
\]

We see that \(l! \equiv 0 \pmod{6}\) for \(l \geq 3\) and

\[
\sum_{j=0}^{2} \binom{6}{2j+1} (4l+2)^{5-2j} \equiv 2(4l+2)^{3} \pmod{6}
\]

\[
\equiv \begin{cases} 
4 \pmod{6} & \text{if } l \equiv 0 \pmod{3}, \\
0 \pmod{6} & \text{if } l \equiv 1 \pmod{3}, \\
2 \pmod{6} & \text{if } l \equiv 2 \pmod{3}.
\end{cases}
\]

Thus (4.1) becomes

\[
6E_{5}^{(-k)} \equiv (-1)^{k} \left( \binom{k}{0} + \binom{k}{2} \right) \pmod{6}
\]

\[
\equiv \begin{cases} 
4 \pmod{6} & \text{if } k = 0, \\
0 \pmod{6} & \text{if } k = 1, \\
(-1)^{k} \binom{k}{2} & \text{if } k \geq 2.
\end{cases}
\]

Therefore, we claim

\[
(4.2) \quad (-1)^{k} \binom{k}{2} \equiv \begin{cases} 
1 \pmod{6} & \text{if } k \text{ is even}, \\
3 \pmod{6} & \text{if } k \text{ is odd}.
\end{cases}
\]

Using the expression \(\binom{k}{2} = 2^{k-1} - 1 \pmod{6}\), we have

\[
\binom{k}{2} = 2^{k-1} - 1 = \sum_{j=0}^{k-2} 2^{j} = 1 + \sum_{j=1, j: \text{odd}}^{k-2} 2^{j} + \sum_{j=2, j: \text{even}}^{k-2} 2^{j}
\]

\[
\equiv 1 + 2[(k-1)/2] + 4[(k-2)/2] \pmod{6},
\]

which gives (4.2). Thus we have Theorem 4.1.

By the same way as above, we can also show the following theorem:
Theorem 4.2. For any non-negative integer $k$, we have

$$12E_{11}^{(-k)} \equiv \begin{cases} 4 \pmod{12} & \text{if } k \text{ is even;} \\ 0 \pmod{12} & \text{if } k \text{ is odd.} \end{cases}$$

In general, to understand the prime factors of the numerator of $E_n^{(-k)}$ is proper. In [8], we prove a congruence relation

$$(n + 1)E_n^{(-k)} \equiv 0 \pmod{p}$$

holds for any odd prime $p$, odd positive integer $n$ and non-positive integer $k$ satisfying $k \equiv p - 2 \pmod{p - 1}$. The second assertion of Theorem 4.1 is also given by combining the above congruence relation with Theorem 3.1.

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References

### Table 1. $E_{n}^{(k)}$ (positive index)

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Table 2. \( E_{n}^{(-k)} \) (negative index)