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REMARKS ON WIENER AMALGAM SPACE TYPE ESTIMATES FOR SCHRÖDINGER EQUATION

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Abstract. We give a new estimate for the solution to the Schrödinger equation with potentials $V(x) = \pm|x|^2$ on the Winer amalgam spaces.

1. Introduction

We shall give a new estimate for the solution to the Schrödinger equation

$$\begin{cases} i\partial_t u + Hu = 0, & (t,x) \in \mathbf{R} \times \mathbf{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^n \end{cases} \tag{1.1}$$

in the framework of the Wiener amalgam spaces. Here $i = \sqrt{-1}$, $u(t,x)$ is a complex valued function of $(t,x) \in \mathbf{R} \times \mathbf{R}^n$, $u_0(x)$ is a complex valued function of $x \in \mathbf{R}^n$ and $\partial_t u = \partial u/\partial t$, and we shall highlight the case $H = \frac{1}{2}\Delta$ or $H = \frac{1}{2}(\Delta \pm |x|^2)$ with $\Delta u = \sum_{i=1}^n \partial^2 u/\partial x_i^2$.

There are a large number of works devoted to study the equation (1.1). Particularly, in the context of the Wiener amalgam spaces $W^{p,q}$ (or the modulation spaces $M^{p,q}$), these types of issues were initiated in the works of Bényi-Gröchenig-Okoudjou-Rogers [1], Wang-Hudzik [14] and Wang-Zhao-Guo [15], and these works have been developed by a number of authors using a large variety of methods (see, for example, Bényi-Okoudjou [2], Kobayashi-Sugimoto [10], Miyachi-Nicola-Rivetti-Tabacco-Tomita [11], Tomita [12], Wang-Huang [13], and the papers [3], [4] cited below). The precise definitions of the Wiener amalgam spaces and the modulation spaces will be given in Section 2, but the main idea of these spaces is to consider the space variable and the variable of its Fourier transform simultaneously.

Concerning the Wiener amalgam spaces, the following theorem is known.

Theorem A (Cordero-Nicola [3], [4]). *Let $2 \le p \le \infty$, $1/p + 1/p' = 1$ and $\varphi_0 \in \mathcal{S}(\mathbf{R}^n) \setminus \{0\}$.*
(i) Suppose $H = \frac{1}{2}\Delta$. Then

$$\|u(t,\cdot)\|_{W^{p,p'}_{\varphi_0}} \lesssim \left(\frac{1+|t|}{t^2}\right)^{n(\frac{1}{2}-\frac{1}{p})} \|u_0\|_{W^{p',p}_{\varphi_0}}$$

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holds for all $u_0 \in \mathcal{S}(\mathbf{R}^n)$.
(*ii*) *Suppose* $H = \frac{1}{2}(\Delta - |x|^2)$. *Then*

$$\|u(t,\cdot)\|_{W^{p,p'}_{\varphi_0}} \lesssim \frac{1}{|\sin t|^{2n(\frac{1}{2}-\frac{1}{p})}} \|u_0\|_{W^{p',p}_{\varphi_0}}$$

holds for all $u_0 \in \mathcal{S}(\mathbf{R}^n)$.
(*iii*) *Suppose* $H = \frac{1}{2}(\Delta + |x|^2)$. *Then*

$$\|u(t,\cdot)\|_{W^{p,p'}_{\varphi_0}} \lesssim \left(\frac{1+|\sinh t|}{\sinh^2 t}\right)^{2n(\frac{1}{2}-\frac{1}{p})} \|u_0\|_{W^{p',p}_{\varphi_0}}.$$

holds for all $u_0 \in \mathcal{S}(\mathbf{R}^n)$.
In (*i*), (*ii*) *and* (*iii*), $u(t,x)$ *denotes the solution of* (1.1) *with* $u(0,x) = u_0(x)$.

The main purpose of the present paper is to give a refinement of Theorem A. More precisely, we give an estimate for $u(t,x)$ with $u(0,x) = u_0(x) \in W^{p,q}_{\varphi_0}(\mathbf{R}^n)$, $1 \le p,q \le \infty$. For the purpose, we introduce the function space $W^{p,q}_{\mathcal{N}}$, which is a generalization of $M^{p,q}$ and $W^{p,q}$ (see Definition 2.2 below). The next theorem plays an essential role.

THEOREM 1.1. *Let* $\varphi_0 \in \mathcal{S}(\mathbf{R}^n) \setminus \{0\}$.
(*i*) *Suppose* $H = \frac{1}{2}\Delta$. *Then*

$$|V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)| = |V_{\varphi_0} u_0(x - t\xi, \xi)|$$

holds for $u_0 \in \mathcal{S}(\mathbf{R}^n)$.
(*ii*) *Suppose* $H = \frac{1}{2}(\Delta - |x|^2)$. *Then*

$$|V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)| = |V_{\varphi_0} u_0(x(t), \xi(t))|$$

holds for $u_0 \in \mathcal{S}(\mathbf{R}^n)$, *where*

$$x(t) = x\cos t - \xi \sin t, \quad \xi(t) = x \sin t + \xi \cos t.$$

(*iii*) *Suppose* $H = \frac{1}{2}(\Delta + |x|^2)$. *Then*

$$|V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)| = |V_{\varphi_0} u_0(x(t), \xi(t))|$$

holds for $u_0 \in \mathcal{S}(\mathbf{R}^n)$, *where*

$$x(t) = x\cosh t - \xi \sinh t, \quad \xi(t) = -x \sinh t + \xi \cosh t.$$

In (*i*), (*ii*) *and* (*iii*), $u(t,x)$ *and* $\varphi(t,x)$ *denote the solutions of* (1.1) *with* $u(0,x) = u_0(x)$ *and* $\varphi(0,x) = \varphi_0(x)$. *Moreover,* $V_{\varphi_0} u_0$ *and* $V_{\varphi(t,\cdot)}(u(t,\cdot))$ *denote the short-time Fourier transform of* u_0 *and* $u(t,\cdot)$ *with respect to the windows* φ_0 *and* $\varphi(t,\cdot)$, *respectively* (*see Section* 2.1).

We remark that Theorem 1.1 also gives the key estimate

$$\|u(t,\cdot)\|_{W^{\infty,1}_{\varphi_0}} \le C(t) \|u_0\|_{W^{1,\infty}_{\varphi_0}}$$

to prove Theorem A (cf. [8, loc. cit.]), and the results in (*i*) and (*ii*) of this theorem were already announced in Kato-Kobayashi-Ito [8], [9].

As a consequence of Theorem 1.1, we have the following estimates.

COROLLARY 1.2. *Let $1 \le p, q \le \infty$ and $\varphi_0 \in \mathcal{S}(\mathbf{R}^n) \setminus \{0\}$.*
(i) Suppose $H = \frac{1}{2}\Delta$. Then

$$\|u_0\|_{W^{p,q}_{\varphi_0}} = \|u(t,\cdot)\|_{W^{p,q}_{\varphi(t,\cdot),\mathcal{A}}}$$

holds for all $u_0 \in \mathcal{S}(\mathbf{R}^n)$, where $\mathcal{A} = \begin{pmatrix} I_n & O_n \\ t \cdot I_n & I_n \end{pmatrix} \in GL(2n, \mathbf{R})$.
(ii) Suppose $H = \frac{1}{2}(\Delta - |x|^2)$. Then

$$\|u_0\|_{W^{p,q}_{\varphi_0}} = \|u(t,\cdot)\|_{W^{p,q}_{\varphi(t,\cdot),\mathcal{B}}}$$

holds for all $u_0 \in \mathcal{S}(\mathbf{R}^n)$, where $\mathcal{B} = \begin{pmatrix} \cos t \cdot I_n & -\sin t \cdot I_n \\ \sin t \cdot I_n & \cos t \cdot I_n \end{pmatrix} \in GL(2n, \mathbf{R})$.
(iii) Suppose $H = \frac{1}{2}(\Delta + |x|^2)$. Then

$$\|u_0\|_{W^{p,q}_{\varphi_0}} = \|u(t,\cdot)\|_{W^{p,q}_{\varphi(t,\cdot),\mathcal{C}}}$$

holds for all $u_0 \in \mathcal{S}(\mathbf{R}^n)$, where $\mathcal{C} = \begin{pmatrix} \cosh t \cdot I_n & \sinh t \cdot I_n \\ \sinh t \cdot I_n & \cosh t \cdot I_n \end{pmatrix} \in GL(2n, \mathbf{R})$.
In $(i), (ii)$ and (iii), $u(t,x)$ and $\varphi(t,x)$ denote the solutions of (1.1) with $u(0,x) = u_0(x)$ and $\varphi(0,x) = \varphi_0(x)$. Here $GL(2n, \mathbf{R})$ is the group of all invertible real matrices of order $2n$. Moreover, I_n and O_n denote the $n \times n$ identity matrix and the $n \times n$ zero matrix, respectively.

We observe that the result in (ii) implies that the operator e^{itH}, defined by $u(t,x) = e^{itH}u_0(x)$, preserves the $W^{p,q}$-norm (and $M^{p,q}$-norm) if $p = q$. In fact, since $\det \mathcal{B} = 1$, we have by the change of variable that

$$\|u_0\|_{W^{p,p}_{\varphi_0}} (= \|u_0\|_{M^{p,p}_{\varphi_0}}) = \|u(t,\cdot)\|_{W^{p,p}_{\varphi(t,\cdot)}}.$$

However, this is false if $p \neq q$. Actually, we notice that

$$|V_{\varphi_0}u_0(x,\xi)| = \left|V_{\varphi(\frac{\pi}{2},\cdot)}(u(\frac{\pi}{2},\cdot))(\xi,-x)\right| \quad \text{and} \quad \|u_0\|_{W^{p,q}_{\varphi_0}} = \left\|u(\frac{\pi}{2},\cdot)\right\|_{M^{q,p}_{\varphi(\frac{\pi}{2},\cdot)}}.$$

Using this and the fact that $W^{p,q} = \mathcal{F}M^{q,p} \not\subset M^{q,p}$ for general p and q, we have the assertion.

2. PRELIMINARIES

The following notation will be used throughout this article. We write $\mathcal{S}(\mathbf{R}^n)$ to denote the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbf{R}^n$ and $\mathcal{S}'(\mathbf{R}^n)$ to denote the space of tempered distributions on $\mathbf{R}^n$, i.e., the topological dual of $\mathcal{S}(\mathbf{R}^n)$. The Fourier transform is defined by $\widehat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbf{R}^n} f(x)e^{-ix\cdot\xi}dx$ and the inverse Fourier transform by $f^\vee(x) = \mathcal{F}^{-1}f(x) = (2\pi)^{-n}\widehat{f}(-x)$. We define

$$||f||_{L^p} = \Big(\int_{\mathbf{R}^n} |f(x)|^p dx\Big)^{1/p}$$

for $1 \le p < \infty$ and $||f||_{L^\infty} = \mathrm{ess.\,sup}_{x\in\mathbf{R}^n} |f(x)|$. We use the notation $I \lesssim J$ if I is bounded by a constant times J, and we denote $I \approx J$ if $I \lesssim J$ and $J \lesssim I$.

2.1. **Short-Time Fourier Transform.** We recall the definitions of the short-time Fourier transform and its adjoint operator. Let $f \in \mathcal{S}'(\mathbf{R}^n)$ and $\phi \in \mathcal{S}(\mathbf{R}^n)$. Then the short-time Fourier transform $V_\phi f$ of f with respect to the window ϕ is defined by

$$V_\phi f(x,\xi) = \langle f(y), \phi(y-x)e^{iy\cdot\xi}\rangle = \int_{\mathbf{R}^n} f(y)\overline{\phi(y-x)}e^{-iy\cdot\xi}dy.$$

Let F be a function on $\mathbf{R}^n \times \mathbf{R}^n$ and $\phi \in \mathcal{S}(\mathbf{R}^n)$. Then the (informal) adjoint operator V_ϕ^* of V_ϕ is defined by

$$V_\phi^* F(x) = \iint_{\mathbf{R}^{2n}} F(y,\xi)\phi(x-y)e^{ix\cdot\xi}dy đ\xi$$

with $đ\xi = (2\pi)^{-n}d\xi$.

If $f, \phi \in \mathcal{S}(\mathbf{R}^n)$, then $V_\phi f \in \mathcal{S}(\mathbf{R}^{2n})$ ([7, Theorem 11.2.5]). For $f \in \mathcal{S}'(\mathbf{R}^n)$ and $\phi \in \mathcal{S}(\mathbf{R}^n)$, $V_\phi f$ is a continuous function on $\mathbf{R}^n \times \mathbf{R}^n$ and

$$|V_\phi f(x,\xi)| \le C(1+|x|+|\xi|)^N \quad \text{for all } (x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n$$

for some constant $C, N \ge 0$ ([7, Theorem 11.2.3]). Moreover, for $\phi, \psi, \gamma \in \mathcal{S}(\mathbf{R}^n)$ satisfying $\langle \psi, \phi\rangle \ne 0$ and $\langle \gamma, \psi\rangle \ne 0$, we have the inversion formula

$$\frac{1}{\langle \psi, \phi\rangle} V_\psi^* V_\phi f = f, \quad f \in \mathcal{S}'(\mathbf{R}^n)$$

([7, Corollary 11.2.7]) and the pointwise inequality

$$|V_\phi f(x,\xi)| \le \frac{C}{|\langle \gamma, \psi\rangle|}(|V_\psi f| * |V_\phi \gamma|)(x,\xi), \quad f \in \mathcal{S}'(\mathbf{R}^n), \tag{2.1}$$

for all $(x,\xi) \in \mathbf{R}^{2n}$ ([7, Lemma 11.3.3]).

2.2. **Modulation Spaces and Wiener Amalgam Spaces.** We review some basic facts concerning the modulation spaces and the Wiener amalgam spaces which will be needed in the following section.

DEFINITION 2.1 (Feichtinger [5], [6]). Let $1 \le p, q \le \infty$ and $\phi \in \mathcal{S}(\mathbf{R}^n) \setminus \{0\}$. Then the modulation space $M_\phi^{p,q}(\mathbf{R}^n) = M^{p,q}$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^n)$ such that the norm

$$\|f\|_{M_\phi^{p,q}} = \left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^n} |V_\phi f(x,\xi)|^p dx\right)^{q/p} d\xi\right)^{1/q} = \|V_\phi f(x,\xi)\|_{L_x^p L_\xi^q}$$

is finite (with usual modifications if $p = \infty$ or $q = \infty$). Furthermore, the Wiener amalgam space $W_\phi^{p,q}(\mathbf{R}^n) = W^{p,q}$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^n)$ such that the norm

$$\|f\|_{W_\phi^{p,q}} = \left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^n} |V_\phi f(x,\xi)|^q d\xi\right)^{p/q} dx\right)^{1/q} = \|V_\phi f(x,\xi)\|_{L_\xi^q L_x^p}$$

is finite (with usual modifications if $p = \infty$ or $q = \infty$).

The space $M_\phi^{p,q}(\mathbf{R}^n)$ is a Banach space, whose definition is independent of the choice of the window ϕ, i.e., $M_\phi^{p,q}(\mathbf{R}^n) = M_\psi^{p,q}(\mathbf{R}^n)$ for all $\phi, \psi \in \mathcal{S}(\mathbf{R}^n) \setminus \{0\}$ ([6, Theorem 6.1]). This property is crucial in the sequel, since we choose a suitable window ϕ to estimate the modulation space norm. If $1 \le p, q < \infty$ then $\mathcal{S}(\mathbf{R}^n)$ is dense in $M^{p,q}$ ([6, Theorem

6.1]). We also note $L^2 = M^{2,2}$, and $M^{p_1,q_1} \hookrightarrow M^{p_2,q_2}$ if $p_1 \le p_2, q_1 \le q_2$ ([6, Proposition 6.5]). Let us define by $\mathcal{M}^{p,q}(\mathbf{R}^n)$ the completion of $\mathcal{S}(\mathbf{R}^n)$ under the norm $\|\cdot\|_{M^{p,q}}$. Then $\mathcal{M}^{p,q}(\mathbf{R}^n) = M^{p,q}(\mathbf{R}^n)$ for $1 \le p, q < \infty$. Moreover, the complex interpolation theory for these spaces reads as follows: Let $0 < \theta < 1$ and $1 \le p_i, q_i \le \infty, i = 1, 2$. Set $1/p = (1-\theta)/p_1 + \theta/p_2$, $1/q = (1-\theta)/q_1 + \theta/q_2$, then $(\mathcal{M}^{p_1,q_1}, \mathcal{M}^{p_2,q_2})_{[\theta]} = \mathcal{M}^{p,q}$ ([6, Theorem 6.1], [13, Theorem 2.3]). We refer to [6] and [7] for more details.

We also remark that

$$|V_\phi f(x,\xi)| = (2\pi)^{-n} |V_{\widehat{\phi}} \widehat{f}(\xi, -x)|$$

by Parseval's formula. Thus

$$\|f\|_{W_\phi^{p,q}} = \|\widehat{f}\|_{M_{\widehat{\phi}}^{q,p}} \approx \|\widehat{f}\|_{M_\phi^{q,p}} = \|f\|_{\mathcal{F}M^{q,p}}$$

and we have $W^{p,q} = \mathcal{F}M^{q,p}$. This implies that the definition of $W^{p,q}$ is independent of the choice of the window, since the modulation space $M^{q,p}$ is independent of the choice of the window. For the same reason, $W^{p,q}$ has other properties similar to those of $M^{p,q}$.

2.3. **Function Space $W_{\phi,\mathcal{N}}^{p,q}$.** Now, we introduce the function space $W_{\mathcal{N}}^{p,q}$, which is a generalization of $M^{p,q}$ and $W^{p,q}$.

Definition 2.2. Let $1 \le p, q \le \infty$ and $\phi \in \mathcal{S}(\mathbf{R}^n) \setminus \{0\}$. For $\mathcal{N} \in GL(2n, \mathbf{R})$, we define the space $W_{\phi,\mathcal{N}}^{p,q}(\mathbf{R}^n) = W_{\mathcal{N}}^{p,q}$ by

$$W_{\phi,\mathcal{N}}^{p,q}(\mathbf{R}^n) = \left\{ f \in \mathcal{S}'(\mathbf{R}^n) \;\Big|\; \|f\|_{W_{\phi,\mathcal{N}}^{p,q}} < \infty \right\},$$

where

$$\|f\|_{W_{\phi,\mathcal{N}}^{p,q}} = \left(\int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} |V_\phi f\left((x,\xi)\mathcal{N}\right)|^q d\xi \right)^{p/q} dx \right)^{1/q}$$

(with usual modifications if $p = \infty$ or $q = \infty$).

Obviously, we have

$$W_{\phi,\mathcal{N}}^{p,q}(\mathbf{R}^n) = W_\phi^{p,q}(\mathbf{R}^n) \quad \text{with} \quad \mathcal{N} = \begin{pmatrix} I_n & O_n \\ O_n & I_n \end{pmatrix}$$

and

$$W_{\phi,\mathcal{N}}^{p,q}(\mathbf{R}^n) = M_\phi^{p,q}(\mathbf{R}^n) \quad \text{with} \quad \mathcal{N} = \begin{pmatrix} O_n & I_n \\ I_n & O_n \end{pmatrix}.$$

In addition, we have $W_{\phi,\mathcal{N}}^{p,p}(\mathbf{R}^n) = W_\phi^{p,p}(\mathbf{R}^n) = M_\phi^{p,p}(\mathbf{R}^n)$ for $1 \le p \le \infty$ and $\mathcal{N} \in GL(2n, \mathbf{R})$. We can easily prove some basic properties similar to those of $M^{p,q}$. For example, we have the following.

Lemma 2.3. *Let $1 \le p, q \le \infty$, $\phi, \psi \in \mathcal{S}(\mathbf{R}^n) \setminus \{0\}$ and $\mathcal{N} \in GL(2n, \mathbf{R})$. Then we have*

$$\|f\|_{W_{\phi,\mathcal{N}}^{p,q}} \approx \|f\|_{W_{\psi,\mathcal{N}}^{p,q}}$$

for $f \in W_{\mathcal{N}}^{p,q}(\mathbf{R}^n)$.

Proof. Applying the inequality (2.1) in Section 2.1, we have

$$\begin{aligned} |V_\phi f\left((x,\xi)\mathcal{N}\right)| &\lesssim \frac{1}{|\langle \gamma, \psi \rangle|} (|V_\psi f| * |V_\phi \gamma|)\left((x,\xi)\mathcal{N}\right) \\ &\approx \frac{1}{|\langle \gamma, \psi \rangle|} \iint_{\mathbf{R}^{2n}} |V_\psi f\left((x-y, \xi-\eta)\mathcal{N}\right)| \, |V_\phi \gamma\left((y,\eta)\mathcal{N}\right)| dy d\eta, \end{aligned}$$

where the last $\approx$ follows from the change of variable $(y,\eta) \to (y,\eta)\mathcal{N}$. Hence, we have

$$\begin{aligned}
\|f\|_{W^{p,q}_{\phi,\mathcal{N}}} &= \|V_\phi f\left((x,\xi)\mathcal{N}\right)\|_{L^q_\xi L^p_x} \\
&\lesssim \frac{1}{|\langle \gamma, \psi\rangle|} \iint_{\mathbf{R}^{2n}} \|V_\psi f\left((x-y,\xi-\eta)\mathcal{N}\right)\|_{L^q_\xi L^p_x} |V_\phi \gamma\left((y,\eta)\mathcal{N}\right)| dy d\eta \\
&= \frac{1}{|\langle \gamma, \psi\rangle|} \|f\|_{W^{p,q}_{\psi,\mathcal{N}}} \|\gamma\|_{W^{1,1}_{\phi,\mathcal{N}}}.
\end{aligned}$$

We note that $\|\gamma\|_{W^{1,1}_{\phi,\mathcal{N}}} < \infty$, since $V_\phi\gamma \in \mathcal{S}(\mathbf{R}^{2n})$ and $\det \mathcal{N} \neq 0$. Interchanging the roles of ϕ and ψ, we obtain the desired result.

□

3. Proof of Theorem 1.1

Proof. We shall prove only (iii), because we can treat (i) and (ii) in the same way as (iii).

Let $u(t,x)$ and $\varphi(t,x)$ be the solutions of (1.1) with $H = \frac{1}{2}(\Delta + |x|^2)$, $u(0,x) = u_0 \in \mathcal{S}(\mathbf{R}^n)$ and $\varphi(0,x) = \varphi_0(x) \in \mathcal{S}(\mathbf{R}^n) \setminus \{0\}$. Using integration by parts, we have

$$\begin{aligned}
&V_{\varphi(t,\cdot)}\left(\frac{1}{2}\Delta u(t,\cdot)\right)(x,\xi) \\
&= \frac{1}{2}\int_{\mathbf{R}^n} \overline{\varphi(t,y-x)} \Delta_y u(t,y) e^{-iy\cdot\xi} dy \\
&= \int_{\mathbf{R}^n} \overline{\frac{1}{2}\Delta_y \varphi(t,y-x)} u(t,y) e^{-iy\cdot\xi} dy + \int_{\mathbf{R}^n} (-i\xi\cdot\nabla_y)\overline{\varphi(t,y-x)} u(t,y) e^{-iy\cdot\xi} dy \\
&\quad - \frac{1}{2}|\xi|^2 \int_{\mathbf{R}^n} \overline{\varphi(t,y-x)} u(t,y) e^{-iy\cdot\xi} dy \\
&= V_{\frac{1}{2}\Delta\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) + \left(i\xi\cdot\nabla_x - \frac{1}{2}|\xi|^2\right) V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi).
\end{aligned}$$

Moreover, we have

$$i\partial_t V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) = V_{-i\partial_t\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) + V_{\varphi(t,\cdot)}(i\partial_t u(t,\cdot))(x,\xi)$$

and

$$\begin{aligned}
&V_{\varphi(t,\cdot)}(|\cdot|^2 u(t,\cdot))(x,\xi) \\
&= \int_{\mathbf{R}^n} \overline{\varphi(t,y-x)} |y|^2 u(t,y) e^{-iy\cdot\xi} dy \\
&= -|x|^2 \int_{\mathbf{R}^n} \overline{\varphi(t,y-x)} u(t,y) e^{-iy\cdot\xi} dy + \int_{\mathbf{R}^n} \overline{|y-x|^2\varphi(t,y-x)} u(t,y) e^{-iy\cdot\xi} dy \\
&\quad + 2\int_{\mathbf{R}^n} \overline{\varphi(t,y-x)} u(t,y)\, x\cdot y\, e^{-iy\cdot\xi} dy \\
&= -|x|^2 V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) + V_{|\cdot|^2\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) + 2ix\cdot\nabla_\xi V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi).
\end{aligned}$$

As $u(t,x)$ and $\varphi(t,x)$ satisfy (1.1), we have

$$\left(i\partial_t + i(\xi\cdot\nabla_x + x\cdot\nabla_\xi) - \frac{1}{2}(|\xi|^2 + |x|^2)\right) V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)$$

$$
\begin{aligned}
&= V_{\varphi(t,\cdot)}\left(i\partial_t u(t,\cdot) + \frac{1}{2}(\Delta + |\cdot|^2)u(t,\cdot)\right)(x,\xi) \\
&\quad - V_{[i\partial_t\varphi(t,\cdot)+\frac{1}{2}(\Delta+|\cdot|^2)\varphi(t,\cdot)]}(u(t,\cdot))(x,\xi) \\
&= 0.
\end{aligned}
$$

Hence the initial value problem (1.1) is transformed via the short-time Fourier transform with respect to the window function $\varphi(t,x)$ to

$$
\begin{cases}
\left(i\partial_t + i(\xi\cdot\nabla_x + x\cdot\nabla_\xi) - \frac{1}{2}(|\xi|^2+|x|^2)\right) V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) = 0, \\
V_{\varphi(0,\cdot)}(u(0,\cdot))(x,\xi) = V_{\varphi_0}u_0(x,\xi).
\end{cases} \tag{3.1}
$$

By the method of characteristics, we have

$$
|V_{\varphi(t,\cdot)}(u(t,\cdot))(x(t),\xi(t))| = |V_{\varphi_0}u_0(x_0,\xi_0)|,
$$

where

$$
x(t) = x_0\cosh t + \xi_0 \sinh t \quad \text{and} \quad \xi(t) = x_0 \sinh t + \xi_0\cosh t.
$$

From this, we immediately have the desired result.

□

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