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MULTILINEAR OPERATORS IN HARMONIC ANALYSIS AND PARTIAL DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

A multilinear operator $T(f_1, \dots, f_m)$ is a linear operator in every variable f_j. Several examples motivate the theory of multilinear operators. We list a few of them:

(1) The m-fold product

$$I(f_1, \dots, f_m) = f_1 \cdots f_m$$

is the easiest example of an m-linear operator. It indicates that natural inequalities between Lebesgue spaces are of the form $L^{p_1} \times \cdots \times L^{p_m} \to L^p$, where $1/p_1 + \cdots + 1/p_m = 1/p$.

(2) A kernel of $m+1$ variables $K(x, y_1, \dots, y_m)$ gives rise to an m-linear operator of the form

$$T(f_1, \dots, f_m)(x) = \int_{\mathbf{R}^{mn}} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \dots dy_m,$$

where the integral may converge in the principal value sense, or even in the sense of distributions.

(3) The special case in which the kernel $K(x, y_1, \dots, y_m)$ in the previous case has the form $K_0(x - y_1, \dots, x - y_m)$ corresponds to the so-called m-linear convolution

$$T_0(f_1, \dots, f_m)(x) = \int_{\mathbf{R}^{mn}} K_0(x - y_1, \dots, x - y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \dots dy_m,$$

in which the integral is taken in the principal value sense. This operator can also be expressed as an m-linear multiplier as follows

$$\int_{\mathbf{R}^{mn}} m_0(\xi_1, \dots, \xi_m) \widehat{f_1}(\xi_1) \cdots \widehat{f_m}(\xi_m) e^{2\pi i x\cdot(\xi_1 + \cdots + \xi_m)}\, d\xi_1 \dots d\xi_m,$$

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where m_0 is the distributional Fourier transform of K_0 on $\mathbf{R}^{mn}$. The Fourier transform of a Schwartz function φ on $\mathbf{R}^n$ is defined by

$$\widehat{\varphi}(\xi) = \int_{\mathbf{R}^n} \varphi(x) e^{-2\pi i x\cdot\xi} dx\,,$$

where $x \cdot \xi = \sum_{j=1}^n x_j \xi_j$ and the distributional Fourier transform is defined analogously via pairing.

(4) The bilinear operator on $\mathbf{R} \times \mathbf{R}$ given by

$$B(f,g)(x) = \int_{-1/2}^{1/2} f(x+t)\, g(x-t)\, dt\,.$$

This operator is *local* in the sense that if f and g are supported in intervals of length *one*, then $B(f,g)$ is supported in the set of halves of the numbers in the algebraic sum of these intervals; this is also an interval of length one.

(5) The truncated bilinear Hilbert transform

$$H(f,g)(x) = \text{p.v} \int_{-1/2}^{1/2} f(x+t)\, g(x-t)\, \frac{dt}{t}$$

in which f, g are functions on the line. The homogeneity of the kernel $\frac{dt}{t}$ makes the study of the truncated and untruncated versions of this operator equivalent. This operator is also local in the previous sense. Its boundedness from $L^p \times L^q \to L^r$ for $1/p + 1/q = 1/r$, $1 < p, q \le \infty$, $2/3 < r < \infty$, was obtained by Lacey and Thiele [18], [19].

(6) The bilinear fractional integral

$$I_\alpha(f,g)(x) = \int_{\mathbf{R}^n} f(x+t)\, g(x-t) |t|^{\alpha-n}\, dt\,,$$

where $0 < \alpha < n$ and f, g are functions on $\mathbf{R}^n$. For this see [10] and [17].

(7) The commutators of A. Calderón [3]: The first commutator is defined as

$$\mathcal{C}_1(f,A)(x) = \text{p.v.}\frac{1}{\pi} \int_{\mathbf{R}} f(y) \frac{A(x)-A(y)}{x-y}\, dy$$

where $A(x)$ is a function on the line; this operator is equal to $[H, M_A]$, where H is the Hilbert transform and M_A is multiplication with A. Using the fundamental theorem of calculus, $\mathcal{C}_1(f,A)$ can be viewed as a bilinear operator of f and A' (the derivative of A) and estimates can be obtained for it in terms of products of norms $\|f\|_{L^{p_1}} \|A'\|_{L^{p_2}}$. The sharpest of these estimates is of the form $L^1(\mathbf{R}) \times L^1(\mathbf{R})$ to $L^{1/2,\infty}(\mathbf{R})$ and was obtained by C. Calderón [3]. The m-th commutator is given by the expression

$$\mathcal{C}_m(f,A)(x) = \text{p.v.}\frac{1}{\pi} \int_{\mathbf{R}} f(y) \left(\frac{A(x)-A(y)}{x-y}\right)^m dy\,.$$

The operator $\mathcal{C}_m$ is not linear in A nor A' but can be multilinearized by considering

$$\widetilde{\mathcal{C}}_m(f, A_1, \dots, A_m)(x) = \text{p.v.}\frac{1}{\pi} \int_{\mathbf{R}} f(y) \frac{A_1(x)-A_1(y)}{x-y} \cdots \frac{A_m(x)-A_m(y)}{x-y}\, dy$$

and expressing each difference $A_j(x) - A_j(y)$ as $\int_y^x A_j'(s)\, ds$ via the fundamental theorem of calculus. The sharpest possible estimate for $\widetilde{\mathcal{C}}_m$ as a function of

$(f, A'_1, \dots, A'_m)$ is the estimate $L^1 \times \cdots \times L^1 \to L^{1/(m+1),\infty}$ obtained by Coifman and Meyer ([5]) when $m = 2$ and Grafakos, Duong, and Yan [8] for $m \geq 3$.

We indicate why the operator in example (4) is bounded from $L^1 \times L^1$ to $L^{1/2}$. For functions f and g supported in intervals of length one, then $B(f,g)$ is also supported in an interval of length one. Then

$$\big\|B(f,g)\big\|_{L^{1/2}} \leq \big\|B(f,g)\big\|_{L^1} \leq \tfrac{1}{2}\,\|f\|_{L^1}\|g\|_{L^1}\,,$$

where the last inequality follows by the change of variables $u = x - t$, $v = x + t$. For general functions f and g write $f_k = f\chi_{[k,k+1]}$ and analogously for g_l. Then $B(f_k, g_l)$ is supported in the interval $[(k+l)/2, (k+l)/2+1]$ and it is equal to zero unless $|k-l| \leq 1$. This reduces things to the case $l = k, k-1, k+1$. For simplicity we consider the case $k = l$. Using the sublinearity of the quantity $\|\cdot\|_{L^{1/2}}^{1/2}$ we obtain

$$\Big\|\sum_k B(f_k, g_k)\Big\|_{L^{1/2}}^{1/2} \leq \sum_k \big\|B(f_k, g_k)\big\|_{L^{1/2}}^{1/2} \leq \sum_k \|f_k\|_{L^1}^{1/2}\|g_k\|_{L^1}^{1/2} \leq \big(\|f\|_{L^1}\|g\|_{L^1}\big)^{1/2},$$

where the last inequality follows by the Cauchy-Schwarz inequality.

We end this introduction by indicating some differences between linear and multilinear operators, even in the simple case of positive kernels of convolution type. First we recall the space weak L^p, or $L^{p,\infty}$ $(0 < p < \infty)$ defined by

$$L^{p,\infty} = \Big\{f : \mathbf{R}^n \to \mathbf{C} :\ \sup_{\lambda>0} \lambda|\{|f| > \lambda\}|^{1/p} = \|f\|_{L^{p,\infty}} < \infty\Big\}\,.$$

This space is normable for $p > 1$ and p-normable for $p < 1$. It is $(1-\varepsilon)$-normable for $p = 1$.

If a linear convolution operator $f \to f * K_0$ with $K_0 \geq 0$, maps $L^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$, then it obviously maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$ and from this it follows that K_0 must be an L^1 (integrable) function. The circle of consequences trivially completes since if K_0 is integrable, then $f \to f * K_0$ maps $L^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$.

In the bilinear case (case $m = 2$ in Example (3)), one also has that if $(f,g) \to T_0(f,g)$ maps $L^1(\mathbf{R}^n) \times L^1(\mathbf{R}^n)$ to $L^{1/2}(\mathbf{R}^n)$, then T_0 maps $L^1(\mathbf{R}^n) \times L^1(\mathbf{R}^n)$ to $L^{1/2,\infty}(\mathbf{R}^n)$, and from this it also follows that the kernel K_0 must be an integrable function. However, it is not the case that for K_0 nonnegative and integrable we have that the corresponding operator T_0 maps $L^1(\mathbf{R}^n) \times L^1(\mathbf{R}^n)$ to $L^{1/2,\infty}(\mathbf{R}^n)$, see Grafakos and Soria [14]. So there are some fundamental differences between linear and bilinear operators even in the simple case of nonnegative kernels.

2. THE KATO-PONCE RULE

Let us start with the classical Leibniz rule saying that for differentiable functions f, g on $\mathbf{R}^n$ and a multiindex $\gamma = (\gamma_1, \dots, \gamma_n)$ we have

$$\partial^\gamma(fg) = \sum_{\beta_j \leq \gamma_j} \binom{\gamma_1}{\beta_1} \cdots \binom{\gamma_n}{\beta_n} (\partial^\beta f)(\partial^{\gamma-\beta} g) \tag{1}$$

where the sum is taken over all multi-indices $\beta = (\beta_1, \dots, \beta_n)$ with $\beta_j \leq \gamma_j$ for all $j = 1, 2, \dots, n$.

One defines the homogeneous fractional differentiation operator D^s, for $s>0$ by

$$D^s(f)(x)=\frac{1}{(4\pi^2)^{s/2}}(-\Delta)^{s/2}(f)(x)=\int_{\mathbf{R}^n}|\xi|^s\widehat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi\,,$$

where $\Delta=\partial_1^2+\cdots+\partial_n^2$ is the usual Laplacian and

$$\widehat{f}(\xi)=\int_{\mathbf{R}^n}f(x)e^{-2\pi i x\cdot\xi}\,dx$$

is the Fourier transform of a Schwartz function f. We also introduce the inhomogeneous fractional differentiation operator $(I-\Delta)^{t/2}$ by

$$(I-\Delta)^{t/2}f(x)=\int_{\mathbf{R}^n}\widehat{f}(\xi)(1+4\pi^2|\xi|^2)^{t/2}e^{2\pi i x\cdot\xi}d\xi\,,$$

where f is a Schwartz function, or in this case, even a tempered distribution and t is a real number. The Sobolev space norm of a tempered distribution f is defined for t real and $1\le p<\infty$ by

$$\big\|f\big\|_{L^p_t}=\big\|(I-\Delta)^{t/2}f\big\|_{L^p}\,.$$

We note that

$$\big\|f\big\|_{L^p}\le\big\|f\big\|_{L^p_t}$$

for any $t>0$ and $1\le p<\infty$. Indeed, this amounts to knowing that the inverse Fourier transform of the multiplier $(1+|\xi|^2)^{-t/2}$ lies in L^1 for all $t>0$. This is a well known fact (see for instance [9]) since this function (called the Bessel potential) is integrable near zero and is bounded by a constant multiple of $e^{-|x|/2}$ as $|x|\to\infty$.

A natural question that arises is whether there is a Leibniz rule analogous to (1) for positive numbers s. Although one may not write an easy explicit Leibniz formula in this case, for the purposes of many applications, it suffices to control a norm of $(I-\Delta)^{s/2}(fg)$ in terms of norms of f and g and their derivatives. This was achieved by Kato and Ponce [16] who proved the following fractional differentiation rule:

$$\big\|fg\big\|_{L^p_s}\le C\Big[\big\|f\big\|_{L^p_s}\big\|g\big\|_{L^\infty}+\big\|f\big\|_{L^p}\big\|(I-\Delta)^{\frac{s}{2}}g\big\|_{L^\infty}+\big\|\nabla f\big\|_{L^\infty}\big\|g\big\|_{L^p_{s-1}}\Big] \tag{2}$$

where $s>0$, $1<p<\infty$. Kato and Ponce used this estimate to obtain commutator estimates for the Bessel operator which in turn they applied to obtain estimates for the Euler and Navier-Stokes equations.

The homogeneous version of this differentiation rule was obtained by Christ and Weinstein [4] who obtained the following inequality for $0<s<1$

$$\big\|D^s(fg)\big\|_{L^r}\le C\Big[\big\|D^s(f)\big\|_{L^{p_1}}\big\|g\big\|_{L^{q_1}}+\big\|f\big\|_{L^{p_2}}\big\|D^s(g)\big\|_{L^{q_2}}\Big] \tag{3}$$

$$\frac{1}{r}=\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{p_2}+\frac{1}{q_2}\,;$$

this rule arose in connection with dispersion of solutions of the generalized Korteweg-De Vries equation. We will refer to both homogeneous and inhomogeneous versions of such inequalities as the *Kato-Ponce differentiation rule.* For related references on this rule see the works of Semmes [22], Bényi, Nahmod, and Torres [1] and Gulisashvili and Kon [15].

We will prove the Kato-Ponce differentiation rule in some cases and we will provide some extensions. Our approach is based on the theory of bilinear singular integrals.

Theorem 1. *Let s be a nonnegative even integer. Suppose that $1 \le p_1, p_2, q_1, q_2 < \infty$ be such that*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}\,.$$

When $1/2 < r < \infty$ and $1 < p_1, p_2, q_1, q_2 < \infty$, there is a constant C such that for all f, g Schwartz functions on $\mathbf{R}^n$ we have

$$\big\|D^s(fg)\big\|_{L^r} \le C\left[\big\|D^s(f)\big\|_{L^{p_1}}\big\|g\big\|_{L^{q_1}} + \big\|f\big\|_{L^{p_2}}\big\|D^s(g)\big\|_{L^{q_2}}\right]. \tag{4}$$

Moreover, when $1/2 \le r < \infty$ and $1 \le p_1, p_2, q_1, q_2 < \infty$, then

$$\big\|D^s(fg)\big\|_{L^{r,\infty}} \le C\left[\big\|D^s(f)\big\|_{L^{p_1}}\big\|g\big\|_{L^{q_1}} + \big\|f\big\|_{L^{p_2}}\big\|D^s(g)\big\|_{L^{q_2}}\right] \tag{5}$$

is valid for all f, g Schwartz functions on $\mathbf{R}^n$. By density these estimates can be extended to all functions for which the right hand side of the inequalities are finite.

Proof. We previously introduced the notion of a bilinear multiplier operator by setting

$$T_\sigma(f,g)(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \widehat{f}(\xi)\widehat{g}(\eta)\sigma(\xi,\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi d\eta$$

where $\sigma(\xi,\eta)$ is some function, called the symbol of T_σ. We now note that $D^s(fg)$ is a bilinear multiplier operator with symbol

$$\sigma(\xi,\eta) = |\xi+\eta|^s\,.$$

It will suffice to express this symbol in terms of $|\xi|^s$ and $|\eta|^s$.

We consider a smooth function ϕ that is equal to 1 on the interval $[0,1]$ and vanishing on the interval $[2,\infty)$. Then we partition $|\xi+\eta|^s$ as follows:

$$|\xi+\eta|^s = \left\{\frac{|\xi+\eta|^s}{|\xi|^s}(1-\phi)\Big(\frac{|\xi|}{|\eta|}\Big)\right\}|\xi|^s + \left\{\frac{|\xi+\eta|^s}{|\eta|^s}\phi\Big(\frac{|\xi|}{|\eta|}\Big)\right\}|\eta|^s$$

and we note that the functions inside the curly brackets are homogeneous of degree 0 and smooth away from the origin $(\xi,\eta) = (0,0)$ (obviously they are singular at the origin). Setting

$$\sigma_1(\xi,\eta) = \frac{|\xi+\eta|^s}{|\xi|^s}(1-\phi)\Big(\frac{|\xi|}{|\eta|}\Big)$$

and

$$\sigma_2(\xi,\eta) = \frac{|\xi+\eta|^s}{|\eta|^s}\phi\Big(\frac{|\xi|}{|\eta|}\Big)$$

we write

$$D^s(fg) = T_{\sigma_1}(D^s f, g) + T_{\sigma_2}(f, D^s g) \tag{6}$$

and matters reduce to the boundedness of the operators T_{σ_1} and T_{σ_2}.

We recall a bilinear multiplier theorem due to Coifman and Meyer [6], [7] (with the extension to indices $p < 1$ by Kenig-Stein [17] and Grafakos-Torres [12]):

Proposition 1. *Suppose that $\sigma(\vec{\xi})$ is a smooth function on $\mathbf{R}^{mn} \setminus \{0\}$ that satisfies*

$$|\partial^\beta \sigma(\vec{\xi})| \le C_\beta |\vec{\xi}|^{-|\beta|} \tag{7}$$

for all multiindices $|\beta| \le mn+1$ and for all $\vec{\xi} \neq 0$. Then the m-linear operator T_σ acting on functions on $\mathbf{R}^n \times \cdots \times \mathbf{R}^n$ with symbol σ is bounded from $L^{p_1}(\mathbf{R}^n) \times \cdots \times L^{p_m}(\mathbf{R}^n)$

to $L^p(\mathbf{R}^n)$ *when* $1 < p_j \le \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, $1/m < p < \infty$ *and is bounded from* $L^{p_1}(\mathbf{R}^n) \times \cdots \times L^{p_m}(\mathbf{R}^n)$ *to* $L^{p,\infty}(\mathbf{R}^n)$ *when* $1 \le p_j \le \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, $1/m \le p < \infty$.

This result was improved by Tomita [23] by weaking assumption (7) to hold only for indices $|\alpha| \le [mn/2] + 1$ when $p > 1$; see also the related work of Grafakos, Miyachi, Tomita [11], Grafakos and Si [13], and Miyachi and Tomita [20].

Now we notice that the functions σ_1 and σ_2 are homogeneous of degree zero and therefore they satisfy (7). Using Proposition 1 (with $m = 2$) and (6), we obtain the Leibniz fractional differentiation rule for the product fg.

□

Corollary 1. *Under the hypotheses of Theorem 1 have the following estimate*

$$\big\|D^s(fg)\big\|_{L^{1/2,\infty}} \le C\Big(\big\|f\big\|_{L^1} + \big\|D^s(f)\big\|_{L^1}\Big)\Big(\big\|g\big\|_{L^1} + \big\|D^s(g)\big\|_{L^1}\Big) \tag{8}$$

whenever s *is a nonnegative even integer.*

We now discuss the inhomogeneous version of the preceding theorem.

Theorem 2. *Suppose that* $1 \le p_1, p_2, q_1, q_2 < \infty$ *and* $1/2 \le r < \infty$ *be such that*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}.$$

Suppose that $s > 2n + 1$. *Then for* f, g *Schwartz functions the inequality*

$$\big\|(I-\Delta)^{s/2}(fg)\big\|_{L^r} \le C\Big[\big\|(I-\Delta)^{s/2}(f)\big\|_{L^{p_1}}\big\|g\big\|_{L^{q_1}} + \big\|f\big\|_{L^{p_2}}\big\|(I-\Delta)^{s/2}(g)\big\|_{L^{q_2}}\Big] \tag{9}$$

is valid when $1 < p_1, p_2, q_1, q_2 < \infty$. *Also the inequality*

$$\big\|(I-\Delta)^{s/2}(fg)\big\|_{L^{r,\infty}} \le C\Big[\big\|(I-\Delta)^{s/2}(f)\big\|_{L^{p_1}}\big\|g\big\|_{L^{q_1}} + \big\|f\big\|_{L^{p_2}}\big\|(I-\Delta)^{s/2}(g)\big\|_{L^{q_2}}\Big] \tag{10}$$

is valid when at least one of p_1, p_2, q_1, q_2 *is equal to* 1.

Proof. The proof proceeds as that one in the previous theorem with the only difference being that the functions σ_1 and σ_2 are defined by

$$\sigma_1(\xi,\eta) = \left(\frac{1+|\xi+\eta|^2}{1+|\xi|^2}\right)^{s/2}(1-\phi)\Big(\frac{|\xi|}{|\eta|}\Big)$$

and

$$\sigma_2(\xi,\eta) = \left(\frac{1+|\xi+\eta|^2}{1+|\eta|^2}\right)^{s/2}\phi\Big(\frac{|\xi|}{|\eta|}\Big)$$

and the key observation is that, although these functions are not homogeneous of degree zero, they still satisfy (7) for all $|\alpha| \le 2n + 1$ (since $s > 2n + 1$) and thus Proposition 1 applies. □

We denote by $L^{r,\infty}_s$ the weak Sobolev space with norm:

$$\big\|f\big\|_{L^{r,\infty}_s} = \big\|(I-\Delta)^{s/2}f\big\|_{L^{r,\infty}}.$$

Corollary 2. *Let $s > 0$. Then for functions f, g in the Schwartz class we have*

$$\|fg\|_{L^r_s} \le \|f\|_{L^p_s}\|g\|_{L^q_s}$$

whenever $1 < p, q < \infty$ and $1/r = 1/p + 1/q$. We also have

$$\|fg\|_{L^{r,\infty}_s} \le \|f\|_{L^p_s}\|g\|_{L^q_s}$$

whenever $1 \le p, q < \infty$ and $1/r = 1/p + 1/q$. These estimates extend by density to all functions in the corresponding Sobolev spaces.

Proof. The important observation is for $s > 2n+1$ and for $s = 0$, the claimed estimate is valid. The case where $0 < s \le 2n+1$ follows by complex interpolation. For purposes of interpolation, one has to reprove the endpoint cases with s replaced by $s + i\theta$ and observe that the estimates obtained are mild in θ. The details are omitted. □

The methods hereby discussed also extend to the case of m-linear operators for $m \ge 3$. Precisely, we have the following estimates:

Theorem 3. *Let $1 \le p_{j,k} < \infty$ for $1 \le j, k \le m$ be such that $1/p = 1/p_{1,i} + \cdots + 1/p_{m,i}$ for all i. Let f_j be functions in the Schwartz class.*

Suppose that s is an even integer. When $p_{j,k} > 1$ we have

$$\|D^s(f_1\cdots f_m)\|_{L^p} \le C\sum_{j=1}^m\Big[\|D^s(f_j)\|_{L^{p_{j,j}}}\prod_{i\neq j}\|f_i\|_{L^{p_{i,j}}}\Big] \tag{11}$$

and when $p_{j,k} \ge 1$ the following estimates are valid:

$$\|D^s(f_1\cdots f_m)\|_{L^{p,\infty}} \le C\sum_{j=1}^m\Big[\|D^s(f_j)\|_{L^{p_{j,j}}}\prod_{i\neq j}\|f_i\|_{L^{p_{i,j}}}\Big].$$

Now suppose that $s > 0$. Then the following estimates are valid for any i:

$$\|f_1\cdots f_m\|_{L^p_s} \le C\,\|f_1\|_{L^{p_{1,i}}_s}\cdots\|f_m\|_{L^{p_{m,i}}_s}$$

whenever and $p_{j,k} > 1$, and

$$\|f_1\cdots f_m\|_{L^{p,\infty}_s} \le C\,\|f_1\|_{L^{p_{1,i}}_s}\cdots\|f_m\|_{L^{p_{m,i}}_s}$$

whenever $p_{j,k} \ge 1$.

The proof of this theorem is obtained via a decomposition analogous to that in Theorem 2.

3. Paraproducts

For continuously differentiable functions on the real line we recall the product rule of differentiation:

$$(fg)' = f'g + fg'$$

from which we obtain the following expression for the product:

$$f(x)g(x) + c = \int_{-\infty}^x f'(t)g(t)\,dt + \int_{-\infty}^x f(t)g'(t)\,dt\,.$$

But the constant c must be zero since both functions on the left and right vanish at infinity. The expressions on the right above can be identified with "half" the product of f and g and for this reason we call them *paraproducts*.

The first paraproduct Π_1 of f and g is a bilinear operator defined as follows:

$$\Pi_1(f,g)(x) = \int_{-\infty}^{x} f'(t)g(t)\,dt\,.$$

Obviously, the perfect reconstruction of the product from the two paraproducts is as follows:

$$fg = \Pi_1(f,g) + \Pi_1(g,f)$$

and the derivative of each term on the right is equal to half the derivative of the product, i.e., $f'g$ and $g'f$ respectively.

Another type of paraproduct was introduced by A. Calderón (1965) for analytic functions F, G defined on the upper half plane $\{x+iy : x \in \mathbf{R},\ y>0\}$ that vanish at infinity. This paraproduct is defined as

$$\mathbf{\Pi}(F,G)(z) = -i\int_0^{\infty} F'(z+iy)G(z+iy)\,dy$$

for complex numbers z with positive imaginary part.

One also has the reconstruction property of Π_1, i.e.,

$$F(z)G(z) = \mathbf{\Pi}(F,G)(z) + \mathbf{\Pi}(G,F)(z)$$

since

$$\begin{aligned}
\mathbf{\Pi}(F,G)(z) + \mathbf{\Pi}(G,F)(z) &= -i\int_0^{\infty}\Big[F'(z+iy)G(z+iy) + F(z+iy)G'(z+iy)\Big]\,dy \\
&= -i\int_0^{\infty}(FG)'(z+iy)\,dy \\
&= -\int_0^{\infty}\frac{d}{dy}(FG)(z+iy)\,dy \\
&= -\big[\lim_{y\to\infty}(FG)(z+iy) - (FG)(z)\big] \\
&= (FG)(z).
\end{aligned}$$

We recall the Poisson kernel

$$P(x) = \frac{1}{\pi}\frac{1}{1+x^2}$$

and the conjugate Poisson kernel

$$Q(x) = \frac{1}{\pi}\frac{x}{1+x^2}$$

and their dilations $P_t(x) = t^{-1}P(x/t)$ and $Q_t(x) = t^{-1}Q(x/t)$. We consider the derivatives of these kernels

$$P'(x) = -\frac{1}{\pi}\frac{2x}{(1+x^2)^2}$$

and

$$Q'(x) = \frac{1}{\pi}\frac{1-x^2}{(1+x^2)^2}$$

and we note that P' and Q' are integrable functions with mean value zero.

If we consider the analytic functions $F(x+iy)=(f*P_y)(x)+i(f*Q_y)(x)$ and $G(x+iy)=(g*P_y)(x)+i(g*Q_y)(x)$ for some real-valued functions f and g, then

$$F'(x+iy)=\frac{d}{dx}F(x+iy)=\frac{d}{dx}\Big[(f*P_y)(x)+i(f*Q_y)(x)\Big]=\frac{1}{y}\Big[(f*P'_y)(x)+i(f*Q'_y)(x)\Big]$$

and $\mathbf{\Pi}(F,G)(x)$ becomes to

$$\int_0^\infty \Big[(f*P'_y)(x)+i(f*Q'_y)(x)\Big]\Big[(g*P_y)(x)+i(g*Q_y)(x)\Big]\frac{dy}{y}$$

for real numbers x. This can be written as a finite sum of integrals of the form

$$\int_0^\infty (g*P_y)(x)\,(f*\mathbb{Q}_y)(x)\frac{dy}{y}$$

and of the form

$$\int_0^\infty (g*\mathbb{Q}_y)(x)\,(f*\mathbb{Q}_y)(x)\frac{dy}{y}$$

where $\mathbb{Q}$ is one of Q,P',Q'.

Using the above, the restriction of $\mathbf{\Pi}(F,G)$ on the real line gives rise to paraproducts of the form

$$\Pi_2(f,\Psi;g,\Phi)(x)=\int_0^\infty (f*\Psi_y)(x)\,(g*\Phi_y)(x)\,\frac{dy}{y}$$

and of the form

$$\Pi_2(f,\Psi;g,\widetilde{\Psi})(x)=\int_0^\infty (f*\Psi_y)(x)\,(g*\widetilde{\Psi}_y)(x)\,\frac{dy}{y}\,,$$

where Φ, Ψ, and $\widetilde{\Psi}$ are integrable functions with mean 1, 0, and 0, respectively. For purposes of most applications, it will suffice to consider functions Φ, Ψ, and $\widetilde{\Psi}$ that are compactly supported in their spatial domains or in frequency.

In the sequel we will consistently denote by $\Psi,\widetilde{\Psi}$ functions whose Fourier transforms are compactly supported and vanish at the origin while we will denote by $\Phi,\widetilde{\Phi}$ functions whose Fourier transforms are compactly supported and are equal to one at the origin.

Finally, we have the discrete versions of Π_2 which we call Π_3

$$\Pi_3(f,\Psi;g,\Phi)=\sum_j S_j^\Phi(g)\Delta_j^\Psi(f)$$

and

$$\Pi_3(f,\Psi;g,\widetilde{\Psi})=\sum_j \Delta_j^{\widetilde{\Psi}}(g)\Delta_j^\Psi(f)\,,$$

where Δ_j^Ψ is the Littlewood-Paley operator given by convolution with $\Psi_{2^{-j}}$ and S_j^Φ is an averaging operator given by convolution with $\Phi_{2^{-j}}$.

We would like to consider boundedness properties of these paraproducts. The easiest one to study is the paraproduct $\Pi_3(f,\Psi;g,\widetilde{\Psi})$ (or analogously its continuous version).

Suppose that Ψ is an integrable $\mathcal{C}^1$ function on $\mathbf{R}^n$ with mean value zero that satisfies

$$|\Psi(x)|+|\nabla\Psi(x)|\le B(1+|x|)^{-n-1}\,. \tag{12}$$

Introducing the square function

$$\mathbf{S}^\Psi(f)=\Big(\sum_{j\in\mathbf{Z}}|\Delta_j^\Psi(f)|^2\Big)^{1/2}$$

we have that

$$\big\|\mathbf{S}^{\Psi}(f)\big\|_{L^p} \le C\big\|f\big\|_{L^p} \tag{13}$$

for $1 < p < \infty$. (The same estimate is valid for $0 < p \le 1$ provided the L^p norm on the right hand side of (13) is replaced by the H^p quasi-norm of f and Ψ is assumed to have more smoothness). We also have that

$$\Pi_3(f, \Psi; g, \widetilde{\Psi}) \le \mathbf{S}^{\Psi}(f)\,\mathbf{S}^{\widetilde{\Psi}}(g)$$

from which several estimates about Π_3 can be obtained. For example, it follows from Hölder's inequality that

$$\big\|\Pi_3(f, \Psi; g, \widetilde{\Psi})\big\|_{L^p} \le C\,\big\|f\big\|_{L^{p_1}}\big\|g\big\|_{L^{p_2}}$$

whenever $1/p_1 + 1/p_2 = 1/p$ and $p_1, p_2 > 1$, with the analogous modification when p_1 or p_2 are less than or equal to 1.

So the most difficult paraproducts are the ones of the form $\Pi_3(f, \Psi; g, \Phi)$. These can be studied easier under the assumption that Ψ and Φ are smooth functions with compactly supported frequency. Also, at this point, the one-dimensional and higher dimensional theory does not present any differences.

For purposes of exposition, we let Ψ be a smooth function with Fourier transform supported in the annulus $1/2 < |\xi| < 2$ on $\mathbf{R}^n$ and satisfying

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi) = 1\,, \qquad \xi \ne 0.$$

We let Φ be a smooth function whose Fourier transform is supported in the ball $|\xi| \le 2$ and that it is equal to 1 on the smaller ball $|\xi| \le 1/2$. The paraproduct Π_3 can be written as

$$\Pi_3(f, \Psi; g, \Phi) = \sum_j \Delta_j^{\Psi}(f) S_{j-3}^{\Phi}(g) + R(f, g)\,,$$

where $R(f, g)$ is another paraproduct of the form $\Pi_3(f, \Psi; g, \widetilde{\Psi})$ that was previously studied.

The main feature of the paraproduct operator $\Pi_3(f, \Psi; g, \Phi)$ is that it is essentially a sum of orthogonal functions in frequency. Indeed, the Fourier transform of the function $\widehat{\Delta_j^{\Psi}(f)}$ is supported in the set

$$\{\xi \in \mathbf{R}^n :\ 2^{j-1} \le |\xi| \le 2^{j+1}\}\,,$$

while the Fourier transform of the function $\widehat{S_{j-3}^{\Phi}(g)}$ is supported in the set

$$\bigcup_{k\le j-3} \{\xi \in \mathbf{R}^n :\ 2^{k-1} \le |\xi| \le 2^{k+1}\}\,.$$

This implies that the Fourier transform of the function $\Delta_j^{\Psi}(f)\, S_{j-3}^{\Phi}(g)$ is supported in the algebraic sum

$$\{\xi \in \mathbf{R}^n :\ 2^{j-1} \le |\xi| \le 2^{j+1}\} + \{\xi \in \mathbf{R}^n :\ |\xi| \le 2^{j-2}\}$$

which is contained in the set

$$\{\xi \in \mathbf{R}^n :\ 2^{j-2} \le |\xi| \le 2^{j+2}\}\,, \tag{14}$$

and the family of sets in (14) is "almost disjoint" as j varies.

We now introduce a function Θ whose Fourier transform is supported in the annulus $\{\xi \in \mathbf{R}^n : 2^{-3} \le |\xi| \le 2^3\}$ and is equal to one on the annulus $\{\xi \in \mathbf{R}^n : 2^{-2} \le |\xi| \le 2^2\}$. Then the Littlewood-Paley operator $\Delta_j^\Theta(h) = h * \Theta_{2^{-j}}$ satisfies $\Delta_j^\Psi(f)\, S_{j-3}^\Phi(g) = \Delta_j^\Theta(\Delta_j^\Psi(f)\, S_{j-3}^\Phi(g))$ and we may write

$$\Pi_3(f, \Psi; g, \Phi) = \sum_{j\in\mathbf{Z}} \Delta_j^\Theta\big(\Delta_j^\Psi(f)\, S_{j-3}^\Phi(g)\big)\,.$$

It is now possible to compute the L^p norm of $\Pi_3(f, \Psi; g, \Phi)$ for $p > 1$ by duality as the supremum of expressions of the form

$$\int_{\mathbf{R}^n} \Pi_3(f, \Psi; g, \Phi)(x)\, h(x)\, dx \tag{15}$$

over functions h with $\big\|h\big\|_{L^{p'}} \le 1$. Using that Δ_j^Θ is self-adjoint we have that the expression in (15) is equal to

$$\int_{\mathbf{R}^n} \sum_{j\in\mathbf{Z}} \Delta_j^\Theta(h)\Delta_j^\Psi(f)\, S_{j-3}^\Phi(g)\, dx$$

which is bounded by

$$\int_{\mathbf{R}^n} \mathbf{S}^\Theta(h)\mathbf{S}^\Psi(f) \sup_j S_j^\Phi(g)\, dx$$

Clearly we have

$$\sup_j S_j^\Phi(g) \le c_\Phi\, M(g),$$

where M is the Hardy-Littlewood maximal function and the required conclusion follows from the boundedness of $\mathbf{S}^\Theta$, $\mathbf{S}^\Psi$, and M on L^q for all $q > 1$.

We have now obtained the following

Proposition 2. *The operator $\Pi_3(f, \Psi; g, \Phi)$ is bounded from $L^{p_1} \times L^{p_2}$ to L^p whenever $1 < p_1, p_2, p < \infty$ and $1/p_1 + 1/p_2 = 1/p$.*

To extend this proposition to the case $p \le 1$ and $p_1, p_2 > 1$, we use the theory of Hardy spaces. First we need the following lemma.

Lemma 1. *Let Δ_k be the Littlewood-Paley operator given by $\Delta_k(g)\widehat{\ }(\xi) = \widehat{g}(\xi)\widehat{\Psi}(2^{-k}\xi)$, $k \in \mathbf{Z}$, where Ψ is a Schwartz function whose Fourier transform is supported in the annulus $\{\xi : 2^{-b} < |\xi| < 2^b\}$, for some $b \in \mathbf{Z}^+$ and satisfies $\sum_{k\in\mathbf{Z}} \widehat{\Psi}(2^{-k}\xi) = c_0$, for some constant c_0. Let $0 < p < \infty$. Then there is a constant $c = c(n, p, c_0, \Psi)$, such that for functions f in $H^p \cap L^2$, and also in $L^p \cap L^2$, we have*

$$\|f\|_{L^p} \le c \Big\| \Big(\sum_{k\in\mathbf{Z}} |\Delta_k(f)|^2 \Big)^{1/2} \Big\|_{L^p}.$$

Proof. Let Φ be a Schwartz function with integral one. Then the following quantity provides a characterization of the H^p norm:

$$\|f\|_{H^p} \approx \big\| \sup_{t>0} |f * \Phi_t| \,\big\|_{L^p}.$$

It follows that for f in $H^p \cap L^2$, which is a dense subclass of H^p, one has the estimate

$$|f| \le \sup_{t>0} |f * \Phi_t|\,,$$

since the family $\{\Phi_t\}_{t>0}$ is an approximate identity. Thus

$$\|f\|_{L^p} \leq c\,\|f\|_{H^p}$$

whenever f is an element of $H^p \cap L^2$.

Keeping this observation in mind we can write:

$$\|f\|_{L^p} \leq c\,\|f\|_{H^p} \leq \Big\|\Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2\Big)^{1/2}\Big\|_{L^p}.$$

By density this estimate is also extends to functions f in $L^p \cap L^2$. $\square$

Using this lemma, we extend Proposition 2 to the case where $p \leq 1$.

Proposition 3. *The operator $\Pi_3(f,\Psi;g,\Phi)$ is bounded from $L^{p_1}\times L^{p_2}$ to L^p whenever $p_1,p_2>1$, $p\leq 1$ and $1/p_1+1/p_2=1/p$.*

Proof. Let Θ be as in the proof of Proposition 2. We split $\Pi_3(f,\Psi;g,\Phi)$ as a sum of ten operators $\Pi_3^s(f,\Psi;g,\Phi)$, $s=0,\dots,9$, where

$$\Pi_3^s(f,\Psi;g,\Phi) = \sum_{j:\ j=10m+s} \Delta_j^\Psi(f) S_{j-3}^\Phi(g)$$

We consider for instance the case where $s=0$. Then in view of Lemma 1 we have

$$\begin{aligned}
\big\|\Pi_3^0(f,\Psi;g,\Phi)\big\|_{L^p} &\leq c'\,\Big\|\Big(\sum_{k\in\mathbf{Z}} |\Delta_k^\Theta(\Pi_3^0(f,\Psi;g,\Phi))|^2\Big)^{1/2}\Big\|_{L^p}\\
&= c'\,\Big\|\Big(\sum_{k\in\mathbf{Z}} |\Delta_k^\Theta\Big(\sum_{j:\ j=10m} \Delta_j^\Psi(f) S_{j-3}^\Phi(g)\Big)|^2\Big)^{1/2}\Big\|_{L^p}\\
&\leq c'\,\Big\|\Big(\sum_{k\in\mathbf{Z}} |\Delta_k^\Theta\Big(\sum_{\substack{|j-k|<4\\ j=10m}} \Delta_j^\Psi(f) S_{j-3}^\Phi(g)\Big)|^2\Big)^{1/2}\Big\|_{L^p}.
\end{aligned}$$

But given a k there is at most one multiple of 10, $j=j(k)$, such that $|j(k)-k|<4$. The function $\widehat{\Theta}(2^{-k}\xi)$ is equal to 1 on the support of the Fourier transform of $\Delta_{j(k)}^\Psi(f)S_{j(k)-3}^\Phi(g)$.

So matters essentially reduce to the study of the L^p boundedness of the square function

$$\Big(\sum_{k\in\mathbf{Z}} |\Delta_{j(k)}^\Psi(f) S_{j(k)-3}^\Phi(g)|^2\Big)^{1/2}$$

where $j(k)$ is a fixed one-to-one function of k. The preceding square function is bounded by

$$\mathbf{S}^\Psi(f)\,\sup_j S_j^\Phi(g) \leq c\,\mathbf{S}^\Psi(f)\,M(g)$$

and an application of Hölder's inequality yields the required assertion.

$\square$

4. Paraproducts and differentiation

In this section we discuss how one can use the previously defined paraproducts to study differentiation, and in particular, the Kato-Ponce differentiation rule. We refer the author to the work of Muscalu, Pipher, Thiele, and Tao [21] for an elegant exposition on the theory of paraproducts, and their connection with the Kato-Ponce differentiation rule.

Given a paraproduct of the form

$$\Pi_3(f,\Psi;g,\Phi)(x)=\sum_{j\in\mathbf{Z}}\Delta_j^{\Psi}(f)(x)\,S_j^{\Phi}(g)(x)\,,$$

for some functions Φ and Ψ as above, we investigate what happens when we differentiate it in the x variable with respect to the differential operator ∂^α. Here $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multiindex and the total order of differentiation is $|\alpha|=\alpha_1+\cdots+\alpha_n$.

The classical Leibniz rule gives a sum of terms, one of which is obtained when ∂^α falls on $\Delta_j^\Psi(f)$, yielding $\Delta_j^\Psi(\partial^\alpha f)$. Then there is a sum of terms of products of derivatives of both $\Delta_j^\Psi(f)$ and $S_j^\Phi(g)$. A typical term of this sort is

$$c_{\alpha,\beta}\,2^{j|\alpha|}\,\Delta_j^{\partial^\beta\Psi}(f)\,S_j^{\partial^{\alpha-\beta}\Phi}(g)\,,$$

where at least one entry of the multiindex β is strictly smaller than the corresponding entry of α. The operator $\Delta_j^{\partial^\beta\Psi}$ is a Littlewood-Paley operator given by convolution with the bump $\partial^\beta\Psi_{2^{-j}}$, while $S_j^{\partial^{\alpha-\beta}\Phi}$ is an averaging operator given by convolution with the bump $\partial^{\alpha-\beta}\Phi_{2^{-j}}$. Since differentiations $\partial^{\alpha-\beta}$ include at least one derivative, one has that both $\partial^{\alpha-\beta}\Phi$ and $\partial^\beta\Psi$ act like "Ψ" functions, i.e., they are smooth functions with compactly supported Fourier transform that vanishes at the origin (they have vanishing integral).

We need to replace $2^{j|\alpha|}\,\Delta_j^{\partial^\beta\Psi}(f)$ by another Littlewood-Paley operator involving a derivative of f. This can be easily achieved by looking at the frequency. We may write

$$2^{j|\alpha|}\,\widehat{\Delta_j^{\partial^\beta\Psi}(f)}=2^{j|\alpha|}\widehat{\partial^\beta\Psi}(2^{-j}\xi)\widehat{f}(\xi)=\frac{\widehat{\partial^\beta\Psi}(2^{-j}\xi)}{(2^{-j}|\xi|)^{|\alpha|}}\,|\xi|^\alpha\widehat{f}(\xi)\,.$$

But the function $\widehat{\psi_{\alpha,\beta}}=|\xi|^{-|\alpha|}\widehat{\partial^\beta\Psi}(\xi)$ is well defined and smooth since $\widehat{\Psi}$ vanishes in a neighborhood of the origin. It follows that

$$2^{j|\alpha|}\,\Delta_j^{\partial^\beta\Psi}(f)=\Delta_j^{\psi_{\alpha,\beta}}(D^{|\alpha|}f)$$

This discussion leads to the conclusion that

$$\partial^\alpha\Pi_3(f,\Psi;g,\Phi)=\Pi_3(\partial^\alpha f,\Psi;g,\Phi)+R(D^{|\alpha|}f,g)\,,$$

where $R(D^{|\alpha|}f,g)$ is a finite sum of paraproducts of the form $\Pi_3(D^{|\alpha|}f,\Psi;g,\widetilde{\Psi})$. We may also replace Ψ by Ψ_α, where

$$\widehat{\Psi_\alpha}(\xi)=\frac{\xi^\alpha}{|\xi|^{|\alpha|}}\widehat{\Psi}(\xi),$$

from which it follows that

$$\partial^\alpha\Pi_3(f,\Psi;g,\Phi)=\Pi_3(D^{|\alpha|}f,\Psi_\alpha;g,\Phi)+R(D^{|\alpha|}f,g)\,,$$

These observations allow us to conclude the interesting fact that differentiating a paraproduct of the form $\Pi_3(f,\Psi;g,\Phi)$ yields a sum of paraproducts in which only the first function is differentiated.

The preceding discussion yields to the proof of another version of the Kato-Ponce differentiation rule: we may write

$$fg = \Pi_3(f,\Psi;g,\Phi) + \Pi_3(g,\Psi;f,\Phi)$$

and differentiating with respect to ∂^α gives

$$\partial^\alpha(fg) = \Pi_3(D^{|\alpha|}f,\Psi_\alpha;g,\Phi) + \Pi_3(D^{|\alpha|}g,\Psi_\alpha;f,\Phi) + R(D^{|\alpha|}f,g) + R(D^{|\alpha|}g,f)\,.$$

Taking L^p norms and using Proposition 3 we obtain a bilinear version of Theorem 3 in which D^s on the left in (11) is replaced by ∂^α (where $s=|\alpha|$).

We now prove a result analogous to Theorem 2 for the homogeneous derivatives D^s.

Theorem 4. *Suppose that* $1\le p_1,p_2,q_1,q_2<\infty$ *and* $1/2\le r<\infty$ *be such that*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}\,.$$

Assume that $s>2n+1$. *Then*

$$\big\|D^s(fg)\big\|_{L^r} \le C\left[\big\|D^s(f)\big\|_{L^{p_1}}\big\|g\big\|_{L^{q_1}} + \big\|f\big\|_{L^{p_2}}\big\|D^s(g)\big\|_{L^{q_2}}\right] \tag{16}$$

is valid when $1<p_1,p_2,q_1,q_2<\infty$. *Also*

$$\big\|D^s(fg)\big\|_{L^{r,\infty}} \le C\left[\big\|D^s(f)\big\|_{L^{p_1}}\big\|g\big\|_{L^{q_1}} + \big\|f\big\|_{L^{p_2}}\big\|D^s(g)\big\|_{L^{q_2}}\right] \tag{17}$$

is valid when at least one of p_1,p_2,q_1,q_2 *is equal to* 1.

Proof. Let us work with Schwartz functions f and g. Introduce a smooth bump Ψ whose Fourier transform is supported in the annulus $6/7<|\xi|<2$ and is equal to one on $1<|\xi|<12/7$ and such that

$$\sum_j \widehat{\Psi}(2^{-j}\xi) = 1$$

for all $\xi\neq 0$. Let Δ_j be the associated Littlewood-Paley operator. Then we have

$$fg = \sum_{j,k} \Delta_j(f)\Delta_k(g)$$

and this identity holds at every point in $\mathbf{R}^n$. We introduce the operator $S_k=\sum_{j\le k}\Delta_j$ which is given by multiplication on the Fourier side by $\widehat{\Phi}(2^{-k}\xi)$.

We write fg as the sum of the following three terms:

$$\Pi_1(f,g) = \sum_{j<k-2} \Delta_j(f)\Delta_k(g)\,,$$

$$\Pi_2(f,g) = \sum_{k<j-2} \Delta_j(f)\Delta_k(g)\,,$$

$$\Pi_3(f,g) = \sum_{|j-k|\le 2} \Delta_j(f)\Delta_k(g)\,.$$

The first and the second of these terms is easy to handle by the standard technique described in Section 2, which also works for all $s>0$:

$$D^s(\Pi_1(f,g))(x) = \sum_k \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{2\pi i x\cdot(\xi+\eta)}\widehat{S_{k-3}}(f)(\xi)\widehat{\Delta_k}(g)(\eta)|\xi+\eta|^s d\xi d\eta\,,$$

which equals

$$D^s(\Pi_1(f,g))(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \widehat{f}(\xi)|\eta|^s\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\Big[\sum_k \widehat{\Phi}(2^{-k+3}\xi)\widehat{\Psi}(2^{-k}\eta)\frac{|\xi+\eta|^s}{|\eta|^s}\Big]d\xi d\eta\,.$$

The expression inside the square brackets is a bilinear Coifman-Meyer multiplier, hence boundedness holds for this term by Proposition 1. We obtain that

$$\big\|D^s(\Pi_1(f,g))\big\|_{L^r} \le C\,\big\|f\big\|_{L^{p_2}}\big\|D^s(g)\big\|_{L^{q_2}} \tag{18}$$

is valid when $1 < p_2, q_2 < \infty$. Also

$$\big\|D^s(\Pi_1(f,g))\big\|_{L^{r,\infty}} \le C\,\big\|f\big\|_{L^{p_2}}\big\|D^s(g)\big\|_{L^{q_2}} \tag{19}$$

when p_2 or q_2 equals 1. The argument for Π_2 is similar, producing estimates analogous to (18) and (19) with the roles of f and g interchanged.

Now we look at Π_3. For simplicity let us only consider the term where $j = k$, i.e.,

$$\sum_j \Delta_j(f)\Delta_j(g).$$

Then we may write

$$\begin{aligned}
D^s(\Pi_3(f,g)) &= \sum_k \Delta_k D^s\Big(\sum_j \Delta_j(f)\Delta_j(g)\Big)\\
&= \sum_k D^s\Delta_k\Big(\sum_{j\ge k-3} \Delta_j(f)\Delta_j(g)\Big)\\
&= \sum_k 2^{ks}\widetilde{\Delta}_k\Big(\sum_{j\ge k-3} \Delta_j(f)\Delta_j(g)\Big)\\
&= \sum_k 2^{ks}\widetilde{\Delta}_k\Big(\sum_{\ell\ge -3} \Delta_{\ell+k}(f)\Delta_{\ell+k}(g)\Big)\\
&= \sum_{\ell\ge -3} 2^{-\ell s}\sum_k \widetilde{\Delta}_k\Big(2^{(\ell+k)s}\Delta_{\ell+k}(f)\Delta_{\ell+k}(g)\Big)\\
&= \sum_{\ell\ge -3} 2^{-\ell s}\sum_k \widetilde{\Delta}_k\Big(\Delta_{\ell+k}(f)\widetilde{\Delta}_{\ell+k}(D^s g)\Big)\,,
\end{aligned}$$

where we set $|\xi|^s\widehat{\Psi}(\xi) = \widehat{\Theta}(\xi)$ and let $\widetilde{\Delta_k}$ be the Littlewood-Paley operator associated with Θ.

The symbol of the preceding bilinear operator is

$$\sum_{\ell\ge -3} 2^{-\ell s}\sum_k \widehat{\Theta}(2^{-k}(\xi+\eta))\widehat{\Theta}(2^{-k-\ell}(\xi))\Psi(2^{-k-\ell}(\eta))$$

Notice that each differentiation in ξ or η produces a factor of $2^\ell(|\xi|+|\eta|)^{-1}$, so to preserve convergence of the series in ℓ, one may only differentiate in ξ and η combined at most β times with $|\beta| < s$. Thus this symbol is of Coifman-Meyer type if $s > 2n+1$, in which case it satisfies (7) for all $|\beta| \le 2n+1$. It follows that Π_3 also satisfies (18) and (19). The proof is now complete. $\square$

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