RIMS Kôkyûroku Bessatsu
B33 (2012), 1–10

A majorant problem for the periodic Schrödinger group

By

Jonathan Bennett and Neal Bez*

§ 1. Introduction

For appropriate sequences $a : \mathbb{Z} \to \mathbb{C}$, let $\mathcal{E}$ be the discrete extension operator given by

$$\mathcal{E}a(x,t) := \sum_{n\in\mathbb{Z}} a_n e(nx + n^2 t)$$

for $(x,t) \in \mathbb{T}^2$. Here we are writing $e(x) := e^{2\pi i x}$ for $x \in \mathbb{T}$, where $\mathbb{T} := [0,1]$. Of course, $u(x,t) = \mathcal{E}a(x,t)$ formally satisfies the free Schrödinger equation $\partial_x^2 u(x,t) = 2\pi i \partial_t u(x,t)$ with periodic initial data $u(0)$ equal to the function whose nth Fourier coefficient is equal to a_n. Moreover, the adjoint of $\mathcal{E}$ is the mapping which restricts the Fourier transform to the discrete parabola $\{(n,n^2) : n \in \mathbb{Z}\}$.

We shall write $a \preccurlyeq A$ when the sequence a is majorised by the sequence A in the sense that $|a_n| \leq A_n$ for each n. Our concern here is to what extent the operator $\mathcal{E}$ satisfies a majorant property on $L^p(\mathbb{T}^2)$; that is, given $a \preccurlyeq A$, in what sense is $\|\mathcal{E}a\|_{L^p(\mathbb{T}^2)}$ majorised by $\|\mathcal{E}A\|_{L^p(\mathbb{T}^2)}$? It appears that this question has not been explicitly posed before and in this note we offer some preliminary results.

The question is of course very reminiscent of the classical Hardy–Littlewood majorant problem for Fourier series, originating in [8], where it is conjectured that for each $p \in [2,\infty)$ there exists a finite constant $\mathbf{B}_p$ such that

$$\Big\| \sum_{n\in\mathbb{Z}} a_n e(nx) \Big\|_{L^p(\mathbb{T})} \leq \mathbf{B}_p \Big\| \sum_{n\in\mathbb{Z}} A_n e(nx) \Big\|_{L^p(\mathbb{T})}$$

whenever $a \preccurlyeq A$. The conjecture was left open by Hardy and Littlewood, but they observed in [8] that one may take $\mathbf{B}_p = 1$ whenever p is an even integer and that

Received September 30, 2011. Revised October 19, 2011.
2000 Mathematics Subject Classification(s): 42B05; 35B10
Key Words: majorant problem; periodic Schrödinger group
The second author was supported by JSPS Invitation Fellowship
*School of Mathematics, The Watson Building, University of Birmingham, Edgbaston, Birmingham, B15 2TT, England.

$\mathbf{B}_3 > 1$. The former claim is straightforward by multiplying out the norm. The latter claim may be extended to $\mathbf{B}_p > 1$ for all $p > 2$ which are not even integers (see, for example, [3]) and using certain re-scaled products of these examples, Bachelis [1] (see also [13]) showed that the Hardy–Littlewood conjecture is in fact false.

A renewed interest stemmed from the observation (see [11]) that a certain quantitative understanding of the Hardy–Littlewood majorant problem would imply the euclidean restriction conjecture for the sphere[1]. In particular, suppose $N \in \mathbb{N}$ and $\Lambda \subseteq \{1, \dots, N\}$, and let $\mathbf{B}_p(\Lambda)$ denote the smallest constant such that

$$\Big\| \sum_{n \in \Lambda} a_n e(nx) \Big\|_{L^p(\mathbb{T})} \leq \mathbf{B}_p(\Lambda) \Big\| \sum_{n \in \Lambda} e(nx) \Big\|_{L^p(\mathbb{T})}$$

holds for all sequences $a : \mathbb{Z} \to \mathbb{C}$ supported in Λ with $a \preccurlyeq 1_\Lambda$. Here, 1_Λ denotes the sequence taking the value one on Λ and zero elsewhere. If it were known that

$$\mathbf{B}_p(\Lambda) \leq C_{p,\varepsilon} N^\varepsilon$$

for a subset Λ which arises from a certain discretisation of the unit sphere, then one may deduce that

$$\|\widehat{g\mathrm{d}\sigma}\|_{L^p(\mathbb{R}^d)} \lesssim \|g\|_{L^\infty(\mathbb{S}^{d-1})} \tag{1.1}$$

for all $g \in L^\infty(\mathbb{S}^{d-1})$, where $d \geq 2$ and $p > \frac{2d}{d-1}$. Whilst this particular quantitative majorant problem appears to be open, it has been shown that whenever $p > 2$ is not even, there exists $\gamma_p > 0$ such that

$$\mathbf{B}_p(N) \geq N^{\gamma_p},$$

where

$$\mathbf{B}_p(N) = \sup_{\Lambda \subseteq \{1, \dots, N\}} \mathbf{B}_p(\Lambda).$$

This may be found in the independent works of Mockenhaupt–Schlag [12] and Green–Ruzsa [7] (in the latter case, explicitly for $p = 3$ but their argument may be modified in general). One may view these arguments as precise versions of those of Bachelis mentioned above.

In the positive direction, we mention the interesting result of Mockenhaupt–Schlag, also in [12], where it is shown that the majorant property is (essentially) true for random

[1]Note that the *continuous* majorant property

$$\|\widehat{g\mathrm{d}\sigma}\|_{L^{\frac{2d}{d-1}}(B(R))} \lesssim \|\widehat{\mathrm{d}\sigma}\|_{L^{\frac{2d}{d-1}}(B(R))}$$

whenever $\|g\|_{L^\infty(\mathbb{S}^{d-1})} \lesssim 1$ is a strong form of the restriction conjecture, which holds when $d = 2$ and is open for $d \geq 3$. Here, $\mathrm{d}\sigma$ is induced Lebesgue measure on $\mathbb{S}^{d-1}$, and throughout we shall use $X \lesssim Y$ and $Y \gtrsim X$ for $X \leq CY$ where the finite constant C depends at most on d and p.

subsets Λ. More precisely, for each $N \in \mathbb{N}$ and $1 \le j \le N$, let $\xi_j = \xi_j(\omega)$ be independent and identically distributed indicator random variables where, for some fixed $\delta \in (0,1)$, ξ_j takes the value one with probability $N^{-\delta}$ and the value zero with probability $1-N^{-\delta}$; thus, $\mathbb{P}[\xi_j = 1] = N^{-\delta}$ and $\mathbb{P}[\xi_j = 0] = 1 - N^{-\delta}$. Setting $\Lambda(\omega) = \{j \in \{1,\dots,N\} : \xi_j(\omega) = 1\}$ to be an associated random subset, it is shown in [12] that for each $\varepsilon > 0$ and $p \ge 2$,

$$\mathbb{P}\Big[\sup_{a: a \preccurlyeq 1} \Big\| \sum_{n \in \Lambda(\omega)} a_n e(nx) \Big\|_{L^p(\mathbb{T})} \ge N^{\varepsilon} \Big\| \sum_{n \in \Lambda(\omega)} e(nx) \Big\|_{L^p(\mathbb{T})} \Big] \to 0$$

as $N \to \infty$. Hence, there is a sense in which the desired majorant bound (for the application to the restriction conjecture) holds for such random sets with probability one as N tends to infinity.

It is our intention in this short and somewhat speculative note to present some of our observations concerning the majorant problem for $\mathcal{E}$. Firstly, whenever $a \preccurlyeq A$,

$$\|\mathcal{E}a\|_{L^p(\mathbb{T}^2)} \le \|\mathcal{E}A\|_{L^p(\mathbb{T}^2)}$$

is true for even integers p and follows by multiplying out the norm (as was the case for Fourier series). Next, we are able to prove that such a perfect majorant property does not hold away from the even integers.

Theorem 1.1. *Suppose $p \in (2,\infty)$ is not an even integer. Then there exist sequences a and A such that $a \preccurlyeq A$ and $\delta_p > 0$ such that*

$$\|\mathcal{E}a\|_{L^p(\mathbb{T}^2)} \ge (1+\delta_p)\|\mathcal{E}A\|_{L^p(\mathbb{T}^2)}.$$

We provide a proof of Theorem 1.1 in Section 3. The following complementary result is completely elementary, but it appears to provide a point of interest because it highlights an essential difference with the majorant problem for Fourier series.

Theorem 1.2. *Suppose $p \in [2,4]$. Then $\|\mathcal{E}a\|_{L^p(\mathbb{T}^2)} \lesssim \|\mathcal{E}A\|_{L^p(\mathbb{T}^2)}$ whenever $a \preccurlyeq A$.*

Thus, at least for $p \in [2,4]$, the analogoue of the original conjecture of Hardy and Littlewood is valid for $\mathcal{E}$. Theorem 1.2 is true because

$$\|\mathcal{E}a\|_{L^4(\mathbb{T}^2)}^4 = \sum_{\substack{m_1+m_2=n_1+n_2 \\ m_1^2+m_2^2=n_1^2+n_2^2}} a_{m_1} a_{m_2} \overline{a_{n_1} a_{n_2}}$$

and the restriction on the support of the summation implies that $\{m_1, m_2\} = \{n_1, n_2\}$. Consequently,

$$\|\mathcal{E}a\|_{L^4(\mathbb{T}^2)} = 2^{1/4}\|a\|_{\ell^2(\mathbb{Z})} \tag{1.2}$$

and, using Hölder's inequality, the restriction on the size of p and Plancherel's theorem,

$$\|\mathcal{E}a\|_{L^p(\mathbb{T}^2)} \leq \|\mathcal{E}a\|_{L^4(\mathbb{T}^2)} \leq 2^{1/4}\|A\|_{\ell^2(\mathbb{Z})} = 2^{1/4}\|\mathcal{E}A\|_{L^2(\mathbb{T}^2)} \leq 2^{1/4}\|\mathcal{E}A\|_{L^p(\mathbb{T}^2)}. \tag{1.3}$$

Thus, implicit constant in the statement of Theorem 1.2 may be taken to be $2^{1/4}$ for each $p \in [2,4]$. An interpolation argument using (1.2) and $\|\mathcal{E}a\|_{L^2(\mathbb{T}^2)} = \|a\|_{\ell^2(\mathbb{Z})}$ allows one to improve this to $2^{1/2-1/p}$, however even this bound does not recover the optimal constant equal to one when $p = 4$. It might be interesting to find the optimal dependence on p of this constant.

The identity (1.2) may be found in the work of Bourgain [5] and one may view this as a statement about the finiteness of the so-called $\Lambda(4)$-constant of the set of parabolic lattice points in the plane, or as a Strichartz-type estimate for periodic solutions of the Schrödinger equation. We elaborate on this connection in the subsequent section.

§ 2. Quantitative formulation

Throughout this section we shall use N to denote a positive integer considered large. For fixed $d \geq 1$, let $S_{d,N}$ be the subset of $\mathbb{Z}^d$ given by

$$S_{d,N} = \{n \in \mathbb{Z}^d : |n_j| \leq N, 1 \leq j \leq N\}$$

and for $p > 2$ let $\mathbf{K}_{d,p}(N)$ denote the smallest constant such that

$$\Big\| \sum_{n \in S_{d,N}} a_n e(n \cdot x + |n|^2 t) \Big\|_{L^p(\mathbb{T}^{d+1})} \leq \mathbf{K}_{d,p}(N)\|a\|_{\ell^2(S_{d,N})}$$

holds for all sequences $a : \mathbb{Z} \to \mathbb{C}$ supported in $S_{d,N}$. In [5] it was conjectured that

$$\mathbf{K}_{d,p}(N) \lesssim N^{\gamma_{d,p}} \tag{2.1}$$

for $p \in (2,\infty] \setminus \{\frac{2(d+2)}{d}\}$, where

$$\gamma_{d,p} = \max\left\{0, \frac{d}{2} - \frac{d+2}{p}\right\}$$

and that for all $\varepsilon > 0$,

$$\mathbf{K}_{d,\frac{2(d+2)}{d}}(N) \leq C_\varepsilon N^\varepsilon \tag{2.2}$$

at the critical exponent.

This conjecture is still open in any dimension. For instance, when $d = 1$, we know from (1.2) that (2.1) holds for $p \leq 4$. At the critical exponent $p = 6$, Bourgain proved in [5] that

$$(\log N)^{1/6} \lesssim \mathbf{K}_{1,6}(N) \lesssim \exp(c \log N / \log\log N) \tag{2.3}$$

and consequently (2.2) is true when $d = 1$, and the ε-loss is truly present[2]. For $p > 6$ by a further argument it is also shown in [5] that (2.1) holds and therefore, when $d = 1$, it remains open whether $\mathbf{K}_{1,p}(N) \lesssim 1$ or not for $p \in (4, 6)$.

In higher dimensions, the best known results are also due to Bourgain. For certain (d, p) these also go back to [5]. It is interesting to note that Bourgain has very recently used the multilinear euclidean restriction theory in [2] and ideas from [6] to improve some of the results in [5]. See also the very recent work of Hu–Li [9] where some of Bourgain's estimates in [5] are recovered using new methods. However, for the rest of this note, we remain in the case $d = 1$.

In order to pose a quantitative version of the majorant problem for $\mathcal{E}$, for each positive integer N, we restrict our attention to subsets $\Lambda \subseteq \{1, \ldots, N\}$ and sequences supported in Λ. Thus, for $p \in [2, \infty)$, let $\mathbf{C}_p(\Lambda)$ denote the smallest constant such that

$$\|\mathcal{E}a\|_{L^p(\mathbb{T}^2)} \le \mathbf{C}_p(\Lambda)\|\mathcal{E}1_\Lambda\|_{L^p(\mathbb{T}^2)}$$

holds for all sequences $a : \mathbb{Z} \to \mathbb{C}$ supported in Λ with $a \preccurlyeq 1_\Lambda$, and

$$\mathbf{C}_p(N) = \sup_{\Lambda \subseteq \{1,\ldots,N\}} \mathbf{C}_p(\Lambda).$$

In the rest of this section, we offer some elementary observations concerning these quantities.

Of course, the argument in (1.3) tells us that $\mathbf{C}_p(N) \lesssim 1$ for each $p \in [2, 4]$. By the same argument, we would be able to extend this to all $p \in [2, 6]$ if Bourgain's conjecture were shown to be true, that is, if we knew that $\mathbf{K}_{1,p}(N) \lesssim 1$ for all $p \in (2, 6)$. The uniform estimates in N for $\mathbf{K}_{1,p}(N)$ cease to be true for $p \ge 6$ and for such p the argument (1.3) as it stands no longer provides uniform estimates for $\mathbf{C}_p(N)$. Somewhat amusingly, the critical exponent is an even integer and here we know $\mathbf{C}_6(N) = 1$ by a different (and trivial) argument. Of course, for $p \in (4, 6)$ we at least know by interpolation that $\mathbf{K}_{1,p}(N) \le C_\varepsilon N^\varepsilon$ and therefore, using (1.3), we get the almost uniform majorant property that $\mathbf{C}_p(N) \le C_\varepsilon N^\varepsilon$.

Now suppose $p \in (6, \infty)$. Using Bourgain's estimate

$$\|\mathcal{E}a\|_{L^p(\mathbb{T}^2)} \lesssim N^{1/2-3/p}|\Lambda|^{1/2}$$

whenever $a \preccurlyeq 1_\Lambda$, and the trivial lower bound

$$\|\mathcal{E}1_\Lambda\|_{L^p(\mathbb{T}^2)} \ge \|\mathcal{E}1_\Lambda\|_{L^p(|x|\lesssim N^{-1},|t|\lesssim N^{-2})} \gtrsim |\Lambda|N^{-3/p} \tag{2.4}$$

it follows that $\mathbf{C}_p(\Lambda) \lesssim 1$ whenever $|\Lambda| \sim N$. Thus, it is natural to consider frequency sets Λ with cardinality N^τ where $\tau \in (0, 1)$. We remark that the majorant property for

[2]it is suggested in [14] that the lower bound in (2.3) should be modified to $(\log N/ \log\log N)^{1/6}$

$\mathcal{E}$ is (essentially) true for random subsets Λ for $\tau \leq 2/p$ thanks to work of Bourgain [4] on $\Lambda(p)$ sets, and this could extend to all τ in the spirit of [12].

We also note that the Hausdorff–Young inequality may be applied to give the upper bound

$$\|\mathcal{E}a\|_{L^p(\mathbb{T}^2)} \lesssim |\Lambda|^{1-1/p}$$

whenever $a \preccurlyeq 1_\Lambda$, which may be combined with either the lower bound $\|\mathcal{E}1_\Lambda\|_{L^p(\mathbb{T}^2)} \geq |\Lambda|^{1/2}$ or (2.4) to obtain certain estimates for $\mathbf{C}_p(\Lambda)$. We remark that, in terms of $\mathbf{C}_p(N)$, such naive considerations lead to the bound $\mathbf{C}_p(N) \lesssim \min\{N^{1/2-3/p}, N^{3/p}\}$ which we observe does not improve on $\mathbf{C}_p(N) \lesssim N^{\eta_p}$, where $\eta_p = 6k(k+1)(\frac{1}{2k} - \frac{1}{p})(\frac{1}{p} - \frac{1}{2(k+1)})$ and $p \in [2k, 2(k+1))$. The latter bound is obtained by interpolating $\mathbf{C}_{2k}(N) = \mathbf{C}_{2(k+1)}(N) = 1$.

§ 3. Proof of Theorem 1.1

The argument in this section is based on ideas of Hardy–Littlewood [8]. The singular nature of the frequencies on the parabolic lattice makes our argument somewhat more involved.

Let k be the unique natural number satisfying

$$k - 2 < p/2 < k - 1.$$

We take a and A to be the sequences given by

$$a_n := \begin{cases} r^{n^2} & \text{if } n = 0, n_1, n_2 \\ -r^{n^2} & \text{if } n = n_3 \\ 0 & \text{otherwise} \end{cases}$$

and

$$A_n := \begin{cases} r^{n^2} & \text{if } n = 0, n_1, n_2, n_3 \\ 0 & \text{otherwise} \end{cases}$$

where $(n_1, n_2, n_3) = (2-k, 2, k)$ and $r \in (0,1)$ is a parameter (considered small) to be chosen later in the proof. The reason for this particular choice for (n_1, n_2, n_3) will become clear in a moment.

Thus, $\mathcal{E}a(x,t) = 1 + b(x,t)$ where

$$b(x,t) := r^{(2-k)^2} e((2-k)x)e((2-k)^2 t) + r^4 e(2x)e(4t) - r^{k^2} e(kx)e(k^2 t).$$

For $r > 0$ sufficiently small it is clear that

$$\mathcal{E}a(x,t)^{p/2} = \sum_{j=0}^{\infty} \binom{p/2}{j} b(x,t)^j,$$

and similarly for $\mathcal{E}A$. To be clear, $\binom{p/2}{0} = 1$ and for $j \in \mathbb{N}$ the generalised binomial coefficient $\binom{p/2}{j}$ is given by

$$\binom{p/2}{j} = \frac{1}{2^j j!} \prod_{\ell=0}^{j-1} (p - 2\ell),$$

which is positive for $j = 0, 1, \ldots, k-1$, negative for $j = k$ and alternating in sign thereafter.

The idea is to expand the powers of b to obtain expansions of the form

$$\mathcal{E}a(x,t)^{p/2} = \sum_{\ell=0}^{\infty} Q_\ell[a](x) e(\ell t) r^\ell$$

and

$$\mathcal{E}A(x,t)^{p/2} = \sum_{\ell=0}^{\infty} Q_\ell[A](x) e(\ell t) r^\ell$$

for certain sequences of trigonometric polynomials $(Q_\ell[a])_{\ell \geq 0}$ and $(Q_\ell[A])_{\ell \geq 0}$, and apply Plancherel's theorem in the t-variable. This will generate expressions for the $L^p(\mathbb{T}^2)$-norms of $\mathcal{E}a$ and $\mathcal{E}A$ which we then view as expansions in the parameter r. Using our choice of (n_1, n_2, n_3), we will see that $Q_\ell[a]$ and $Q_\ell[A]$ coincide for $\ell = 0, \ldots, k^2 - 1$. Furthermore, our choice will also yield $\|Q_{k^2}[a]\|_{L^2(\mathbb{T})} > \|Q_{k^2}[A]\|_{L^2(\mathbb{T})}$, and here we capitalise on the fact that $\binom{p/2}{k} < 0$. This means the coefficient of the leading term in r of $\|\mathcal{E}a\|^p_{L^p(\mathbb{T}^2)} - \|\mathcal{E}A\|^p_{L^p(\mathbb{T}^2)}$ is negative, which gives the desired conclusion by choosing $r > 0$ sufficiently small.

Since

$$\left| \mathcal{E}a(x,t)^{p/2} - \sum_{j=0}^{2k^2+1} \binom{p/2}{j} b(x,t)^j \right| \lesssim r^{2k^2+2}$$

and

$$b(x,t)^j = \sum_{\substack{0 \leq \lambda,\mu,\nu \leq j \\ \lambda+\mu+\nu=j}} (-1)^\nu \frac{j!}{\lambda!\mu!\nu!} r^{(2-k)^2\lambda + 4\mu + k^2\nu}$$
$$e(((2-k)\lambda + 2\mu + k\nu)x) e(((2-k)^2\lambda + 4\mu + k^2\nu)t)$$

it follows that

$$\mathcal{E}a(x,t)^{p/2} = \sum_{\ell=0}^{k^2} Q_\ell[a](x) e(\ell t) r^\ell + R[a](x,t).$$

Here

$$Q_\ell[a](x) = \sum_{j=0}^{2k^2+1} \sum_{\substack{0 \leq \lambda,\mu,\nu \leq j \\ \lambda+\mu+\nu=j \\ \lambda n_1^2 + \mu n_2^2 + \nu n_3^2 = \ell}} \binom{p/2}{j} (-1)^\nu \frac{j!}{\lambda!\mu!\nu!} e(((2-k)\lambda + 2\mu + k\nu)x)$$

and $R[a]$ is the remainder term satisfying $|R[a](x,t)| \lesssim r^{k^2+1}$ and

$$|\langle R[a](x,\cdot), e(\ell\cdot)\rangle_{L^2(\mathbb{T})}| \lesssim r^{2k^2+2}$$

for each $x \in \mathbb{T}$, $\ell = 0, \ldots, k^2$. Writing

$$\|\mathcal{E}a\|_{L^p(\mathbb{T}^2)}^p = \int_{\mathbb{T}^2} \mathcal{E}a(x,t)^{p/2}\overline{\mathcal{E}a(x,t)}^{p/2}\,\mathrm{d}t\mathrm{d}x$$

and using orthogonality and the above features of $R[a]$ we obtain

$$\left| \|\mathcal{E}a\|_{L^p(\mathbb{T}^2)}^p - \sum_{\ell=0}^{k^2} \|Q_\ell[a]\|_{L^2(\mathbb{T})}^2 r^{2\ell} \right| \lesssim r^{2k^2+2}. \tag{3.1}$$

A similar argument shows that (3.1) is true with A replacing a, where

$$Q_\ell[A](x) = \sum_{j=0}^{2k^2+1} \sum_{\substack{0\leq\lambda,\mu,\nu\leq j\\ \lambda+\mu+\nu=j\\ \lambda n_1^2+\mu n_2^2+\nu n_3^2=\ell}} \binom{p/2}{j}\frac{j!}{\lambda!\mu!\nu!} e(((2-k)\lambda+2\mu+k\nu)x).$$

Notice that if $0 \leq \ell \leq k^2-1$ and $\lambda n_1^2 + \mu n_2^2 + \nu n_3^2 = \ell$ then necessarily $\nu = 0$. Therefore $Q_\ell[a] = Q_\ell[A]$ for such ℓ. To determine $Q_{k^2}[a]$ and $Q_{k^2}[A]$, we need to find all solutions $(j,\lambda,\mu,\nu) \in (\mathbb{N}_0)^4$ satisfying

$$\lambda+\mu+\nu = j$$

and

$$(2-k)^2\lambda + 4\mu + k^2\nu = k^2.$$

Obviously $\nu \in \{0,1\}$ from the latter equation and consequently the *only* solutions are

$$(j,\lambda,\mu,\nu) \in \{(1,0,0,1),(k,1,k-1,0)\}.$$

Hence, we have

$$Q_{k^2}[a](x) = \left(-\frac{p}{2} + k\binom{p/2}{k}\right)e(kx)$$

and

$$Q_{k^2}[A](x) = \left(\frac{p}{2} + k\binom{p/2}{k}\right)e(kx).$$

Note that by choosing $(n_1,n_2,n_3) = ((2-k)^2, 2, k)$ we have obtained a solution with $j = k$ which allows us to use $\binom{p/2}{k} < 0$. In particular, using (3.1),

$$\left| \|\mathcal{E}a\|_{L^p(\mathbb{T}^2)}^p - \|\mathcal{E}A\|_{L^p(\mathbb{T}^2)}^p + 2kp\binom{p/2}{k} r^{2k^2} \right| \lesssim r^{2k^2+2}$$

and by choosing $r > 0$ sufficiently small, it is clear that there exists $\delta_p > 0$ such that

$$\|\mathcal{E}a\|_{L^p(\mathbb{T}^2)} \geq (1 + \delta_p)\|\mathcal{E}A\|_{L^p(\mathbb{T}^2)}.$$

This completes the proof of Theorem 1.1.

We remark that, as it stands, the above example cannot be used to deduce that $\mathbf{C}_p(N) > 1$ for any $p > 2$ which is not an even integer since our argument relies on choosing $r < 1$. For such p, it would be appealing to find $a \preccurlyeq A$ with $a_n \in \{-1, 0, 1\}$ and $A_n \in \{0, 1\}$ for each n such that $\|\mathcal{E}a\|_{L^p(\mathbb{T}^2)} > \|\mathcal{E}A\|_{L^p(\mathbb{T}^2)}$. It would also be interesting to establish the minimal size of frequency sets for which such a property may hold. In the problem for Fourier series, a constructive argument of Mockenhaupt–Schlag [12] shows that, for any such p, four term idempotents are sufficient, and it is shown in [10] using numerical integration and error estimates that certain three-term idempotents are sufficient for $p < 6$ (this is optimal in the sense that two-term idempotents will not provide examples).

References

[1] G. F. Bachelis, *On the upper and lower majorant properties in $L^p(G)$*, Quart. J. Math. **24** (1973), 119–128.

[2] J. Bennett, A. Carbery, T. Tao, *On the multilinear restriction and Kakeya conjectures*, Acta Math. **196** (2006), 261–302.

[3] R. P. Boas, *Entire functions*, Academic Press, New York 1954.

[4] J. Bourgain, *Bounded orthogonal systems and the $\Lambda(p)$-set problem*, Acta Math. **162** (1989), 227–245.

[5] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations*, Geom. Funct. Anal. **3** (1993), 107–156.

[6] J. Bourgain, L. Guth, *Bounds on oscillatory integral operators obtained from multilinear estimates*, Geom. Funct. Anal., to appear.

[7] B. Green, I. Z. Ruzsa, *On the Hardy–Littlewood majorant problem*, Math. Proc. Cambridge Philos. Soc. **137** (2004) 511–517.

[8] G. H. Hardy, J. E. Littlewood, *Notes on the theory of series (XIX): A problem concerning majorants of Fourier series*, Quart. J. Math. **6** (1935), 304–315.

[9] Y. Hu, X. Li, *Discrete Fourier restriction associated with Schrödinger equations*, arXiv:1108.5164.

[10] S. Krenedits, *Three-term idempotent counterexamples in the Hardy–Littlewood majorant problem*, arXiv:1006.0409.

[11] G. Mockenhaupt, *Bounds in Lebesgue spaces of oscillatory integrals*, Habiliation thesis.

[12] G. Mockenhaupt, W. Schlag, *On the Hardy–Littlewood majorant problem for random sets*, J. Funct. Anal. **256** (2009), 1189–1237.

[13] H. L. Montgomery, *Ten lectures on the interface between analytic number theory and harmonic analysis*, CBMS Regional Conference Series in Mathematics **84**, AMS, Providence, RI, 1994.

[14] T. Oh, *Note on a lower bound of the Weyl sum in Bourgain's NLS paper (GAFA '93)*, available at http://www.math.princeton.edu/∼hirooh/WeylSum.pdf.