RIMS Kôkyûroku Bessatsu
B33 (2012), 49–57

An endpoint Strichartz estimate in spherical coordinates

Yonggeun Cho

Department of Mathematics, and Institute of Pure and Applied Mathematics

Chonbuk National University

Jeonju 561-756, Republic of Korea

e-mail: changocho@jbnu.ac.kr

Gyeongha Hwang

Department of Mathematics, POSTECH

Pohang 790-784, Republic of Korea

e-mail: gyeonghahwang@gmail.com

Sanghyuk Lee

Department of Mathematical Sciences, Seoul National University

Seoul 151-747, Republic of Korea

e-mail: shklee@snu.ac.kr

Abstract

We study Strichartz estimates in spherical coordinates for dispersive equations which are defined by spherically symmetric pseudo-differential operators. We extend the recent results in [7, 11] to include more general class of dispersive equations. We use a bootstrapping argument based on various weighted Strichartz estimates.

Received September 28, 2011. Revised November 23, 2011.
2000 *Mathematics Subject Classification.* 35J10, 35B45
Key words and phrases. dispersive equations, Strichartz estimates in polar coordinates, weighted Strichartz estimates, bootstrapping argument.

1 Introduction

In this paper we consider the Cauchy problem of linear dispersive equations:

$$iu_t - \omega(|\nabla|)u = 0 \ \text{ in } \ \mathbb{R}^{1+n}, \quad u(0) = \varphi \ \text{ in } \ \mathbb{R}^n, \ n \geq 2 \tag{1.1}$$

where $|\nabla| = \sqrt{-\Delta}$ and the operator $\omega(|\nabla|)$ is the pseudo-differential operator of which multiplier is $\omega(|\xi|)$. We will work with $\omega \in C[0,\infty) \cap C^\infty(0,\infty)$ which satisfies the following properties:

(*i*) $\omega'(\rho) > 0$, and either $\omega''(\rho) > 0$ or $\omega''(\rho) < 0$ for all $\rho > 0$,

(*ii*) $\omega^{(k)}(\rho_1) \sim \omega^{(k)}(\rho_2)$, $k = 1, 2$ for $0 < \rho_1 < \rho_2 < 2\rho_1$,

(*iii*) $\rho|\omega^{(k+1)}(\rho)| \lesssim |\omega^{(k)}(\rho)|$ for all $k \geq 1$ and $\rho > 0$.

Typical examples of ω are $\rho^a (0 < a \neq 1)$, $\sqrt{1+\rho^2}$, $\rho\sqrt{1+\rho^2}$, and $\frac{\rho}{\sqrt{1+\rho^2}}$ which describe the Schrödinger type equations (see [12] for $a < 2$), Klein-Gordon or semirelativistic [8], iBq, and imBq equations. (For the last two see [3] and references therein.)

The solution can formally be written by

$$u(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t\omega(|\xi|))} \widehat{\varphi}(\xi)\, d\xi.$$

Here $\widehat{\varphi}$ is the Fourier transform of φ defined by $\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\,dx$. In [6] the standard Strichartz estimate in $L_t^q L_x^p$ was considered with ω satisfying (*i*), (*ii*), (*iii*) and the following was shown: if $n \geq 1$ and the pair (q,p) satisfies that $2 \leq q, p \leq \infty$, $\frac{1}{q} \leq \frac{n}{2}(\frac{1}{2} - \frac{1}{p})$ and $(q,p) \neq (2,\infty)$, then

$$\||\nabla|^s \mathcal{D}_\omega^{s_1, s_2} u\|_{L_t^q L_x^p} \lesssim \|\varphi\|_{L_x^2} \tag{1.2}$$

for $s_1 = \frac{1}{q} - s_2$, $s_2 = \frac{1}{nq}$ and $s = \frac{2}{q} - n(\frac{1}{2} - \frac{1}{p})$, where $\mathcal{D}_\omega^{s_1,s_2}$ is a pseudo-differential operator whose symbol is

$$\left(\frac{\omega'(|\xi|)}{|\xi|}\right)^{s_1} |\omega''(|\xi|)|^{s_2}.$$

In this paper we study the estimate (1.2) by making use of mixed norm spaces given in the spherical coordinates. For this purpose we use the time-space norm given by

$$\|f\|_{L_r^p L_\sigma^\ell} = \Big(\int_0^\infty \Big(\int_{S^{n-1}} |f(r\sigma)|^\ell \, d\sigma\Big)^{\frac{p}{\ell}} r^{n-1}\, dr\Big)^{\frac{1}{p}}, \ \ 1 \leq p, \ell \leq \infty.$$

For simplicity we denoted the spaces $L^p(r^{n-1}dr)$ by L_r^p. Clearly $\|f\|_{L_x^p} = \|f\|_{L_r^p L_\sigma^p}$. Then let us define several function spaces of Sobolev type. Let Δ_σ be the Laplace-Beltrami operator defined on the unit sphere and set $D_\sigma = \sqrt{1-\Delta_\sigma}$. For $|s| < n/p$, $\gamma \in \mathbb{R}$, we denote by $\dot H_r^{s,p} H_\sigma^{\gamma,\ell}$ the space

$$\Big\{ f \in \mathcal{S}' : \|f\|_{\dot H_r^{s,p} H_\sigma^{\gamma,\ell}} \equiv \| \, |\nabla|^s D_\sigma^\gamma f \|_{L_r^p L_\sigma^\ell} < \infty \Big\}.$$

It should be noted that C_0^∞ is dense in $\dot H_r^{s,p} H_\sigma^{\gamma,\ell}$ since $|s| < n/p$ (see [2]). We also use spaces equipped with the time-space norm

$$\|v\|_{L_t^q \dot H_r^{s,p} H_\sigma^{\gamma,\ell}} = \Big(\int_{\mathbb{R}} \|v(\cdot,t)\|^q_{\dot H_r^{s,p} H_\sigma^{\gamma,\ell}} \, dt \Big)^{\frac1q}, \ \ 1 \le q \le \infty.$$

If $\ell = 2$, we use a simplified notation $\dot H_r^s H_\sigma^\gamma$ for $\dot H_r^{s,2} H_\sigma^{\gamma,2}$.

It is well known that the range of p, q for (1.2) can not be extended as it can be shown by Knapp's example. There have been results [7, 15, 16] which extend the range by allowing loss of angular regularity (also see [9, 13, 14] for related results). Such results have been proven to be useful in the study of various nonlinear equations [1, 7]. Recently in [11] the authors showed that if $n \ge 2$, $\frac1q < (n-1)(\frac12 - \frac1p)$ or $(q,p) = (\infty, 2)$ for $p, q \ge 2$ and $(q,p) \ne (\infty,\infty), (2,\infty)$, then

$$\| |\nabla|^s u \|_{L_t^q L_x^p} \lesssim \|\varphi\|_{L_r^2 H_\sigma^\gamma} \tag{1.3}$$

for $\omega(\rho) = \rho^a, a > 0$, $s = \frac{a}{q} - n(\frac12 - \frac1p)$ and $\gamma \ge 1/q$. They utilized Rodnianski's argument in [15] and weighted Strichartz estimates (see [5, 7, 1]).

In this short note we show that the estimate (1.3) can be extended to include more general ω and the angular regularity can be improved (see Proposition 3.1 below). For simplicity we consider only the endpoint case $q = 2$ since the full estimate can be obtained by interpolation with the trivial estimate $\|u\|_{L_t^\infty L_x^2} \lesssim \|\varphi\|_{L_x^2}$ or the estimates in Theorem 1.7 of [11]. The novelty here is the use of bootstrapping to extend the range of (1.4).

The following is our main result.

Theorem 1.1. *Suppose that ω satisfies the conditions $(i)-(iii)$. Let $n \ge 3$, $\frac{2(n-1)}{n-2} < p < \frac{2n}{n-2}$ and $s_0 = \frac12 - \frac{n-1}{(n-2)p}$. Then for sufficiently small $\varepsilon > 0$ we have*

$$\| |\nabla|^s \mathcal{D}_\omega^{s_1,s_2} u \|_{L_t^2 L_x^p} \lesssim \|\varphi\|_{L_r^2 H_\sigma^\gamma} \tag{1.4}$$

for $s = \frac{n}{p} - \frac{n-2}{2}$, $s_1 = \frac12 - s_0 - \varepsilon$, $s_2 = s_0 + \varepsilon$ and $\gamma > \frac12 - ns_0$.

If $\omega(\rho) = \rho^a, 0 < a \ne 1$, then since $\omega'(\rho)/\rho \sim \omega''(\rho) \sim \rho^{a-2}$, we get (1.3) with γ as in Theorem 1.1. We will not pursue the optimality of angular regularity, which is another interesting issue.

For the proof of the theorem we use bootstrapping argument based on the Sobolev inequality and weighted Strichartz estimates in spherical coordinates. We will start bootstrapping from the endpoint Strichartz estimate. Once we have an endpoint estimate (1.4) for $p \neq \infty$[1], making use of Sobolev inequalities (2.1), (2.2) and (2.3), we get the estimate (1.4) for $p = p_k$, $k = 1, 2, 3, \dots$, successively. The sequence p_k decreasingly converges to $\frac{2(n-1)}{n-2}$. Regardless of ω, by this argument we can get estimate (1.4) arbitrarily close to $p = \frac{2(n-1)}{n-2}$ and thus via interpolation we also get the estimate $\||\nabla|^s \mathcal{D}_\omega^{s_1,s_2} u\|_{L_t^q L_x^p} \lesssim \|\varphi\|_{L_r^2 H_\sigma^\gamma}$ for $p, q \geq 2$ satisfying $\frac{1}{q} < (n-1)(\frac{1}{2} - \frac{1}{p})$. There is an ε-loss involved in s_1, s_2 which results from interpolations of the estimates for $p < \frac{2(n-1)}{n-2}$. But there is no loss if we impose additional condition that $\omega'(\rho) \sim \rho|\omega''(\rho)|$. If one can obtain an estimate on the critical line $L = \{(1/p, 1/q) : \frac{1}{q} = (n-1)(\frac{1}{2} - \frac{1}{p})\}$, such loss can be removed.

Finally, we make a remark on the case of the wave equation in which $\omega(\rho) = \rho$. Using the known endpoint Strichartz estimate for $n \geq 4$ ([10]) for the wave equation, one may apply the bootstrapping argument of this note to get the same result as of Sterbenz [15]. However it is not possible to remove the ε-loss in the angular regularity this way. For this it seems that one needs to obtain estimates along the critical line L.

Acknowledgements

Y. Cho and G. Hwang were supported by NRF grant No. 2010-0007550 and S. Lee was supported in part by NRF grant No. 2011-0001251 (Korea).

2 Weighted estimates

We first recall Sobolev inequality which was introduced in [4] and extended in [7]. Let $0 < a < \frac{n-1}{2}$ and $\alpha \leq \frac{n-1}{2} - a$. Then

$$\||x|^a |\nabla|^{a-\frac{n}{2}} D_\sigma^\alpha f\|_{L_r^\infty L_\sigma^2} \lesssim \|f\|_{L_x^2}, \tag{2.1}$$

In [2], the cases $f \in L_x^p$, $1 \leq p < 2$ were treated. Using complex interpolation between (2.1) and the trivial estimates $\|f\|_{L_r^\infty L_\sigma^\infty} \lesssim \|f\|_{L_r^\infty L_\sigma^\infty}$ and $\|f\|_{L_r^\infty L_\sigma^2} \lesssim \|f\|_{L_r^\infty L_\sigma^2}$, we get for

[1]This is why we work with $n \geq 3$. Actually if $n = 2$, then Strichartz estimate in spherical coordinates is worse than the standard one.

$2 \le \ell \le \infty$

$$\||x|^{\frac{2a}{\ell}}|\nabla|^{\frac{2}{\ell}(a-\frac{n}{2})}D_\sigma^{\frac{2\alpha}{\ell}} f\|_{L_r^\infty L_\sigma^\ell} \le C_0\|f\|_{L_x^\ell}, \tag{2.2}$$

$$\||x|^{\frac{2a}{\ell}}|\nabla|^{\frac{2}{\ell}(a-\frac{n}{2})}D_\sigma^{\frac{2\alpha}{\ell}} f\|_{L_r^\infty L_\sigma^2} \le C_0\|f\|_{L_r^\ell L_\sigma^2}. \tag{2.3}$$

Replacing f of (2.2) with u and applying endpoint Strichartz estimate (1.2), we obtain the following.

Lemma 2.1. *Let $n \ge 3$, $0 < a < \frac{n-1}{2}$ and $\alpha \le \frac{n-1}{2} - a$. Then*

$$\||x|^{a(1-\frac{2}{n})}|\nabla|^{(a-\frac{n}{2})(1-\frac{2}{n})}\mathcal{D}_\omega^{\frac{n-1}{2n},\frac{1}{2n}}D_\sigma^{\alpha(1-\frac{2}{n})}u\|_{L_t^2L_r^\infty L_\sigma^{\frac{2n}{n-2}}} \le C_0'\|\varphi\|_{L_x^2}. \tag{2.4}$$

Proof. From (2.2) with $\ell = \frac{2n}{n-2}$ and Hölder's inequality it follows that

$$\begin{aligned}
&\||x|^{a(1-\frac{2}{n})}|\nabla|^{(a-\frac{n}{2})(1-\frac{2}{n})}\mathcal{D}_\omega^{\frac{n-1}{2n},\frac{1}{2n}}D_\sigma^{\alpha(1-\frac{2}{n})}u\|_{L_t^2L_r^\infty L_\sigma^{\frac{2n}{n-2}}}\\
&\le C_0\|\mathcal{D}_\omega^{\frac{n-1}{2n},\frac{1}{2n}}u\|_{L_t^2L_r^{\frac{2n}{n-2}}(L_\sigma^2\cap L_\sigma^{\frac{2n}{n-2}})} \le c_nC_0\|\mathcal{D}_\omega^{\frac{n-1}{2n},\frac{1}{2n}}u\|_{L_t^2L_x^{\frac{2n}{n-2}}},
\end{aligned}$$

where c_n depends only on n. Then the estimate (1.2) with $\ell = \frac{2n}{n-2}$ gives (2.4). □

We will use the following $L_t^2L_x^2$ estimate.

Lemma 2.2. *Let $-n/2 < b < -1/2$ and $\beta \le -\frac{1}{2} - b$. Then we have*

$$\||x|^b|\nabla|^{b+1}\mathcal{D}_\omega^{\frac{1}{2},0}D_\sigma^\beta u\|_{L_t^2L_x^2} \le C_1\|\varphi\|_{L_x^2}. \tag{2.5}$$

For (2.5) we refer to [1] and also to [5, 7] for earlier versions.

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1 by showing Proposition 3.1. It will be shown via bootstrapping argument which makes use of weighted Strichartz estimates introduced in the previous section.

Proposition 3.1. *Let ω satisfy $(i)-(iii)$ and $n \ge 3$. Then, for $\frac{2(n-1)}{n-2} < p < \infty$,*

$$\||\nabla|^{-\frac{n-2}{2}+\frac{n}{p}}\mathcal{D}_\omega^{\frac{1}{2},0}D_\sigma^{\frac{n-2}{2}-\frac{n-1}{p}}u\|_{L_t^2L_r^pL_\sigma^2} \lesssim \|\varphi\|_{L_x^2}. \tag{3.1}$$

Assuming this for the moment, we prove Theorem 1.1. In fact, using (3.1) and Sobolev embedding ($H_\sigma^{\ell,(n-1)(1/2-1/p)} \hookrightarrow L^p_\sigma$) on the unit sphere, we have for $\frac{2(n-1)}{n-2} < p < \infty$

$$\||\nabla|^s \mathcal{D}_\omega^{\frac12,0} u\|_{L^2_t L^p_x} \lesssim \|\varphi\|_{L^2_r H_\sigma^{\frac12}},$$

where $s = \frac{n}{p} - \frac{n-2}{2}$. Now, from the endpoint estimate of (1.2) we have

$$\||\nabla|^s \mathcal{D}_\omega^{\frac12-\frac{1}{2n},\frac{1}{2n}} u\|_{L^2_t L_x^{\frac{2n}{n-2}}} \lesssim \|\varphi\|_{L^2_r H^0_\sigma}.$$

Interpolation between these tow estimates gives the desired result.

Now it remains to prove Proposition 3.1 .

3.1 Bootstrapping

We start with interpolating (2.4) with $\alpha = (n-1)/2 - a$ and (2.5) with $\beta = -1/2 - b$. So, we have for $2 \le p_1 \le \infty$

$$\||x|^{c_1} |\nabla|^{d_1} \mathcal{D}_\omega^{\delta_1,\delta_1'} D_\sigma^{\gamma_1} u\|_{L^2_t L^{p_1}_r (L^2_\sigma \cap L^{\ell_1}_\sigma)} \le \mathcal{C}_1 \|\varphi\|_{L^2_x}, \tag{3.2}$$

where $\mathcal{C}_1 = {C_0'}^{1-\theta_1} {C_1}^{\theta_1}$, $\theta_1 = 2/p_1$, and

$$\begin{aligned}
c_1 &= a(1-2/n)(1-2/p_1) + 2b/p_1,\\
d_1 &= c_1 - (n-2)/2 + n/p_1,\\
\delta_1 &= \frac{1}{n}(\frac{n-1}{2} + \frac{1}{p_1}), \quad \delta_1' = \frac{1}{n}(\frac12 - \frac{1}{p_1}),\\
\gamma_1 &= \frac{n^2-3n+2}{2n} - \frac{n^2-2n+2}{np_1} - c_1,\\
\frac{1}{\ell_1} &= \frac{n-2}{2n} + \frac{1}{np_1}.
\end{aligned}$$

We call it the first stage estimate and will proceed similarly by combining the resulting estimate and (2.5). At every stage we choose the indices $c_k, k \ge 1$ to be 0. In view of the range of a and b we can take $c_1 = 0$ when p_1 satisfies that $2 + \frac{2n}{(n-1)(n-2)} = p_0 < p_1 < \infty$. In particular, such p_1 gives

$$\||\nabla|^{d_1} \mathcal{D}_\omega^{\delta_1,\delta_1'} D_\sigma^{\gamma_1} u\|_{L^2_t L^{p_1}_r (L^2_\sigma \cap L^{\ell_1}_\sigma)} \le \mathcal{C}_1 \|\varphi\|_{L^2_x}. \tag{3.3}$$

We use it in the place of Strichartz estimate (1.2). Now, from the estimates (2.3) and (3.3) it follows that

$$\begin{aligned}
&\||x|^{\frac{2a}{p_1}} |\nabla|^{\frac{2}{p_1}(a-\frac{n}{2})+d_1} \mathcal{D}_\omega^{\delta_1,\delta_1'} D_\sigma^{\frac{2\alpha}{p_1}+\gamma_1} u\|_{L^2_t L^\infty_r L^2_\sigma}\\
&\le C_0 \||\nabla|^{d_1} \mathcal{D}_\omega^{\delta_1,\delta_1'} D_\sigma^{\gamma_1} u\|_{L^2_t L^{p_1}_r L^2_\sigma}\\
&\le C_0 \mathcal{C}_1 \|\varphi\|_{L^2_x},
\end{aligned}$$

where a and α are given in Lemma 2.1. Interpolating this with (2.5) for $\alpha = \frac{n-1}{2} - a$ and $\beta = -b - \frac{1}{2}$, we have the second stage estimate: for $2 \le p_2 \le \infty$

$$\| |x|^{c_2} |\nabla|^{d_2} \mathcal{D}_\omega^{\delta_2, \delta_2'} D_\sigma^{\gamma_2} u \|_{L_t^2 L_r^{p_2} L_\sigma^2} \le \mathcal{C}_2 \|\varphi\|_{L_x^2},$$

where $\mathcal{C}_2 = (C_0 \mathcal{C}_1)^{1-\theta_2} C_1^{\theta_2}$, $\theta_2 = 2/p_2$ and

$$\begin{aligned} c_2 &= \frac{2a}{p_1}(1 - \frac{2}{p_2}) + \frac{2b}{p_2} < \frac{2a}{p_0}(1 - \frac{2}{p_2}) + \frac{2b}{p_2}, \\ d_2 &= c_2 + (-\frac{n}{p_1} + d_1)(1 - \frac{2}{p_2}) + \frac{2}{p_2}, \\ \delta_2 &= \delta_1(1 - \frac{2}{p_2}) + \frac{1}{p_2}, \quad \delta_2' = \delta_1'(1 - \frac{2}{p_2}), \\ \gamma_2 &= (\gamma_1 + \frac{n-1}{p_1})(1 - \frac{2}{p_2}) - \frac{1}{p_2} - c_2. \end{aligned}$$

If $2 + \frac{p_0}{n-1} < p_2 < \infty$, then by suitable choices of a, b, we can make $c_2 = 0$. Thus $d_2 = -\frac{n-2}{2} + \frac{n}{p_2}$, $\gamma_2 > 0$ and we also have

$$\| |\nabla|^{d_2} \mathcal{D}_\omega^{\delta_2, \delta_2'} D_\sigma^{\gamma_2} u \|_{L_t^2 L_r^{p_2} L_\sigma^2} \le \mathcal{C}_2 \|\varphi\|_{L_x^2}.$$

Repeating this procedure k times, we obtain

$$\| |x|^{c_k} |\nabla|^{d_k} \mathcal{D}_\omega^{\delta_k, \delta_k'} D_\sigma^{\gamma_k} u \|_{L_t^2 L_r^{p_k} L_\sigma^2} \le \mathcal{C}_k \|\varphi\|_{L_x^2},$$

where $\mathcal{C}_k = (C_0 \mathcal{C}_{k-1})^{1-\theta_k} C_1^{\theta_k}$, $\theta_k = 2/p_k$ and

$$\begin{aligned} c_k &= \frac{2a}{p_{k-1}}(1 - \frac{2}{p_k}) + \frac{2b}{p_k}, \\ d_k &= c_k + (-\frac{n}{p_{k-1}} + d_{k-1})(1 - \frac{2}{p_k}) + \frac{2}{p_k}, \\ \delta_k &= \delta_{k-1}(1 - \frac{2}{p_k}) + \frac{1}{p_k}, \quad \delta_k' = \delta_{k-1}'(1 - \frac{2}{p_k}), \\ \gamma_k &= (\gamma_{k-1} + \frac{n-1}{p_{k-1}})(1 - \frac{2}{p_k}) - \frac{1}{p_k} - c_k. \end{aligned}$$

If p_k satisfies that $\widetilde{p}_k < p_k < \infty$, where $\widetilde{p}_k = 2 + \frac{2}{n-1} + \frac{2}{(n-1)^2} + \cdots + \frac{2}{(n-1)^{k-2}} + \frac{p_0}{(n-1)^{k-1}}$, then we can choose a, b, such that $c_k = 0$ and $d_k = -\frac{n-2}{2} + \frac{n}{p_k}, \gamma_k > 0$. Thus we have

$$\| |\nabla|^{d_k} \mathcal{D}_\omega^{\delta_k, \delta_k'} D_\sigma^{\gamma_k} u \|_{L_t^2 L_r^{p_k} L_\sigma^2} \le \mathcal{C}_k \|\varphi\|_{L_x^2}. \tag{3.4}$$

3.2 A limiting argument

We first observe that $\widetilde{p}_k$ is decreasing and $\widetilde{p}_k \to \frac{2(n-1)}{n-2} \equiv \widetilde{p}$ as $k \to \infty$. Now we fix p with $\widetilde{p} < p < p_0$ and let $k \to \infty$ to get the desired estimate (3.1).

Since $\widetilde{p} < p < p_0$, $1-\theta_k$ is bound away from 0 and 1, there exists $0 < \lambda < 1$ such that

$$\mathcal{C}_k \le C_0^{1-\theta_k+(1-\theta_k)(1-\theta_{k-1})+\cdots\prod_{2\le j\le k}(1-\theta_j)} C_0'\mathcal{C}_1 \le C_0^{\sum_{j\ge 1}\lambda^j} C_0'\mathcal{C}_1 = C_0^{\frac{\lambda}{1-\lambda}} C_0'\mathcal{C}_1.$$

If p is fixed near $\frac{2(n-1)}{n-2}$, then we can pick p_k such that $p_k = p$ for all $k \ge k_0$ for some large k_0. Since $\delta_k = \delta_{k-1}(1-2/p) + 1/p$ for $k \ge k_0$, δ_k is increasing and bounded and thus $\lim_{k\to\infty}\delta_k = \frac{1}{2}$. Also $\delta_k'/\delta_{k-1}' = (1-2/p)$ implies $\delta_k' \to 0$ as $k \to \infty$. On the other hand, γ_k is decreasing and bounded below so that $\gamma_k \to \frac{n-2}{2} - \frac{n-1}{p}$. This limit goes to 0 as $p \to \frac{2(n-1)}{n-2}$.

By the usual density argument, for the proof of (3.1) we may assume $\widehat{\varphi} \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. Since $|\nabla|^{d_k}\mathcal{D}_\omega^{\delta_k,\delta_k'}D_\sigma^{\gamma_k}u$ and $|\nabla|^{-\frac{n-2}{2}+\frac{n}{p}}\mathcal{D}_\omega^{\frac{1}{2},0}D_\sigma^{\frac{n-2}{2}-\frac{n-1}{p}}u$ are smooth and have the same compact Fourier support, it is easy to see that both are $O((|x|+1)^{-M}(|t|+1)^{-M})$ for any large M. Then it is obvious that

$$\lim_{k\to\infty}|\nabla|^{d_k}\mathcal{D}_\omega^{\delta_k,\delta_k'}D_\sigma^{\gamma_k}u = |\nabla|^{-\frac{n-2}{2}+\frac{n}{p}}\mathcal{D}_\omega^{\frac{1}{2},0}D_\sigma^{\frac{n-2}{2}-\frac{n-1}{p}}u.$$

Hence by taking limit

$$\lim_{k\to\infty}\||\nabla|^{d_k}\mathcal{D}_\omega^{\delta_k,\delta_k'}D_\sigma^{\gamma_k}u\|_{L_t^2L_r^{p_k}L_\sigma^2} = \||\nabla|^{-\frac{n-2}{2}+\frac{n}{p}}\mathcal{D}_\omega^{\frac{1}{2},0}D_\sigma^{\frac{n-2}{2}-\frac{n-1}{p}}u\|_{L_t^2L_r^pL_\sigma^2}.$$

Since $\mathcal{C}_k$ is uniform on k, from (3.4) we get (3.1) provided that p is fixed near $\widetilde{p}$. This completes the proof of Proposition 3.1.

References

[1] Y. Cho, S. Lee and T. Ozawa, On Hartree equations with derivatives, *Nonlinear Analysis* ***74*** (2011), 2094-2108.

[2] Y. Cho and K. Nakanishi, On the global existence of semirelativistic Hartree equations, *RIMS Kokyuroku Bessatsu* ***B22*** (2010), 145-166.

[3] Y. Cho and T. Ozawa, On small amplitude solutions to the generalized Boussinesq equations, *DCDS-A* ***17*** (2007), 691-711.

[4] Y. Cho and T. Ozawa, Sobolev inequalities with symmetry, *Commun. Contem. Math.* ***11*** (2009), 355-365.

[5] Y. Cho, T. Ozawa, H. Sasaki and Y. Shim, Remarks on the semirelativistic Hartree equations, *DCDS-A* ***23*** (2009), 1273-1290.

[6] Y. Cho, T. Ozawa and S. Xia, Remarks on some dispersive estimates, *Comm. Pure Appl. Anal.* ***10*** (2011), 1121 - 1128.

[7] D. Fang and C. Wang, Weighted Strichartz estimates with angular regularity and their applications, *Forum Math.* ***23*** (2011), 181-205.

[8] J. Fröhlich and E. Lenzmann, Mean-Field limit of quantum Bose gases and non-linear Hartree equation, *Semin. Equ. Deriv. Partielles, Exp. no. XIX, 26pp.* Ecole Polytechnique Palaiseau (2004).

[9] T. Hoshiro, On weighted L^2 estimates of solutions to wave equations, *J. Anal. Math.* ***72*** (1997), 127-140.

[10] M. Keel and T. Tao, Endpoint Strichartz estimates, *Amer. J. Math.* ***120*** (1998), 955-980.

[11] J.-C. Jiang, C. Wang and X. Yu, Generalized and weighted Strichartz estimates, in preprint.

[12] N. Laskin, Fractional quantum mechanics, *Phys. R. E* ***62*** (2002), 3135-3145.

[13] S. Machihara, M. Nakamura, K. Nakanishi, and T. Ozawa, Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation, *J. Funct. Anal. 219* (2005), 1-20.

[14] M. Sugimoto, A smoothing property of Schrodinger equations along the sphere, *J. Anal. Math. 89* (2003), 15-30.

[15] J. Sterbenz, Angular regularity and Strichartz estimates for the wave equation, With an appendix by Igor Rodnianski, *Int. Math. Res. Not.* ***2005*** *No.4,* 187-231.

[16] T. Tao, Spherically averaged endpoint Strichartz estimates for the two-dimensional Schrödinger equation, *Commun. Partial Differential Equations* ***25*** (2000), 1471-1485.