RIMS Kôkyûroku Bessatsu
B33 (2012), 59–85

A description of Bourgain-Pavlović's ill-posedness theorem of the Navier-Stokes equations in the critical Besov space

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Abstract

It is investigated the border in between the time-local well-posedness and the ill-posedness of the Navier-Stokes equations in the whole space with the initial data in the critical spaces. It is known by Koch-Tataru that the time-local existence theorem of mild solutions in $BMO^{-1} = \dot{F}^{-1}_{\infty,2}$. Besides, Bejenaru-Tao and Bourgain-Pavlovic proved that the equicontinuity is not equipped when the initial data belong to $\dot{B}^{-1}_{\infty,\infty} = \dot{F}^{-1}_{\infty,\infty}$. In addition, the term-wise estimates for the successive approximation of the mild solutions and its convergence or divergence are established.

1 Introduction

1.1 Problem

This note is a brief survey of the results related to [47], mainly.

We consider the Cauchy problem of the nonstationary incompressible viscous flow of the ideal fluid in the whole space $\mathbf{R}^n$ with $n \geq 2$. This is mathematically described as the Navier-Stokes equations:

$$\text{(NS)} \qquad \begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = 0 & \text{in } \mathbf{R}^n \times (0,T), \\ \nabla \cdot u = 0 & \text{in } \mathbf{R}^n \times (0,T), \\ u|_{t=0} = u_0 & \text{in } \mathbf{R}^n. \end{cases}$$

Received September 29, 2011. Revised June 2, 2012.
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This initial value problem is called (NS). We define the notations of derivatives as follows: $u_t := \partial_t u := \partial u/\partial t$, $\partial_j := \partial/\partial x_j$ for $j = 1, \dots, n$, $\nabla := (\partial_1, \dots, \partial_n)$, $\Delta := \sum_{j=1}^n \partial_j^2$. Here, for vectors $a = (a^1, \dots, a^n)$ and $b = (b^1, \dots, b^n)$, $a \cdot b$ or (a, b) denotes $\sum_{j=1}^n a^j b^j$. The velocity $u = (u^1, \dots, u^n) = (u^1(x,t), \dots, u^n(x,t))$ and the pressure $p = p(x,t)$ are unknown functions. The problem is to determine the solution (u, p) to (NS) uniquely from the given initial velocity u_0 in some function space. It is natural to impose the compatibility condition on u_0, that is, $\nabla \cdot u_0 = 0$ holds for all $x \in \mathbf{R}^n$.

The mathematical analysis of mechanics of viscous fluid has a long history. Mathematical studying of (NS) was started by Oseen [43] who established the time-local existence of a classical solution to (NS) with a regular initial datum. One of the most important results on (NS) is obtained by Leray [35, 36] in 1930's. In [35] Leray showed that for $n = 2$ there exists a unique time-global classical solution, when the initial velocity u_0 is square-integrable with $\nabla \cdot u_0 = 0$ in the distribution sense. He also constructed the time-global weak solutions for $n = 3$. See also Hopf [22] and Masuda [37]. It is a famous open problem whether one can obtain the uniqueness and smoothness of Leray's weak solutions, that is, (NS) admits a time-global unique solution in $L^2(\mathbf{R}^3)$. That is different to our aim, so we do not penetrate into its detail.

By the Duhamel principle we derive the integral equation from (NS)

$$\text{(INT)} \qquad u(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-\tau)\Delta} \mathbf{P}(u(\tau), \nabla) u(\tau) d\tau.$$

See e.g. Fujita-Kato [14, 28]. We call the solution of (INT) a **mild solution**. This derivation is understood via the following abstract equation of value in a Banach space:

$$\text{(ABS)} \qquad u' = \Delta u - \mathbf{P}(u, \nabla) u, \quad u(0) = u_0.$$

Here, we denote the heat semigroup $e^{t\Delta} := G_t *$, the Gauss kernel $G_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$, convolution with respect to spatial variables $f * g(x) := \int_{\mathbf{R}^n} f(x-z) g(z) dz$, the Helmholtz projection $\mathbf{P} := (\delta_{ij} + R_i R_j)_{i,j=1,\dots,n}$, Kronecker's delta $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$, the Riesz transform $R_i := \partial_i (-\Delta)^{-1/2} := \mathcal{F}^{-1} \frac{\sqrt{-1}\xi_i}{|\xi|} \mathcal{F}$. The Fourier transform is defined by

$$\mathcal{F} f(\xi) := \hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-\sqrt{-1} x \cdot \xi} f(x) dx,$$

and $\mathcal{F}^{-1}$ is its inverse;

$$\mathcal{F}^{-1} f(x) := \check{f}(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{\sqrt{-1} x \cdot \xi} f(\xi) d\xi.$$

If u is a mild solution, (u, p) is expected to be a classical solution, i.e. u is in C^1 in t and C^2 in x at each time $t > 0$ and location $x \in \mathbf{R}^n$. This formal equivalency between

(INT) and (NS) can be justified when u has a sufficient regularity, provided if p is defined suitably, for example,

$$p = \sum_{i,j=1}^{n} R_i R_j u^i u^j. \tag{1.1}$$

We rather discuss (INT) and mild solutions than (NS) and classical solutions. Mild solutions are usually constructed as the limit of the successive approximation (or, its subsequence)

$$u_1(t) := e^{t\Delta} u_0 \qquad \text{and} \qquad u_{j+1}(t) := u_1 - \mathcal{B}(u_j) \quad \text{for} \quad j \in \mathbf{N} \tag{1.2}$$

in $C([0,T];X)$ for $u_0 \in X$ with a Banach space X, where

$$\mathcal{B}(u,v) := \int_0^t e^{(t-\tau)\Delta} \mathbf{P}(u(\tau),\nabla) v(\tau) d\tau \qquad \text{and} \qquad \mathcal{B}(u) := \mathcal{B}(u,u). \tag{1.3}$$

In this note we discuss on the time-local solvability, time-global solvability for small data, uniqueness and ill-posedness of the Navier-Stokes equations in the whole space with initial data in critical spaces, due to the analysis of mild solutions and approximation. We will refer to the definition of function spaces and their properties, in particular, the important facts concerning with the mild solutions. This note is contributed to understand of the positive results by Koch-Tataru [30], and the negative one by the technique due to Bejenaru-Tao [3], Bourgain-Pavlovic [8] and the author of this note [47].

1.2 Motivation

We now refer to the motivation of recent works on (NS) in several function spaces. To solve (NS) uniquely and time-globally in 3-dimension, one may consider the following steps: firstly the smooth time-local solution is constructed; secondly the solution is extended uniquely and time-globally. Along this strategy, Kato-Fujita [28] introduced the notion of mild solutions, and proved that (NS) admits a unique time-local smooth solution, when $u_0 \in H^{\frac{n}{2}-1}(\mathbf{R}^n)$. They actually discussed that the approximation sequence $\{u_j\}$ uniquely converges to the mild solution in the class $C([0,T]; H^{\frac{n}{2}-1})$. Although they established this result in smooth bounded domains with non-slip boundary conditions originally, the proof can be applied to the whole space problem without any difficulty. The detail was shown in [14].

Since the results of Kato-Fujita are splendid, there are a lot of papers of the applications of their method in many directions. Some researcher wanted to eliminate the smoothness on the initial data, since the smoothness of the solutions is automatically obtained by the usual smoothing effect of solutions to equations of parabolic type. For this purpose Kato [27] (in the whole space) and Giga-Miyakawa [21] (in a bounded domain) studied the properties of the heat semigroup in the Lebesgue spaces, using $L^p - L^q$

smoothing estimates, and they proved that (NS) admits a time-local unique smooth solution in $L^n(\mathbf{R}^n)$ for all $n \geq 2$. Giga [16] also obtained the time-local existence with initial data in $L^p(\mathbf{R}^n)$ for $n \leq p < \infty$. The time-local existence for L^∞ initial data is also constructed by Cannon-Knightly [10], Cannone [11], Giga-Inui-Matsui [19] in general dimension.

We explain the scaling invariant space. For $\lambda > 0$, put

$$u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) := \lambda^2 p(\lambda x, \lambda^2 t).$$

If (u,p) is a solution to (NS), then (u_λ, p_λ) also satisfies (NS), automatically. If $(u_\lambda, p_\lambda) = (u,p)$, then that is called a **self-similar solution**. A study on the self-similar solutions plays an important role for mathematical investigation on partial differential equations. Meyer [38] proposed the notion of the **scaling invariant** spaces with respect to x as follows: we regard X as a scaling invariant space if $\|u\|_X = \|\lambda u(\lambda \cdot + a)\|_X$ for all $\lambda > 0$ and $a \in \mathbf{R}^n$. Concretely, $L^n(\mathbf{R}^n)$ is scaling invariant; in fact, one may easily check $\|u\|_{L^n_x} = \|\lambda u(\lambda \cdot + a)\|_{L^n_x}$. Once the initial velocity u_0 belongs to a scaling invariant space, and small enough with respect to the norm, there is a chance to obtain the existence of a time-global smooth unique solution. In 1984 Kato [27] figured out this fact, he showed it when $u_0 \in L^n(\mathbf{R}^n)$. So, we call this fact Kato's principle or, time-global well-posedness for small data (GWSD). This immediately implies that $u = 0$ is a stable stationary solution to (NS) in a small ball of L^n, that is to say, the local stability. We intend to say that, around 1981, Giga-Miyakawa [21, 41, 42] also noticed this fact independently of Kato. Moreover, Giga [16] and von Wahl [54] pointed out that Kato's principle is applicable whence the function space of initial data is scaling invariant. This means that one may not make sense the smallness in not scaling invariant spaces. After [21, 27], there are a lot of contribution works on Kato's principle in several scaling invariant spaces. Actually, Kato-Ponce did it in $\dot{H}_2^{\frac{n}{2}-1}$ in [29], Kozono-Yamazaki showed it in $\dot{B}^{-1+n/p}_{p,\infty}$ for $p \in (n, \infty)$ in [32], or Cannone et al. in [11, 12, 13, 44]. In addition, the weak-$L^n(\mathbf{R}^n)$ space (which is equivalent to the Lorentz space $L^{n,\infty}$) is also considered by Kozono-Yamazaki [33]. In 2001 Koch-Tataru proved it by [30] in BMO^{-1}. The function spaces which are concerned are wider and wider:

$$\dot{H}_2^{n/2-1} \subset L^n \subset \dot{B}^{-1+n/p}_{p,\infty} \subset BMO^{-1} = \dot{F}^{-1}_{\infty,2} \subset \dot{F}^{-1}_{\infty,\infty} = \dot{B}^{-1}_{\infty,\infty}$$

for $p \in (n, \infty)$. These embeddings are continuous (in norms), and $\dot{B}^{-1}_{\infty,\infty}$ is the biggest function space in scaling invariant spaces. In fact, Meyer showed that all scaling invariant space is a subspace of $\dot{B}^{-1}_{\infty,\infty}$. This implies that all self-similar solution belongs to $\dot{B}^{-1}_{\infty,\infty}$. Therefore, from view point of pure mathematical interests, many researchers tried (still try) to investigate (NS) in such function spaces. Bourgain-Pavlovic [8] finally showed the negative results in $\dot{B}^{-1}_{\infty,\infty}$, namely, Kato's principle is not applicable in $\dot{B}^{-1}_{\infty,\infty}$. This note is a contribution for better understanding of their results, and we also argue the convergence or divergence of successive approximation.

Furthermore, there are some results on the local existence of mild solutions in the subcritical spaces (not scaling invariant, for example, the Besov space $B^{-\alpha}_{p,q}$ with $\alpha < 1 - n/p$). See e.g. [46]. Besides, in the case of supercritical spaces ($\alpha > 1 - n/p$) it seems to be tough to construct mild solutions by successive approximation, in general. Nevertheless, using L^2-theory by Leray-Hopf, one can obtain the existence of time-global weak solutions when $u_0 \in L^p(\mathbf{R}^3)$ for $p \in (2, 3)$; see e.g. Calderón [9].

Before closing this section, we will refer to notation. Hereafter, we denote the numerical constants by C, which may differ to the others in lines, likely. We do not distinguish scalar valued functions and vector valued, as well as the function spaces, if no confusion occurs. We use Bourgain's notation of an equivalency $A \sim B$, which means that there is a constant C such that $C^{-1}A \leq B \leq CA$ as well as the norm equivalency $\|\cdot\|_A \sim \|\cdot\|_B$ by $C^{-1}\|f\|_A \leq \|f\|_B \leq C\|f\|_A$ for all f; we use it when we do not have interests in the constant C, particularly. We also use a notation of the almost equivalency $A \simeq B$, which means that $A = B + R$ such that $\|R\| \leq \frac{1}{3}\|B\|$ with some norm $\|\cdot\|$.

2 Function spaces

2.1 Sobolev space

Let us introduce the function spaces in this section. Let $n \in \mathbf{N}$, $s \in \mathbf{R}$ and let $1 \leq p, q \leq \infty$. The set of test functions is denoted by $\mathcal{D}$ or, $C_c^\infty(\mathbf{R}^n)$. Its topological dual stands for $\mathcal{D}'$, which is the set of distributions. The set of rapidly decreasing functions (in the sense of Schwartz) is written as $\mathcal{S}$; the set of tempered distributions is $\mathcal{S}'$. For $p \in [1, \infty]$, $L^p := L^p(\mathbf{R}^n) := \{f \in L^1_{loc}; \|f\|_p < \infty\}$ is the Lebesgue space of p-th integrable functions whose norm denotes

$$\begin{aligned} \|f\|_p :=& \ \left(\int_{\mathbf{R}^n} |f(x)|^p dx\right)^{1/p} \quad \text{if} \quad p < \infty, \\ \|f\|_\infty :=& \ \operatorname{ess\,sup}_{x \in \mathbf{R}^n} |f(x)| \quad \text{if} \quad p = \infty. \end{aligned}$$

We often omit the notation of the domain $(\mathbf{R}^n)$. Note that $\mathcal{S} \subset L^p \subset \mathcal{S}'$, and the first inclusion is dense when $p \in [1, \infty)$. So, we may define the operators ($\mathcal{F}$, $e^{t\Delta}$, R_i, $\mathbf{P}$, etc.) as a tempered distribution.

The solenoidal subspace stands for $L^p_\sigma := \{f \in L^p;\ \nabla \cdot f = 0\}$, where $\nabla \cdot f = 0$ means in the distribution sense. For $p \in (1, \infty)$ one may see

$$L^p_\sigma = \overline{C^\infty_{c,\sigma}}^{\|\cdot\|_p} := \text{closure of } \{f \in C_c^\infty;\ \nabla \cdot f = 0\} \text{ in } \|\cdot\|_p.$$

Let $m \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}$, $p \in [1, \infty]$, the Sobolev space $W^{m,p}$ denotes by

$$\begin{aligned} W^{m,p} :=& \ \Big\{f \in L^p; \|f\|_{W^{m,p}} < \infty\Big\}, \\ \|f\|_{W^{m,p}} :=& \ \textstyle\sum_{|\alpha| \leq m} \|\partial^\alpha f\|_p. \end{aligned}$$

Here $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}_0^n$ is a multi-index; $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Usually, m is called the differentiability exponent, and p is called the integrability exponent. We also use this terminology, throughout of this note. The inhomogeneous Bessel-potential space is defined by

$$H_p^s := (1-\Delta)^{-s/2} L^p := \{\mathcal{F}^{-1}(1+|\xi|^2)^{-s/2} \hat{f}\,;\, f \in L^p\}$$

with $s \in \mathbf{R}$ and $p \in [1, \infty]$. Note that $W^{m,p} = H_p^m$ for $m \in \mathbf{N}_0$.

The homogeneous Sobolev space is defined by

$$\begin{aligned} \dot{W}^{m,p} &:= \Big\{ f \in L_{loc}^p ; \|f\|_{\dot{W}^{m,p}} < \infty \Big\}, \\ \|f\|_{\dot{W}^{m,p}} &:= \textstyle\sum_{|\alpha|=m} \|\partial^\alpha f\|_p. \end{aligned}$$

We denote $\dot{H}_p^s := (-\Delta)^{-s/2} L^p := \{\mathcal{F}^{-1}|\xi|^{-s}\hat{f}\,;\, f \in L^p\}$ by the homogeneous Bessel-potential space. One can also see that $\dot{W}^{m,p} = \dot{H}_p^m$ for $m \in \mathbf{N}_0$.

Concerning to the fractional order of Sobolev space, we analogously define the Slobodeckij space $W^{s,p}$ with the norm

$$\|f\|_{W^{s,p}} := \|f\|_{W^{[s],p}} + \sum_{|\alpha|=[s]} \left(\int\!\!\int \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|^p}{|x-y|^{n+\{s\}p}} dxdy \right)^{1/p}$$

for $s \in \mathbf{R}_+ \setminus \mathbf{N}$ and $p \in (1, \infty)$. Here we have used the Gauss notation; $s = [s] + \{s\}$ and $[s] \in \mathbf{N}_0$ and $\{s\} \in (0,1)$. There are many characterization of these function spaces, in particular, using the interpolation theory; see e.g. [4, 52]. However, we omit the details.

2.2 BMO, Besov and Triebel-Lizorkin spaces

Now we consider BMO (Bounded Mean Oscillation) functions:

$$\begin{aligned} BMO &:= \{f \in L_{loc}^1 ; [[f]]_{BMO} < \infty\}, \\ [[f]]_{BMO} &:= \textstyle\sup_{Q \subset \mathbf{R}^n} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy, \\ f_Q &:= \textstyle\frac{1}{|Q|} \int_Q f(z) dz. \end{aligned}$$

Clearly, $[[\cdot]]_{BMO}$ is a seminorm, however, not a norm. In fact, $[[f]]_{BMO} = 0$ if and only if f is constant. We should note that $BMO/\mathbf{R}$ (or, $BMO/\mathbf{C}$ if we deal with complex valued functions) is a normed space, then a Banach space. Obviously, $L^\infty \subset BMO \subset \mathcal{S}'$. Notice that $[[f]]_{BMO} \leq 2\|f\|_\infty$. We now introduce the Carleson measure due to Strichartz [51] and this leads us the equivalent norms:

$$[[f]]_{BMO} \sim \sup_{x \in \mathbf{R}^n, R>0} \left(\frac{1}{|B_R(x)|} \int_{B_R(x)} \int_0^{R^2} |e^{t\Delta} f(y)|^2 \frac{dt}{t} dy \right)^{1/2} \tag{2.1}$$

for $f \in BMO/\mathbf{R}$. Here we have used $\sim$ the notation of a norm-equivalency.

To define the Besov spaces and Triebel-Lizorkin spaces we now introduce the Paley-Littlewood decomposition. Let us call $\{\phi_j\}_{j=-\infty}^{\infty}$ the Paley-Littlewood decomposition if $\hat{\phi}_0 \in C_c^\infty(\mathbf{R}^n)$, $\operatorname{supp}\hat{\phi}_0 \subset \{\xi;\ 1/2 \le |\xi| \le 2\}$, $\hat{\phi}_j(\xi) = \hat{\phi}_0(2^{-j}\xi)$ and $\sum_{j=-\infty}^{\infty}\hat{\phi}_j(\xi) = 1$ except for $\xi = 0$. Also, let us denote $\psi = \mathcal{F}^{-1}(1 - \sum_{j=1}^{\infty}\hat{\phi}_j)$, so $\{\hat{\psi}, \hat{\phi}_1, \hat{\phi}_2, \ldots\}$ is a dyadic decomposition of the unity in the phase space.

Notice that $\psi, \phi_j \in \mathcal{S}$. We can easily verify by dilation argument that

$$\|\phi_j\|_1 = \|\phi_0\|_1, \qquad j \in \mathbf{Z} \tag{2.2}$$

independently in j. Obviously, $\int \phi_j = 0$ for all $j \in \mathbf{Z}$. Also,

$$\mathcal{F}^{-1}(\hat{\phi}_j \cdot \hat{\phi}_k) = \phi_j * \phi_k = 0 \qquad \text{if} \quad |j-k| \ge 2,$$

this fact is called Bony's paraproduct lemma due to [5]. This yields that

$$\phi_j * f = \phi_j * \Big(\sum_{k=j-1}^{j+1}\phi_k\Big) * f. \tag{2.3}$$

In the same way, we have $\psi * \phi_j = 0$ for $j \ge 2$. By (2.2) and (2.3) it holds true that for $s \in \mathbf{R}$ there exists a positive constant C such that

$$\begin{aligned} \|(1-\Delta)^{s/2}\phi_j * f\|_p &\le C2^{sj}\|\phi_j * f\|_p \quad \text{for} \quad j \in \mathbf{N}, \\ \|(-\Delta)^{s/2}\phi_j * f\|_p &\le C2^{sj}\|\phi_j * f\|_p \quad \text{for} \quad j \in \mathbf{Z}, \end{aligned}$$

which is a sort of Bernstein's inequality; see e.g. [4].

Definition 2.1. *Let $s \in \mathbf{R}$, $p \in [1,\infty]$ and $q \in [1,\infty]$. An inhomogeneous Besov space is defined by*

$$\begin{aligned} B^s_{p,q} &:= \big\{f \in \mathcal{S}';\ \|f\|_{B^s_{p,q}} < \infty\big\}, \\ \|f\|_{B^s_{p,q}} &:= \Big[\|\psi * f\|_\infty + \textstyle\sum_{j=1}^{\infty} 2^{jsq}\|\phi_j * f\|_p^q\Big]^{1/q} \qquad \text{if} \quad q < \infty, \\ \|f\|_{B^s_{p,\infty}} &:= \|\psi * f\|_\infty + \sup_{1\le j\le\infty} 2^{js}\|\phi_j * f\|_p \qquad \text{if} \quad q = \infty. \end{aligned}$$

This Besov norm is understood as $\|\cdot\|_{l^q(L^p)}$ in the sense that $\{\|f_j\|_p\}_{j=0}^\infty \in l^q$, where $f \mapsto \{\psi * f, 2^s\phi_1 * f, 2^{2s}\phi_2 * f, \ldots\} =: \{f_j\}_{j=0}^\infty$. Following Johnsen [25], we call s the differentiability-exponent, p the integral-exponent and q the sum-exponent.

Definition 2.2. *An inhomogeneous Triebel-Lizorkin space is defined by*

$$\begin{aligned} F^s_{p,q} &:= \big\{f \in \mathcal{S}';\ \|f\|_{F^s_{p,q}} < \infty\big\}, \\ \|f\|_{F^s_{p,q}} &:= \Big\||\psi * f| + \big(\textstyle\sum_{j=1}^{\infty} 2^{jsq}|\phi_j * f|^q\big)^{1/q}\Big\|_p \qquad \text{if} \quad p,q < \infty, \\ \|f\|_{F^s_{p,\infty}} &:= \Big\||\psi * f| + \sup_{1\le j\le\infty} 2^{js}|\phi_j * f|\Big\|_p, \\ \|f\|_{F^s_{\infty,q}} &:= \sup_{k\in\mathbf{N}_0, x\in\mathbf{R}^n} \tfrac{1}{|B^{2^{-k}}(x)|}\int_{B_{2^{-k}}(x)} \Big(\textstyle\sum_{j\ge k} 2^{sjq}|\phi_j * f(y)|^q\Big)^{1/q} dy, \\ \|f\|_{F^s_{\infty,\infty}} &:= \sup_{k\in\mathbf{N}_0, x\in\mathbf{R}^n} \tfrac{1}{|B^{2^{-k}}(x)|}\int_{B_{2^{-k}}(x)} \sup_{j\ge k} 2^{sj}|\phi_j * f(y)|dy. \end{aligned}$$

Similarly to the Besov norm, this Triebel-Lizorkin norm is understood as $\|\cdot\|_{L^p(l^q)}$ in the sense that $\big\|\|f_j\|_{l^q}\big\|_{L^p}$.

Note. (a) $B^s_{p,q}$ and $F^s_{p,q}$ are Banach spaces. One can easily check that the Cauchy sequence converges. Clearly, $\mathcal{S}$ is a subset of $B^s_{p,q}$ and $F^s_{p,q}$ for all $s \in \mathbf{R}$ and $p, q \in [1, \infty]$; and dense if $p < \infty$ and $q < \infty$.

(b) $B^s_{p,p} = F^s_{p,p}$. Moreover, $B^s_{p,p} = F^s_{p,p} = W^{s,p}$ if $s \in \mathbf{R}_+ \setminus \mathbf{N}$.

(c) The embeddings hold from Minkowski's inequality ($l^q \subset l^r$ for $q \leq r$):

$$\begin{aligned} &B^s_{p,1} \subset B^s_{p,p} \subset H^s_p \subset B^s_{p,\infty} \quad \text{if} \quad p \leq 2, \\ &B^s_{p,1} \subset H^s_p \subset B^s_{p,p} \subset B^s_{p,\infty} \quad \text{if} \quad p \geq 2, \\ &F^s_{p,1} \subset H^s_p = F^s_{p,2} \subset F^s_{p,\infty} \quad \text{if} \quad p \in (1, \infty). \end{aligned}$$

The last one follows from the fact that $F^0_{p,2} = L^p$ (equivalent norms) and the Mikhlin-Hörmander multiplier theory.

(d) The embeddings of Sobolev type

$$B^{s_1}_{p_1,q_1} \subset B^{s_2}_{p_2,q_2} \quad \text{and} \quad F^{s_1}_{p_1,q_1} \subset F^{s_2}_{p_2,q_2}$$

hold if either "$s_1 > s_2$ and $p_1 = p_2$" or "$s_1 - n/p_1 = s_2 - n/p_2$, $s_1 > s_2$ and $p_1 < p_2$" without any restriction on the sum-exponents q_1 and q_2.

(e) The equivalency between the Besov space and the Hölder class:

$$B^s_{\infty,\infty} = C^s \quad \text{if} \quad s \in \mathbf{R}_+ \setminus \mathbf{N}$$

holds. For $s \in \mathbf{N}$ the Besov space $B^s_{\infty,\infty}$ is equivalent to the Zygmund class $\mathcal{C}^s$, which is a natural extension for all $s > 0$ of Hölder class; see e.g. the book of Triebel [52].

(f) We easily see that

$$B^0_{\infty,1} \subset BUC \subset L^\infty \subset B^0_{\infty,\infty}.$$

Here BUC stands for the space of bounded and uniformly continuous functions. Only one typographical error in the book of Triebel [52] appears in here: $B^0_{\infty,1}$ seems to be a Banach algebra with respect to the point-wise multiplication. However, that is not true. This fact was pointed out by Yamazaki; the concrete explaining is found in Runst-Sickel [45].

(g) For the cases $p \in (0, 1)$ or $q \in (0, 1)$, one can analogously define $B^s_{p,q}$ and $F^s_{p,q}$ as quasi-Banach spaces, corresponding quasi-norms; the triangle inequality does not hold, in general. We do not penetrate this situation, since we always need the triangle inequality with almost every calculation in this note, for instance, to construct mild solutions by iteration arguments.

We are now in a position to define the homogeneous Besov and Triebel-Lizorkin spaces. Let $\mathcal{Z}'$ be the topological dual space of

$$\mathcal{Z} := \Big\{ f \in \mathcal{S}\,;\, \partial^\alpha \hat{f}(0) = 0,\ \forall \alpha \in \mathbf{N}^n_0 \Big\}.$$

Definition 2.3. *For $s \in \mathbf{R}$, $1 \le p \le \infty$ and $1 \le q \le \infty$, we define the homogeneous Besov space by (see [4, 45, 52, 53]):*

$$\begin{aligned} \dot{B}^s_{p,q} &:= \Big\{ f \in \mathcal{Z}' ; \|f\|_{\dot{B}^s_{p,q}} < \infty \Big\}, \\ \|f\|_{\dot{B}^s_{p,q}} &:= \Big[\textstyle\sum_{j=-\infty}^{\infty} 2^{jsq} \|\phi_j * f\|_p^q \Big]^{1/q} \quad \text{if} \quad q < \infty, \\ \|f\|_{\dot{B}^s_{p,\infty}} &:= \sup_{-\infty \le j \le \infty} 2^{js} \|\phi_j * f\|_p \quad \text{if} \quad q = \infty. \end{aligned}$$

Also, we define the homogeneous Triebel-Lizorkin space by

$$\begin{aligned} \dot{F}^s_{p,q} &:= \Big\{ f \in \mathcal{Z}' ; \|f\|_{\dot{F}^s_{p,q}} < \infty \Big\}, \\ \|f\|_{\dot{F}^s_{p,q}} &:= \Big\| \Big[\textstyle\sum_{j=-\infty}^{\infty} 2^{jsq} |\phi_j * f|^q \Big]^{1/q} \Big\|_p \quad \text{if} \quad p, q < \infty, \end{aligned}$$

and define it for the cases $p = \infty$ or $q = \infty$ by the same modification of inhomogeneous Triebel-Lizorkin spaces.

Note. (h) By the definition of ϕ_j it is clear that $\|f\|_{\dot{B}^s_{p,q}} = 0$ if $f \in \mathcal{P} := \{\text{polynomials}\}$. Thus, $\|\cdot\|_{\dot{B}^s_{p,q}}$ and $\|\cdot\|_{\dot{F}^s_{p,q}}$ are seminorms. The quotient spaces divided by polynomials $\dot{B}^s_{p,q}/\mathcal{P}$ and $\dot{F}^s_{p,q}/\mathcal{P}$ are Banach spaces.
(i) Clearly, $\mathcal{Z}$ is a subset of $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$, and dense if $p, q < \infty$.
(j) $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ are subsets of $\mathcal{S}'$ if the exponents satisfy

$$\text{either } \text{“}s < n/p\text{”} \quad \text{or} \quad \text{“}s = n/p \text{ and } q = 1\text{”}. \tag{2.4}$$

Under this conditions, the operators $\mathcal{F}$, $e^{t\Delta}$, $\mathbf{P}$, R_i can be defined on the homogeneous spaces as the tempered distribution sense. Also, it is natural to select the representative element such that

$$f = \sum_{j=-\infty}^{\infty} \phi_j * f \quad \text{in} \quad \mathcal{S}'. \tag{2.5}$$

See the details in Bourdaud [6] or Kozono-Yamazaki [32]. Throughout of this note, we basically treat the homogeneous space under the exponents satisfying (2.4) only.
(k) The following equivalencies are known:

$$\mathcal{H}^1 = \dot{F}^0_{1,2} \quad \text{and} \quad BMO = \dot{F}^0_{\infty,2},$$

which are equivalent norms. Here $\mathcal{H}^1 := \{f \in L^1 ; R_i f \in L^1, {}^\forall i = 1, \dots, n\}$ is the Hardy space. It holds true that $\dot{B}^s_{p,p} = \dot{F}^s_{p,p}$. Also, the homogeneous versions of the embeddings as the same to (c) and (d) hold.
(l) We are mainly interested in the case $p = \infty$, and following continuous embeddings are easily seen:

$$\dot{B}^0_{\infty,1} \subset BUC \subset L^\infty \subset BMO \subset \dot{B}^0_{\infty,\infty}.$$

Typically, thanks to (2.5), we get

$$\|f\|_\infty = \|\sum_{j=-\infty}^{\infty} \phi_j * f\|_\infty \le \sum_{j=-\infty}^{\infty} \|\phi_j * f\|_\infty = \|f\|_{\dot{B}^0_{\infty,1}}.$$

(m) By dilation for any integer j there exists a positive constant C_0 (independent of k and j) such that $\|R_k\phi_j\|_1 \le C_0$. Hence, we see that the Riesz transform is bounded in the homogeneous spaces as subspaces of $\mathcal{S}'$ when the exponents satisfy (2.4).

In this note we mainly deal with the case $p=\infty$. Define $\dot{\mathcal{B}}^{-1}_{\infty,\infty}$ by

$$\begin{aligned} \dot{\mathcal{B}}^{-1}_{\infty,\infty} = & \left\{ f \in \mathcal{S}'; \|f\|_{\dot{\mathcal{B}}^{-1}_{\infty,\infty}} < \infty \right\}, \\ \|f\|_{\dot{\mathcal{B}}^{-1}_{\infty,\infty}} := & \; \|\!|f|\!\| := \sup_{\rho>0} \sqrt{\rho}\|e^{\rho\Delta} f\|_\infty. \end{aligned}$$

The equivalency $\dot{\mathcal{B}}^{-1}_{\infty,\infty} = \dot{B}^{-1}_{\infty,\infty}/\mathbf{R}$ and $\|\cdot\|_{\dot{\mathcal{B}}^{-1}_{\infty,\infty}} \sim \|\cdot\|_{\dot{B}^{-1}_{\infty,\infty}}$ hold true except for constant functions; see e.g. [1, 34]. Indeed, for a non-zero constant function $f_c \equiv c \in \mathbf{R}^n \setminus \{0\}$ we see that

$$\sup_{\rho>0} \sqrt{\rho}\|e^{\rho\Delta} f_c\|_\infty = \sup_{\rho>0} \sqrt{\rho}|c| = \infty \ne \sup_{j\in\mathbf{Z}} 2^{-j}\|\phi_j * f_c\|_\infty = 0.$$

The reader should notice that the non-zero f_c do not satisfy (2.5).

3 Local well-posedness in $\dot{f}^{-1}_{\infty,2}$

3.1 Well-posedness in the sense of Hadamard

In this section we explain the results of Koch-Tataru [30], briefly. They constructed time-local unique mild solutions with initial data in vmo^{-1}, and mild solutions can be extended time-globally if BMO^{-1}-norm of the initial velocity is small sufficiently. Before stating their results, we now recall the notion of well-posedness in the sense of Hadamard.

Definition 3.1. *We say that the Cauchy problem is* (WP) *well-posed in X if the following three conditions are satisfied:*

(i) *A solution exist.*

(ii) *The solution is unique.*

(iii) *The solution equips the equicontinuity.*

The property (iii) means that the solution depends on the initial data continuously in some reasonable topology e.g. $C([0,\infty);X)$, that is, for all $t>0$ and $\varepsilon>0$ there exists $\delta>0$ such that $\|u_0-\tilde{u}_0\|_X<\delta$ then $\|u(t)-\tilde{u}(t)\|_X<\varepsilon$. Here $u(t)$ and $\tilde{u}(t)$ are solutions at time t with initial data u_0 and $\tilde{u}_0$, respectively. If we only get the time-local existence of unique solution, replacing ∞ by T for some $T\in(0,\infty)$ and $t\in(0,T)$ at (i) and (iii), then it is called (TLWP) time-local well-posed. For the case one can obtain the well-posedness if the initial data are small enough, it is called (GWSD) time-global well-posedness for small data. From view point of the dynamical system, (GWSD) implies the local stability of the trivial solution $u = 0$. We call (IP) ill-posed if one of (i) $-$ (iii) is failed. This usual terminology is used throughout this note.

Leray [35] showed that (NS) is (WP) in $L^2_\sigma(\mathbf{R}^2)$. The famous problem is to show whether (NS) is (WP) in $L^2_\sigma(\mathbf{R}^3)$, or not. Kato [27], Giga-Miyakawa [21] proved that (NS) is (TLWP) and (GWSD) in $L^n_\sigma(\mathbf{R}^n)$.

We will discuss well-posedness of (NS) in the Besov or Triebel-Lizorkin spaces closed and related to L^∞, due to the mild solutions. We now focus into the continuity of solutions in time at the initial time. Dealing with L^∞-initial data, we have to take care about the following fact:

Lemma 3.2. *Let $f \in L^\infty$. Then $e^{t\Delta} f \to f$ in L^∞ as $t \to 0$ if and only if $f \in BUC$.*

In other words, $e^{t\Delta}$ is strongly continuous in BUC, but not in L^∞. Or, $e^{t\Delta}$ is (C_0)-semigroup in BUC. Concerning the Heavyside function, $h(x) = 1$ for $x \geq 0$ and $h(x) = 0$ for $x < 0$, it is easy to see that $\|e^{t\Delta}h - h\|_\infty = \frac{1}{2}$ for all $t > 0$. The proof of this lemma is found in e.g. [19].

Recall the integral equation (INT). It is clear that the second terms of right-hand-side vanish as $t \to 0$ whence it is integrable. So, in order to get the continuity of solutions in time up to initial time, it is naturally required the restriction on $u_0 \in X$ satisfying that $e^{t\Delta}$ is strongly continuous at $t = 0$ in X. For this purpose we now introduce the little Besov space and little Triebel-Lizorkin space.

Definition 3.3. *Let $s \in \mathbf{R}$, $1 \leq p, q \leq \infty$. Subspaces of $B^s_{p,q}$ and $F^s_{p,q}$ are defined by*

$$\begin{aligned} b^s_{p,q} &:= \left\{ g \in B^s_{p,q};\, e^{t\Delta} g \to g \text{ in } B^s_{p,q} \text{ as } t \to 0 \right\}, \\ f^s_{p,q} &:= \left\{ g \in F^s_{p,q};\, e^{t\Delta} g \to g \text{ in } F^s_{p,q} \text{ as } t \to 0 \right\}. \end{aligned}$$

Assume, in addition, that exponents satisfy (2.4)*, the homogeneous version is defined by*

$$\begin{aligned} \dot{b}^s_{p,q} &:= \left\{ g \in \dot{B}^s_{p,q};\, g = \textstyle\sum_{j=-\infty}^{\infty} \phi_j * g \text{ in } \mathcal{S}',\, e^{t\Delta} g \to g \text{ in } \dot{B}^s_{p,q} \text{ as } t \to 0 \right\}, \\ \dot{f}^s_{p,q} &:= \left\{ g \in \dot{F}^s_{p,q};\, g = \textstyle\sum_{j=-\infty}^{\infty} \phi_j * g \text{ in } \mathcal{S}',\, e^{t\Delta} g \to g \text{ in } \dot{F}^s_{p,q} \text{ as } t \to 0 \right\}. \end{aligned}$$

They are closed subspace of usual Besov or Triebel-Lizorkin spaces, so Banach spaces. It is easy to check that

$$\overline{C^\infty_c}^{\|\cdot\|_{B^s_{p,q}}} \subset b^s_{p,q} = \overline{B^{s+1}_{p,q}}^{\|\cdot\|_{B^s_{p,q}}} \subset B^s_{p,q}.$$

Also, one may see that $b^s_{p,q} = B^s_{p,q}$ if and only if $q < \infty$. See more details of little Besov spaces in [2, 46].

Next, we refer to the function spaces which are used by Koch-Tataru [30]. Let BMO^{-1} be

$$\begin{aligned} BMO^{-1} :=& \ \Big\{f \in \mathcal{S}' \,;\, \|f\|_{BMO^{-1}} < \infty\Big\}, \\ \|f\|_{BMO^{-1}} :=& \ \sup_{x\in\mathbf{R}^n, R>0} \left(\tfrac{1}{|B_R(x)|}\int_0^{R^2}\int_{B_R(x)} |e^{t\Delta}f(y)|^2 dydt\right)^{1/2}, \end{aligned}$$

where $B_R(x)$ is an open ball radius $R > 0$ centered at $x \in \mathbf{R}^n$. One may see that BMO^{-1} is equivalent to the set of first derivatives of BMO functions, and also they coincide the specific homogeneous Triebel-Lizorkin space:

$$BMO^{-1} = \partial BMO = \dot{F}^{-1}_{\infty,2}.$$

Recall (2.1). The reader may find the details of basic properties of BMO or $\dot{F}^s_{p,q}$ in e.g. [30, 45, 50, 51, 56].

One can see that the several interesting functions belong to BMO^{-1} (and then $\dot{B}^{-1}_{\infty,\infty}$), for example, the trigonometric functions e.g. $[x \mapsto \sin x]$ which are not decaying at space infinity, $[x \mapsto \sin x + \sin(\sqrt{2}x)]$ is an almost periodic function, $[x \mapsto e^x \sin(e^x)]$ is a growing and oscillating function, $[x \mapsto p.v.\frac{1}{x}]$ has a singularity.

For $T \in (0, \infty]$ we denote the norm of BMO_T^{-1} by

$$\|f\|_{BMO_T^{-1}} := \sup_{x\in\mathbf{R}^n, R\in(0,\sqrt{T})} \left(\frac{1}{|B_R(x)|}\int_0^{R^2}\int_{B_R(x)} |e^{t\Delta}f(y)|^2 dydt\right)^{1/2}.$$

Let us now define the bmo^{-1} and vmo^{-1}.

$$\begin{aligned} bmo^{-1} :=& \ \{f \in \mathcal{S}';\, \|f\|_{bmo^{-1}} := \|f\|_{BMO_1^{-1}} < \infty\} = F^{-1}_{\infty,2} \supset BMO^{-1}, \\ vmo^{-1} :=& \ \{f \in bmo^{-1};\, \lim_{T\to 0} \|f\|_{BMO_T^{-1}} = 0\} \\ =& \ \{f \in bmo^{-1};\, \lim_{t\to 0} \|e^{t\Delta}f - f\|_{bmo^{-1}} = 0\} = f^{-1}_{\infty,2}. \end{aligned}$$

Here vmo is the localized version of VMO the space of vanishing mean oscillation functions. In the book of Stein [50] VMO functions are required the vanishing in both $\lim_{T\to 0}\|f\|_{BMO_T} = \lim_{T\to\infty}\|f\|_{BMO_T} = 0$, which is slightly different to above.

Let $T \in (0, \infty]$, the function v of x and t we define $\mathcal{E}_T$-norm by

$$\begin{aligned} \|v\|_{\mathcal{E}_T} :=& \ \sup_{0<t<T} \sqrt{t}\|v(t)\|_\infty \\ & + \sup_{x\in\mathbf{R}^n, R\in(0,\sqrt{T})} \Big(\tfrac{1}{|B_R(x)|}\int_0^{R^2}\int_{B_R(x)} |v(y,t)|^2 dydt\Big)^{1/2}. \end{aligned}$$

This norm is associated to the natural class of solutions of the heat equation as well as the Navier-Stokes equations. Actually, let $v = e^{t\Delta}v_0$ with $v_0 \in BMO^{-1}$, we see that

$$\begin{aligned} \|e^{t\Delta}v_0\|_{\mathcal{E}_T} &= \sup_{0<t<T} \sqrt{t}\|e^{t\Delta}v_0\|_\infty \\ &\quad + \sup_{x\in\mathbf{R}^n, R\in(0,\sqrt{T})} \Big(\tfrac{1}{|B_R(x)|}\int_0^{R^2}\int_{B_R(x)} |e^{t\Delta}v_0(y)|^2 dydt\Big)^{1/2} \\ &\le C\|v_0\|_{\dot{B}^{-1}_{\infty,\infty}} + C\|v_0\|_{\dot{F}^{-1}_{\infty,2}} \le C\|v_0\|_{\dot{F}^{-1}_{\infty,2}} < \infty, \end{aligned}$$

the first inequality obviously holds for taking $T = \infty$. The discovering $\mathcal{E}_T$-norm is crucial.

3.2 In $vmo^{-1} = f^{-1}_{\infty,2}$

We give the main results of Koch-Tataru:

Theorem 3.4 (Koch-Tataru [30]). (NS) *is* (TLWP) *in* vmo^{-1}, *i.e.*, ${}^{\forall}u_0 \in vmo^{-1}$, ${}^{\exists}T > 0$ *and mild solution* ${}^{\exists 1}u \in \mathcal{E}_T \cap C([0,T]; vmo^{-1})$. *Moreover,* (GWSD) *in* BMO^{-1}, *i.e., if we assume, in addition, that* $\|u_0\|_{BMO^{-1}}$ *is small enough, then* ${}^{\exists 1}u \in \mathcal{E}_\infty \cap C([0,\infty); vmo^{-1})$.

Remark 3.5. (i) *When* $u_0 \in bmo^{-1}$, *there is a lack of continuity of mild solutions at* $t = 0$. *Also, uniqueness is not known; see Miura [39].*
(ii) *By definition of* $\mathcal{E}_T$*-norm it is shown that the mild solution* $u(t) \in L^\infty$ *for any small* $t > 0$. *Thus, the pressure giving by* (1.1) *makes sense of value in* BMO *for* $t > 0$. *Under this setting* (u,p) *satisfies* (NS) *in the classical sense, and the solution is uniquely determined by* u_0*; see Kato [26].*
(iii) *By smoothing effect for any small* $t_0 > 0$ *the mild solution* $u(t_0) \in W^{1,\infty}(\mathbf{R}^n)$. *So, this* t_0 *can be regarded as a new initial time, and* $u(t_0)$ *as a new bounded and smooth initial velocity. By analysis in* L^∞*-framework by e.g. [19] we may observe the properties of obtained mild solutions, more precisely. For example, the propagation speed of Koch-Tataru's solutions is infinity; see e.g. [40].*
(iv) *It is well-known that Serrin's class* $L^s(0,T;L^r)$ *with* $\frac{2}{s} + \frac{n}{r} \le 1$ *satisfying* $s > 2$ *and* $r \in (n,\infty)$ *produces the regularity of solutions to* (NS). *Since the embedding* $L^s(0,T;L^r) \subset \mathcal{E}_T$ *holds, one can argue the solution in Serrin's class as a Koch-Tataru's solution.*

We use the iteration scheme (so-called successive approximation or fixed point argument) for the proof of Theorem 3.4. In fact, we successively define $\{u_j\}$ by (1.2) with (1.3). One can see that the approximation sequence $\{u_j\}_{j=1}^\infty$ is a Cauchy sequence in $\mathcal{E}_T \cap C([0,T]; vmo^{-1})$. The Key of the proof is the inequality for estimating to the bilinear terms: there exists a positive constant C such that

$$\|\mathcal{B}(u,v)\|_{\mathcal{E}_T} \le C\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T} \qquad \text{for} \quad u,v \in \mathcal{E}_T. \tag{3.1}$$

This inequality holds true even for $T = \infty$. One may find the proof of (3.1) due to the point-wise estimates of the heat kernel in [30], and the estimates involving the higher order differentiation in [40].

It is not known the benefit bilinear estimates in neither $C([0,T]; vmo^{-1})$ to which the solutions naturally belong, as long as the author knows. Remark that the function space which contain non-decaying functions is usually not a Banach algebra with respect to point-wise multiplications, e.g. $\dot{B}^0_{\infty,1}$, vmo^{-1}, BMO^{-1} and $\dot{B}^{-1}_{\infty,\infty}$, except for L^∞. Thus, it seems to be difficult to make sense the bilinear terms in such function spaces, basically.

4 Ill-posedness in $\dot{f}^{-1}_{\infty,\infty} = \dot{b}^{-1}_{\infty,\infty}$

4.1 Norm inflation

In this section we will give a rigorous proof of [8], that is, (NS) is (IP) in $\dot{f}^{-1}_{\infty,\infty} = \dot{b}^{-1}_{\infty,\infty}$ in $\mathbf{R}^3$. Firstly, it is shown a lack of equicontinuity of mild solutions. Also, we see that it seems to be difficult to construct a unique time-local mild solution.

Theorem 4.1 (Bourgain-Pavlovic [8]). *Let $n = 3$. For $\delta \in (0,1)$ and $T \in (0,1)$, there exists a $u_0 \in \dot{b}^{-1}_{\infty,\infty}(\mathbf{R}^3)$ such that $\|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} < \delta$ with $\nabla \cdot u_0 = 0$, there exists a mild solution u in $C([0,T]; \dot{b}^{-1}_{\infty,\infty})$ and $\|u(T)\|_{\dot{B}^{-1}_{\infty,\infty}} > 1/\delta$.*

Remark 4.2. (i) *This assertion indicates that in the class $C([0,T]; \dot{b}^{-1}_{\infty,\infty})$ to which mild solutions ought to belong, mild solutions do not have the equicontinuity. Thus, this assertion is to be said* **ill-posedness theorem***. Namely,* (NS) *is not* (TLWP) *in $\dot{b}^{-1}_{\infty,\infty}$ and wider spaces, for example, $b^{-1}_{\infty,\infty}$ and the supercritical spaces $b^{-\alpha}_{\infty,\infty}$ with $\alpha > 1$. Also,* (NS) *is not* (WPSD) *in $\dot{b}^{-1}_{\infty,\infty}$, even though $\dot{b}^{-1}_{\infty,\infty}$ is scaling invariant. Furthermore, to show the uniqueness of mild solutions in this class seems to be difficult in the sense that the usual arguments due to Gronwall type inequality do not work.*
(ii) *One can also prove the lack of equicontinuity by the technique of Bejenaru-Tao [3]; see subsection* 4.3.
(iii) *This assertion is still true for the case $n \geq 4$ by the simple modification of the proof. However, in the case $n = 2$ it is not clear whether the same results can be proved, or not.*
(iv) *It is supposed that one can also obtain the same statement in other function spaces. Yoneda wrote [55] for ill-posedness in $\dot{F}^{-1}_{\infty,q}$ with $q \in (2,\infty)$, using the same argument of [8]. Moreover, the author thinks that the similar results can be obtained for strong solutions to other equations of parabolic type, particularly, the Keller-Segel equations; see e.g. Iwabuchi [23].*

4.2 Initial datum

Theorem 4.1 follows from the technique of Bourgain [7] for establishing the similar ill-posedness theorem for the KdV equation. His method is so-called "norm inflation". Before stating the outline of the proof, we now fix the initial velocity, concretely. In what follows, the initial velocity is fixed to be of the form

$$u_0(x) := \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} h_s \big[e_2 \cos(k_s \cdot x) + e_3 \cos(l_s \cdot x) \big], \tag{4.1}$$

that is to say,

$$u_0(x) = \Big(0, \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} h_s \cos(h_s x_1), \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} h_s \cos(h_s x_1 - x_2) \Big)$$

with parameters $Q > 0$ and large $r \in \mathbf{N}$; other notations are as follows:

$$\begin{aligned}
e_2 &:= \overrightarrow{e}_2 := (0,1,0) && (= v_s),\\
e_3 &:= \overrightarrow{e}_3 := (0,0,1) && (= v_s'),\\
h_s &:= h(s) := 2^{s(s-1)/2}\gamma^{s-1}\eta && \text{for } s \in \mathbf{N},\\
k_s &:= (h_s, 0, 0), &&\\
l_s &:= (h_s, -1, 0) && (= k_s').
\end{aligned}$$

Here $\gamma, \eta \in \mathbf{N}$ are also parameters; v_s, v_s', k_s' are the notation in [8]. The specific time T when the inflation occurs can be regarded as a parameter, replacing the time variable $[t \mapsto \lambda t]$ with some $\lambda > 0$. Using this scaling argument, we can relax the restriction $T < 1$. However, for the sake of simplicity of the proof, and for the readers' convenience, T remains as a given small number.

It is clear by definition that $u_0(x) = (0, u_0^2(x_1), u_0^3(x_1, x_2))$ and $u_0 \in \dot{B}^{-1}_{\infty,\infty}$ by the simple calculation below. Moreover, u_0 is a uniformly continuous function, so $u_0 \in \dot{b}^{-1}_{\infty,\infty}$; see [46]. It should be emphasized that we are able to fix the directions of $v_s = e_2$ and $v_s' = e_3$ without loss of generality, since (NS) is invariant under the Galilei transformation. In addition, it should be more emphasized that the selections of v_s and v_s' are slightly different to those of [8]; that is a crucial point noticed by Yoneda.

The proof of Theorem 4.1 is realized by the suitable selection of the parameters (Q, r, γ, η) for each $\delta, T \in (0,1)$. Since

$$h_{s+1}/h_s = 2^{(2s+1)/2}\gamma, \tag{4.2}$$

it follows that $h_s << h_{s+1}$ for large s or γ; this property is so-called 'lacunary'. For the sake of simplicity, $h(z) := 2^{z(z-1)/2}\gamma^{z-1}\eta$ denotes the function of $z > 0$. The compatibility condition $\nabla \cdot u_0 = 0$ is satisfied by $e_2 \cdot k_s = 0$ and $e_3 \cdot l_s = 0$, obviously. It is clear that u_0 is a smooth periodic function (thus bounded) with the period $2\pi/h$ in x_1 and 2π in x_2. This implies that the mild solution is also periodic with the period 2π, regarded as a function on the torus $(2\pi\mathbf{T})^3$, as long as the mild solution exists. So, the kinematic energy is bounded by the initial energy $\frac{1}{2}\|u_0\|^2_{L^2((2\pi\mathbf{T})^3)}$; this is huge but finite. And also, $\hat{u}_0$ is a sum of Dirac's delta functions, therefore, $u_0 \in FM_0$;

$$FM_0 := \{\mathcal{F}^{-1}v \in \mathcal{S}';\ v = \text{sum of finite Radon measures},\ v(0) = 0\}.$$

We refer to the detail of FM_0 in [17, 18].

Let u_1 be the first approximation of iteration, that is, the solution to the heat equation with initial datum given by (4.1):

$$u_1(x,t) = \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} h_s \big[e_2 e^{-h_s^2 t} \cos(h_s x_1) + e_3 e^{-(h_s^2+1)t} \cos(h_s x_1 - x_2) \big].$$

For $t > 0$ we obtain that $u_1(t) := u_1(\cdot, t) \in L^\infty \cap BMO^{-1}$, even though these norms are large. The function u_1 is of the form

$$u_1 = (0, u_1^2(x_1, t), u_1^3(x_1, x_2, t)). \tag{4.3}$$

It is well-known that one can construct the unique mild solution with initial velocity given by (4.1) in the L^∞-framework. Moreover, the estimate for the possible existence time T_* (until the mild solution is constructed in $C([0, T_*]; L^\infty)$ by the usual iteration scheme) is obtained by [19] to be bounded from below: $T_* \geq C/\|u_0\|_\infty^2 \sim h_r^{-2}$ with the universal constant $C > 0$. Indeed, by $h_r >> r$ we see that

$$\|u_0\|_\infty \sim \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} h_s \sim h_r >> 1 \qquad \text{if} \quad r >> 1.$$

Therefore, T_* might be very tiny. Also, one may see that

$$\|u_0\|_{BMO^{-1}} \sim Q\sqrt{r} >> 1 \qquad \text{if} \quad r >> 1.$$

However, we observe the homogeneous Besov norm $\|\cdot\|_{\dot{B}^{-1}_{\infty,\infty}} \sim |\!|\!| \cdot |\!|\!| = \sup_{\rho>0} \sqrt{\rho}\|e^{\rho\Delta} \cdot \|_\infty$ as

$$\|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} \sim |\!|\!| u_0 |\!|\!| \sim \frac{Q}{\sqrt{r}} << 1 \qquad \text{if} \quad r >> 1. \tag{4.4}$$

In fact, by the definition of Besov norm from Paley-Littlewood decomposition it holds true that

$$\begin{aligned} \|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} &\sim \sup_{j\in\mathbf{Z}} \|\phi_j * \nabla^{-1}u_0\|_\infty \\ &\leq \sqrt{2}\tfrac{Q}{\sqrt{r}} \sup_j \sup_x |\phi_j * \textstyle\sum_{s=1}^r \cos(k_s \cdot)(x)| \\ &\leq 2\sqrt{2} \sup_j \|\phi_j\|_1 \tfrac{Q}{\sqrt{r}} << 1 \qquad \text{if} \quad r >> 1. \end{aligned}$$

Here we have used the two facts that $\|\phi_j\|_1 = \|\phi_0\|_1$ for $j \in \mathbf{Z}$ and for each $s \in \{1, \dots, r\}$ there are at most 2 indices $j \in \mathbf{Z}$ such that $\phi_j * \cos(k_s \cdot x) \neq 0$. For reader's convenience we now give an elementally proof of (4.4) as follows; we will explain the details later:

$$\begin{aligned} |\!|\!| u_0 |\!|\!| &= \sup_{\rho>0} \sqrt{\rho}\|e^{\rho\Delta}u_0\|_\infty \\ &= \sup_\rho \sqrt{\rho}\tfrac{Q}{\sqrt{r}} \sup_x \left| \textstyle\sum_{s=1}^r h_s[e_2 e^{-h_s^2\rho} \cos(k_s \cdot x) + e_3 e^{-(h_s^2+1)\rho} \cos(l_s \cdot x)] \right| \\ &\leq \sqrt{2}\tfrac{Q}{\sqrt{r}} \sup_{\rho>0} \textstyle\sum_{s=1}^r \sqrt{\rho} h_s e^{-h_s^2\rho} \\ &\leq \sqrt{2}\tfrac{Q}{\sqrt{r}} \left[\sup_{0<\rho<h_r^{-2}} \textstyle\sum_{s=1}^r \sqrt{\rho} h_s e^{-h_s^2\rho} + \sup_{\rho\geq h_r^{-2}} \textstyle\sum_{s=1}^r \sqrt{\rho} h_s e^{-h_s^2\rho} \right] \\ &\leq \sqrt{2}\tfrac{Q}{\sqrt{r}} \left[\sqrt{h_r^{-2}}(2h_r) + \sup_{\rho\geq h_r^{-2}} \left\{ \textstyle\sum_{s=1}^{s_\rho} + \textstyle\sum_{s=s_\rho+1}^r \right\} \sqrt{\rho} h_s e^{-h_s^2\rho} \right] \\ &\leq \sqrt{2}\tfrac{Q}{\sqrt{r}} \Big[2 + \sup_{\rho\geq h_r^{-2}} \textstyle\int_0^{s_\rho-1} \sqrt{\rho} h(z+1) e^{-h(z+1)^2\rho} dz \\ &\qquad\qquad + \tfrac{e^{-1/2}}{\sqrt{2}} + \sup_{\rho\geq h_r^{-2}} \textstyle\int_{s_\rho}^r \sqrt{\rho} h(z) e^{-h(z)^2\rho} dz \Big] \\ &\leq \sqrt{2}\tfrac{Q}{\sqrt{\rho}} \left[2 + \tfrac{e^{-1/2}}{\sqrt{2}} + \sup_{\rho\geq h_r^{-2}} \textstyle\int_1^r \sqrt{\rho} h(z) e^{-h(z)^2\rho} dz \right] \leq C_* \tfrac{Q}{\sqrt{r}} \end{aligned}$$

with the numerical constant C_* independent of parameters. We take $g(\varsigma) := \varsigma e^{-\varsigma^2}$, then g is monotone increasing when $\varsigma < 1/\sqrt{2}$, and monotone decreasing when $\varsigma > 1/\sqrt{2}$. Thus, we choose $s_\rho \in \{1, \dots, r\}$ such that

$$\sqrt{\rho} h_s \le \sqrt{\rho} h_{s+1} \quad \text{if} \quad s < s_\rho, \qquad \sqrt{\rho} h_s \ge \sqrt{\rho} h_{s+1} \quad \text{if} \quad s \ge s_\rho.$$

The maximal value of g is taken as $\max g = g(1/\sqrt{2}) = 1/\sqrt{2e}$. By the monotonicity we derive the estimate replaced from sum by integration. The last inequality follows from the fact that the derivation of $\int_1^r \cdots$ with respect to ρ is positive when ρ is small, besides this is negative when ρ is large. In this section we often use the fact that $\sup_{\rho>0} \sum_{s=1}^r \sqrt{\rho} h_s e^{-h_s^2 \rho} \le C_*$ bounded uniformly in r.

Recall the successive approximation and its modification of convergence version. Let us put the sequence $\{v_k\}_{k=1}^\infty$ as

$$\begin{aligned} v_1(t) &:= u_1(t) := e^{t\Delta} u_0, \\ v_{k+1}(t) &:= u_{k+1}(t) - u_k(t) = -\mathcal{B}(u_k) + \mathcal{B}(u_{k-1}) \end{aligned}$$

for $k \in \mathbf{N}$. Therefore, we rewrite u_j and the mild solution $u = \lim_{j\to\infty} u_j$ as

$$u_j(t) = \sum_{k=1}^{j} v_k(t) \qquad \text{and} \qquad u(t) = \sum_{k=1}^{\infty} v_k(t), \tag{4.5}$$

as long as the mild solution exists. In what follows, we calculate $v_k(t)$ and estimate the Besov norm of them at $t = T$. Moreover, we notice that

$$v_k = (0, 0, v_k^3(x_1, x_2, t)) \qquad \text{for} \quad k \ge 2. \tag{4.6}$$

Gathering (4.3) with (4.6), the mild solution should be of the form

$$u = (0, u_1^2(x_1, t), u^3(x_1, x_2, t)) \quad \text{with} \quad u^3 = \sum_{k=1}^{\infty} v_k^3. \tag{4.7}$$

4.3 Term-wise estimates

For all $\delta, T \in (0,1)$, we correctly select parameters (Q, r, γ, η) to see that

$$\|v_1(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \le \|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq C_* \frac{Q}{\sqrt{r}} =: S < \delta. \tag{4.8}$$

Also, $v_2 = M_2 + R_2$ and $M_2 := e_3 \frac{Q^2}{4} e^{-t} \sin x_2$ with

$$\|v_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq \|M_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}} = C_\flat Q^2 =: L \ge \frac{2}{\delta} \tag{4.9}$$

in the next subsection. Here $A \simeq B$ means the almost equal, that is, $A = B + R$ such that $|R| < \frac{1}{3}|B|$ for the scalar valued, and $\|R\|_{\dot{B}^{-1}_{\infty,\infty}} < \frac{1}{3}\|B\|_{\dot{B}^{-1}_{\infty,\infty}}$ for functions; $C_\flat > 0$ is a numerical constant. We will see that M_2 is the major term of v_2 at $t \simeq T$ in the next subsection. Conversely, R_2 is the collection of the remainder terms of v_2 at $t \simeq T$. It is remarkable that $M_k(t)$ no longer might be the leading term if we take neither a different norm nor $t << T$. We further prove that $v_3 = M_3 + R_3$ with

$$M_3 := -\frac{Q^3}{8\sqrt{r}} te^{-t} \sum_{s=1}^{r} h_s e^{-h_s^2 t} \{\cos(h_s x_1 + x_2) + \cos(h_s x_1 - x_2)\} e_3$$

and

$$\|v_3(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq \|M_3(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq \frac{Q^3\sqrt{T}}{8\sqrt{2er}} \simeq \frac{Q^2}{4\eta} S \tag{4.10}$$

for $t \simeq T \simeq \eta^{-2}$. Moreover, we see that for v_4

$$v_4(T) = M_4(T) + R_4(T), \quad M_4(T) = -KM_2(T), \quad K := \frac{(1-3e^{-2})Q^2}{8r\eta^2} > 0$$

and the estimate

$$\|v_4(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq \|M_4(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq KL. \tag{4.11}$$

By induction one may also show that $v_k(T) = M_k(T) + R_k(T)$ and

$$M_{2k-1}(T) = (-K)^{k-2} M_3(T) \quad \text{and} \quad M_{2k}(T) = (-K)^{k-1} M_2(T) \tag{4.12}$$

for $k \geq 2$ with $t \simeq T \simeq \eta^{-2}$. For the proof of Theorem 4.1, taking parameters such that $K < 1/12$, we may appeal to rough estimates for the remainder terms in the following way. Since the number of terms of v_k is 2^k, and the biggest term in the Besov norm of the components of v_k is that of M_k, it is allowed to compute

$$\begin{aligned} \|v_{2k-1}(T)\|_{\dot{B}^{-1}_{\infty,\infty}} &\leq (\sharp\,\text{terms}) \cdot \|M_{2k-1}(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \leq (4K)^{k-2} S, \\ \|v_{2k}(T)\|_{\dot{B}^{-1}_{\infty,\infty}} &\leq (\sharp\,\text{terms}) \cdot \|M_{2k}(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \leq (4K)^{k-1} L \end{aligned}$$

for $k \geq 3$. Once we obtain these estimates, it follows from (4.5):

$$\|u(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \geq \|v_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}} - \sum_{k=2}^{\infty} (4K)^{k-1} \|v_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \geq \frac{L}{2},$$

if $K < 1/12$. We simply discard the sum of odd numbers above, since S is very small compared with L. Finally, the choice of parameters yields that $S \simeq \delta$ and $L \simeq \frac{2}{\delta}$, this completes the proof of Theorem 4.1.

We see the proofs of (4.8) – (4.11) in [47]. So, we omit the details in here.

Choice of parameters We now mention the selection of the parameters (Q, r, γ, η) for the proof of Theorem 4.1. Firstly, we always fix $\gamma := 3$. We impose that $\eta \in \mathbf{N}$ with $\eta \geq 2$ large such that $\eta \sim T^{-1/2}$ for $T \in (0, 1)$. For any $\delta \in (0, 1)$, we fix $Q > 1$ large such that $Q > \sqrt{\frac{3}{C_\flat \delta}}$. Finally, we choose $r \in \mathbf{N}$ large such that $r > 4C_*^2\delta^{-4}$, $T > h_r^{-2}$ and $K < \frac{1}{12}$.

No blow-up From the choice of initial velocity, the gradient of pressure terms are always annihilated, that is to say, $\nabla p = 0$ due to (1.1) and (4.7). This yields that the mild solution exists uniquely and time-globally in $C([0,\infty); L^\infty(\mathbf{R}^3))$, since it is known that the mild solution exists uniquely and time-locally in this class, and the a priori estimate $\|u(t)\|_{L^\infty} \le \|u_0\|_{L^\infty}$ holds for all $t > 0$ by the maximal principle for the solution u to the following Burgers type equations

$$u_t - \Delta u + (u, \nabla)u = 0, \qquad u|_{t=0} = u_0 \in L^\infty.$$

Therefore, the blow-up does not occur.

Role of Q The choice of the parameter Q (depending on r and η) is essential. As the conclusion, we investigate the following four cases:

1. It is possible to show that $\|u_j(T)\|_{\dot{B}^{-1}_{\infty,\infty}}$ does not converge as $j \to \infty$ when Q is large so that $K > 4$. This implies that one should take a subsequence to proceed the iteration scheme (1.2) to construct the mild solution up to time T from the initial datum u_0 given by (4.1), even though $\{u_j\}$ is a Cauchy sequence in $C([0,T_*]; L^\infty)$ with $T_* \sim h_r^{-2}$.

2. If Q is large, but not so large compared with r and η such that $K < 1/12$, then the norm inflation occurs, likely. The author guesses that the norm inflation solution can be extended time-global one with exponential decay as $t \to \infty$, since M_k is always the major part of v_k and the estimates above are valid for all $t > T$.

3. On the other hand, if Q is small such that $C_\sharp Q < 1$, where $C_\sharp$ is the universal constant related to the constant C in (3.1), then the norm inflation does not occur. In this case the unique time-global mild solution exists due to Theorem 3.4 and subsection 4.7.

4. One can prove that there exists a unique time-global mild solution in the certain class e.g. $C([0,\infty); L^\infty)$ if $Q << 1$ due to the argument of [18]. In this case the mild solution decays exponentially as $t \to \infty$.

Once we get (4.9) with large L, it seems to be difficult to apply the fixed point argument, directly. More precisely, the mapping from the initial data to the mild solutions seems to be not of class C^2; Germain intended to show it in [15]. The proof of Theorem 4.1 is slightly different to that of [8]. They actually intended to show

$$\|y(T)\|_{\dot{B}^{-1}_{\infty,\infty}} << 1, \tag{4.13}$$

where $y := u - u_2 := u - u_1 + \mathcal{B}(u_1) = \sum_{k=3}^{\infty} v_k$. Bourgain-Pavlovic [8] proceeded on the investigation to get (4.13). However, it is not clear to the author how to choose the parameters such that (4.13) and (4.9) are satisfied along their strategy, simultaneously. Although they mentioned the way-out by new techniques (slicing the time-interval into many parts) due to Koch-Tzvetkov e.g. [31], it is unlikely to get some advantage by their method in the situation $u, y \in C([0,T]; \dot{b}^{-1}_{\infty,\infty})$. Remark that it seems to be hard to find the associate norm like $\|\cdot\|_{\mathcal{E}_T}$ for Koch-Tataru's solution. As seen in (4.11), it is difficult to show (4.13) without smallness of Q directly, even if y is relatively smaller than v_2. Besides, if Q is small, then the norm inflation does not occur.

Technique of Bejenaru-Tao We explain the technique by Bejenaru-Tao [3]. They also proved a lack of equicontinuity to the nonlinear Schrödinger equation in some supercritical spaces. Their proof is based on the scaling argument and the contradiction arguments as follows. Thanks to Theorem 3.4, there exists ε such that there exists a unique time-global mild solution if $\|u_0\|_{BMO^{-1}} < \varepsilon$. Consider the family of initial velocity λu_0 for $0 < \lambda \le 1$. Find $u_0 \in BMO^{-1}$ such that the $\dot{B}^{-1}_{\infty,\infty}$-norms of u, u_1, y have the order $o(\lambda)$ as $\lambda \to 0$ for all $t > 0$; besides, $\lambda^{-1}\|v_2\|_{\dot{B}^{-1}_{\infty,\infty}} > C$ with some constant C with some $\lambda > 0$ and t. The scaling argument works well, since the bilinear estimates (3.1) do not depend on T. Actually, u_0 is taken as similar to (4.1). This contradicts to the expansion $u = u_1 + v_2 + y$.

The author thinks that their proof is good, however, there is no information about the behavior of mild solutions. While, Theorem 4.1 is a constructive and concrete assertion for the ill-posedness.

4.4 First and second approximation

In this subsection we only calculate the forms of v_2. Divide v_2 into three parts. Let us see

$$
\begin{aligned}
&(u_1(\tau), \nabla) u_1(\tau) \\
&\quad = \sum_{m=1}^{3} \tfrac{Q}{\sqrt{r}} \sum_{s=1}^{r} h_s [e_2^m e^{-h_s^2 \tau} \cos(k_s \cdot x) + e_3^m e^{-(h_s^2+1)\tau}] \cos(l_s \cdot x)] \\
&\qquad \times \partial_m \Big(\tfrac{Q}{\sqrt{r}} \sum_{q=1}^{r} h_q [e_2 e^{-h_q^2 \tau} \cos(k_q \cdot x) + e_3 e^{-(h_q^2+1)\tau}] \cos(l_q \cdot x)] \Big) \\
&\quad = \tfrac{Q^2}{r} \sum_{s=1}^{r} \sum_{q=1}^{r} h_s h_q e_3 e^{-(h_s^2+h_q^2+1)\tau} \cos(k_s \cdot x) \sin(l_q \cdot x) \\
&\quad = \tfrac{Q^2}{r} \sum_{s=1}^{r} h_s^2 e_3 e^{-(2h_s^2+1)\tau} \left(-\tfrac{1}{2}\right) \sin x_2 \\
&\qquad + \tfrac{Q^2}{r} \sum_{s=1}^{r} h_s^2 e_3 e^{-(2h_s^2+1)\tau} \tfrac{1}{2} \sin(2h_s x_1 - x_2) \\
&\qquad + \tfrac{Q^2}{r} \sum_{s,q=1,\, s\neq q}^{r} h_s h_q e_3 e^{-(h_s^2+h_q^2+1)\tau} \cos(k_s \cdot x) \sin(l_q \cdot x) \\
&\quad =: N_1 + N_2 + N_3.
\end{aligned}
$$

For each $\ell = 1, 2, 3$ we set

$$U_\ell := U_\ell(t) := -\int_0^t e^{(t-\tau)\Delta} \mathbf{P} N_\ell(\tau) d\tau.$$

Thus, $v_2 = \sum_{\ell=1}^{3} U_\ell$. In the conclusion U_1 happens "inflation", as the contrast to that U_2 and U_3 are small, when Q and r are large. Notice that v_2 satisfies (4.6). Therefore,

$\nabla \cdot N_\ell = 0$ and $\mathbf{P}N_\ell = N_\ell$ as well as $\nabla \cdot v_2 = 0$.

The estimates for v_2 are shown in [47]. The original proof of [8] has a gap at the estimate for U_3, which seems to be balanced to U_1. So, it was not clear whether the norm inflation actually occurs, or not. Yoneda found the modification of the initial datum such that U_3 can be regarded as small terms (compared with U_1) as the same as U_2.

4.5 Calculi for v_3 and v_4

In this subsection the forms of v_3 and v_4 are derived, concretely. We invoke

$$\begin{aligned} v_3 = u_3 - u_2 &= u_1 - \mathcal{B}(u_2) - \{u_1 - \mathcal{B}(u_1)\} \\ &= -\mathcal{B}(v_1 + v_2, v_1 + v_2) + \mathcal{B}(v_1, v_1) \\ &= -\mathcal{B}(v_1, v_2) \\ &= -\int_0^t e^{(t-\tau)\Delta}\mathbf{P}e_3 v_1^2(\tau)\partial_2 v_2^3(\tau)d\tau \\ &= -\int_0^t e^{(t-\tau)\Delta} v_1^2(\tau)\partial_2 v_2^3(\tau)d\tau e_3. \end{aligned}$$

Since v_1 and v_2 are functions independent of x_3, the fourth equality holds by $v_2 = (0, 0, v_2^3)$, and the last equality holds by divergence-free of the integrant. Clearly, v_3 satisfies (4.6) with $k = 3$; $v_3^1 = v_3^2 = 0$. Analogously, we observe that (4.6) are valid for all $k \geq 4$.

We now calculate the concrete expression of v_3^3 at $t \simeq T \simeq \eta^{-2}$:

$$\begin{aligned} v_3^3 &= -\int_0^t e^{(t-\tau)\Delta}\Bigg[\left\{\tfrac{Q}{\sqrt{r}}\textstyle\sum_{s=1}^r h_s e^{-h_s^2\tau}\cos(h_s x_1)\right\}\cdot\tfrac{Q^2}{4r} \\ &\quad\cdot\Big\{\textstyle\sum_{q=1}^r (1 - e^{-2h_q^2\tau})e^{-\tau}\cos x_2 \\ &\qquad + \textstyle\sum_{q=1}^r (e^{2h_q^2\tau} - 1)e^{-4(h_q^2+1)\tau}\cos(2h_q x_1 - x_2) \\ &\qquad + \textstyle\sum_{q\neq p}(e^{2h_q h_p\tau} - 1)e^{-(h_q^2+2h_q h_p+h_p^2+1)\tau}\cos(h_q x_1 + h_p x_1 - x_2) \\ &\qquad + \textstyle\sum_{q\neq p}(1 - e^{-2h_q h_p\tau})e^{-(h_q^2-2h_q h_p+h_p^2+1)\tau}\cos(h_q x_1 - h_p x_1 + x_2)\Big\}\Bigg]d\tau \\ &= \tfrac{-Q^3}{4\sqrt{r}}\int_0^t \textstyle\sum_{s=1}^r h_s e^{-(h_s^2+1)\tau}e^{(t-\tau)\Delta}\{\cos(h_s x_1)\cos x_2\}d\tau + (remainder) \\ &= \tfrac{-Q^3}{8\sqrt{r}} t e^{-t}\textstyle\sum_s h_s e^{-h_s^2 t}\{\cos(h_s x_1 + x_2) + \cos(h_s x_1 - x_2)\} + (remainder) \\ &=: M_3(t) + R_3(t). \end{aligned}$$

Here and hereafter, we do not distinguish the vector valued M_k and its third component if no confusion occurs likely, since $M_k = (0, 0, M_k^3)$ for all $k \geq 2$ as well as $R_k = (0, 0, R_k^3)$. It is easy to see the estimate $\|M_3(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \sim Q^3/\sqrt{r} << 1$ and the remainder term R_3 is small compared with M_3 as the similar to the estimates for U_1, U_2 and U_3.

Next, we compute v_4. It follows that at $t \simeq T \simeq \eta^{-2}$

$$\begin{aligned} v_4^3 &= -\int_0^t e^{(t-\tau)\Delta}\Bigg[\left\{\tfrac{Q}{\sqrt{r}}\textstyle\sum_{s=1}^r h_s e^{-h_s^2\tau}\cos(h_s x_1)\right\}\cdot\left(-\tfrac{Q^3}{8\sqrt{r}}\right)\tau e^{-\tau} \\ &\qquad\cdot\textstyle\sum_{q=1}^r h_q e^{-h_q^2\tau}\{-\sin(h_q x_1 + x_2) + \sin(h_q x_1 - x_2)\}\Bigg]d\tau + R_4 \\ &= -\tfrac{Q^4}{8r}\textstyle\sum_{s=1}^r h_s^2 e^{-t}\int_0^t \tau e^{-2h_s^2\tau}d\tau \sin x_2 + R_4 \\ &= -\tfrac{Q^4}{32r}e^{-t}\sin x_2\left[\textstyle\sum_{s=1}^r \tfrac{1}{h_s^2}\left\{1 - e^{-2h_s^2 t}(1 + 2h_s^2 t)\right\}\right] + R_4 \\ &= -\tfrac{(1-3e^{-2})Q^4}{32r\eta^2}e^{-t}\sin x_2 + R_4 = -KM_2 + R_4. \end{aligned}$$

Here we move the summation over $s \geq 2$ to the remainder terms; the remainder term R_4 might differ to the others in lines, likely. Also, it is easy to see that the Besov norm of R_4 is relatively small compared with that of $M_4 = -KM_2$. Then, we have $\|v_4(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq KL$. The same argument indicates (4.12), as long as $T \simeq \eta^{-2}$. Therefore, the proof of Theorem 4.1 now completes. $\square$

4.6 No convergence of approximation

For the case of huge Q, the successive approximation is no longer a good approximation in $C(0,T;\dot{B}^{-1}_{\infty,\infty})$.

Theorem 4.3 (S. [47]). *For $T > 0$ there exists a u_0 such that $\|u_j(T)\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbf{R}^3)}$ does not converge.*

Proof. Let us assume $T < 1/4$ without loss of generality. We choose the initial datum u_0 given by (4.1). We will prove that

$$\|R_k(T)\|_{\dot{B}^{-1}_{\infty,\infty}} < \frac{1}{3}\|M_k(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \tag{4.14}$$

for $k \in \mathbf{N}$. Once we get (4.14), one sees

$$\|u_{4j+2}(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \geq \sum_{k=1}^{j}(K/4)^{k-1}\|M_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \to \infty \quad \text{as} \quad j \to \infty \tag{4.15}$$

when $K > 4$. It suffices to show (4.14) under the suitable choice of parameters with $k \geq 3$. Determine $\gamma = 3$ and $r = 2$. Select $\eta \in \mathbf{N}$ as $\eta \geq 2$ and $\eta \simeq T^{-1/2}$. Let Q be taken large such that $K > 4$. If ℓ is odd and $\ell \geq 3$, then we see

$$\frac{\|R_\ell(T)\|_{\dot{B}^{-1}_{\infty,\infty}}}{\|M_\ell(T)\|_{\dot{B}^{-1}_{\infty,\infty}}} \leq 2^\ell\Big(\sum_{s=1}^{2} e^{-h_s^2 T}\Big)^\ell = 2^\ell(e^{-1} + e^{-36})^\ell < \frac{1}{3}.$$

Analogously as the estimates for R_2, in the case for even $\ell \geq 4$ one can prove the similar inequality. This completes the proof of Theorem 4.3. $\square$

The calculation above implies that it seems hard to show the convergence of $\{u_j\}$ in the class $C(0,T;\dot{b}^{-1}_{\infty,\infty})$. It is not clear whether the other successive approximation, for example,

$$w_1 := u_1 \qquad \text{and} \qquad w_{j+1} := w_1 - \mathcal{B}(w_j, w_{j+1}),$$

does converge, or not. Although one can easily observe that $\|u_{4j+2}(T)\|_{BMO^{-1}}$ tends to infinity as $j \to \infty$ by the continuous embedding $BMO^{-1} \subset \dot{B}^{-1}_{\infty,\infty}$, it is not clear to the author whether another norms e.g. $\|\cdot\|_{L^3((2\pi\mathbf{T})^3)}$ or $\|\cdot\|_{L^\infty}$ of $\{u_j(T)\}$ diverge as $j \to \infty$, or not. It is obvious that the proofs of above theorems do not fit the situation in two-dimension.

4.7 Estimate for y

In the end of this note we express the remainder $y = u - u_2$ and its property whence Q is small enough for readers' convenience. Before computing the norms, we establish a proposition for embedding type for periodic functions.

Proposition 4.4. *Let $\kappa > 0$ and $n \in \mathbf{N}$. Assume that $v \in L^\infty \cap \dot{B}^{-1}_{\infty,\infty}(\mathbf{R}^n)$ is periodic with period $2\pi/\kappa$ in x_j for all $j = \{1, \dots, n\}$. Then*

$$\|v\|_{\dot{B}^{-1}_{\infty,\infty}} \le C\kappa^{-n}\|v\|_\infty \tag{4.16}$$

holds true with constant C depending only on n.

The proof is easy, so we omit. It is easy to see that for $v \in \mathcal{E}_T$ enjoying the period $2\pi/\kappa$ in $\mathbf{R}^3$

$$\|v(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \le C\kappa^{-n}\|v(T)\|_\infty = C\kappa^{-3}T^{-\frac{1}{2}}\|T^{\frac{1}{2}}v(T)\|_\infty \le C\kappa^{-3}T^{-\frac{1}{2}}\|v\|_{\mathcal{E}_T}$$

by (4.16) and the definition of $\mathcal{E}_T$-norm. This technique leads us to show the smallness of Besov norm of functions at T by the smallness of $\mathcal{E}_T$-norm, even though it seems to be tough to compute the Besov norm directly.

Now we estimate y. Let u be a mild solution, and let $y := u - v_1 - v_2$. A formal calculation yields that

$$\begin{aligned} y_t &= u_t - (v_1)_t - (v_2)_t \\ &= u_t - \Delta e^{t\Delta}u_0 - \mathbf{P}(u_1(t), \nabla)u_1(t) - \int_0^t \Delta e^{(t-\tau)\Delta}\mathbf{P}(u_1(\tau), \nabla)u_1(\tau)d\tau. \end{aligned}$$

Subtracting this to $\Delta y = \Delta u - \Delta v_1 - \Delta v_2$, we have

$$\begin{aligned} y_t - \Delta y &= -\mathbf{P}(u, \nabla)u + \mathbf{P}(u_1, \nabla)u_1 \\ &= G_1 + G_2 + G_3 =: G. \end{aligned}$$

Here we set

$$\begin{aligned} G_1 &:= G_1(t) := -\mathbf{P}\big\{(y, \nabla)(u_1 - v_2) + (u_1 - v_2, \nabla)y\big\}, \\ G_2 &:= G_2(t) := -\mathbf{P}(y, \nabla)y, \\ G_3 &:= G_3(t) := -\mathbf{P}(u_1, \nabla)v_2. \end{aligned}$$

Since

$$u_1 = (0, u_1^2(x_1, x_2, t), u_1^3(x_1, x_2, t)) \quad \text{and} \quad v_2 = (0, 0, v_2^3(x_1, x_2, t)),$$

it is noticed that $(v_2, \nabla)u_1 = 0$ and $(v_2, \nabla)v_2 = 0$, easily. Furthermore, from $\mathcal{B}(u_1)(0) = 0$ it deduces that $y(x, 0) \equiv 0$. By Duhamel's principle y can be regarded as the solution to the following equation of integral form:

$$y(t) = \int_0^t e^{(t-\tau)\Delta}G(\tau)d\tau. \tag{4.17}$$

In terms of $\mathcal{B}$, we rewrite it by

$$y = -\mathcal{B}(y, u_1 - v_2) - \mathcal{B}(u_1 - v_2, y) - \mathcal{B}(y) - \mathcal{B}(u_1, v_2).$$

Moreover, we may obviously seek that $y = (0, 0, y^3(x_1, x_2, t))$ as well as (4.6).

It is straightforward to prove the estimates for v_1 and v_2 in $\mathcal{E}_T$-norm. We thus choose T small such that $C\sqrt{T}Q^2 < 1/4$ as well as r large such that $CQ/\sqrt{r} < 1/4$. Compute (4.17) in $\mathcal{E}_T$-norm by (3.1) and the triangle-inequality to have

$$\begin{aligned}
\|y\|_{\mathcal{E}_T} &= \|\mathcal{B}(y, u_1 - v_2) + \mathcal{B}(u_1 - v_2, y) + \mathcal{B}(y) + \mathcal{B}(u_1, v_2)\|_{\mathcal{E}_T} \\
&\leq C\left\{(\|u_1\|_{\mathcal{E}_T} + \|v_2\|_{\mathcal{E}_T} + \|y\|_{\mathcal{E}_T})\,\|y\|_{\mathcal{E}_T} + \|u_1\|_{\mathcal{E}_T}\|v_2\|_{\mathcal{E}_T}\right\} \\
&\leq C\left\{\left(Q + \sqrt{T}Q^2 + \|y\|_{\mathcal{E}_T}\right)\|y\|_{\mathcal{E}_T} + \frac{\sqrt{T}Q^3}{\sqrt{r}}\right\} \\
&\leq (C_\sharp Q + C\|y\|_{\mathcal{E}_T})\,\|y\|_{\mathcal{E}_T} + \frac{1}{4C}
\end{aligned} \tag{4.18}$$

with some positive constant $C_\sharp$. It is not difficult to show that y is small in this way if $C_\sharp Q < 1$.

In [8] Bourgain and Pavlovic compute $\|y\|_{\mathcal{E}_T}$, dividing the time-interval into many parts. Although the author thinks that it is unnecessary to employ their method, it is supposed that their technique leads us to some new idea and inspiration. So, the author would give an explaining of their method.

Let $T_0 \in (0, T)$ be fixed, and let T_0 be assumed as a new initial time for the equation (4.17) with initial datum $y(T_0)$. That is to say, for $t > T_0$

$$y(t) = e^{(t-T_0)\Delta}y(T_0) + \int_{T_0}^{t} e^{(t-\tau)\Delta}G(\tau)d\tau. \tag{4.19}$$

One can rewrite the second terms in the right hand side of (4.19) by

$$\begin{aligned}
\int_{T_0}^{t} e^{(t-\tau)\Delta}G(\tau)d\tau &= \int_{0}^{t} e^{(t-\tau)\Delta}G(\tau)\chi_{[T_0,t]}(\tau)d\tau \\
&= -\mathcal{B}(y^\sharp, u_1^\sharp - v_2^\sharp) - \mathcal{B}(u_1^\sharp - v_2^\sharp, y^\sharp) - \mathcal{B}(y^\sharp) - \mathcal{B}(u_1^\sharp, v_2^\sharp)
\end{aligned}$$

in terms of $\mathcal{B}$. Here we have denoted $^\sharp$ by

$$y^\sharp := y^\sharp(t) := \begin{cases} 0 & if \quad t < T_0, \\ y(t) & if \quad t \geq T_0. \end{cases}$$

Analogously, we define $u_1^\sharp$ and $v_2^\sharp$. By semigroup property we also rewrite the first terms in the right hand side of (4.19) by

$$e^{(t-T_0)\Delta}y(T_0) = e^{(t-T_0)\Delta}\int_0^{T_0} e^{(T_0-\tau)\Delta}G(\tau)d\tau$$

$$
\begin{aligned}
&= \int_0^t e^{(t-\tau)\Delta} G(\tau)\chi_{[0,T_0]}(\tau)d\tau \\
&= -\mathcal{B}(y^\flat, u_1^\flat - v_2^\flat) - \mathcal{B}(u_1^\flat - v_2^\flat, y^\flat) - \mathcal{B}(y^\flat) - \mathcal{B}(u_1^\flat, v_2^\flat).
\end{aligned}
$$

Here $^\flat$ denotes

$$
y^\flat := y^\flat(t) := \begin{cases} y(t) & if \quad t < T_0, \\ 0 & if \quad t \geq T_0. \end{cases}
$$

Analogously, we define $u_1^\flat$ and $v_2^\flat$. When we settle $T_1 \in (T_0, T)$ small again to deduce that $\|y\|_{\mathcal{E}_{T_1}}$ have a better estimate. By (3.1) and so on, we see

$$
\begin{aligned}
\|y\|_{\mathcal{E}_{T_1}} \leq & C(\|u_1^\sharp\|_{\mathcal{E}_{T_1}} + \|v_2^\sharp\|_{\mathcal{E}_{T_1}} + \|y^\sharp\|_{\mathcal{E}_{T_1}})\|y^\sharp\|_{\mathcal{E}_{T_1}} + C\|u_1^\sharp\|_{\mathcal{E}_{T_1}}\|v_2^\sharp\|_{\mathcal{E}_{T_1}} \\
& + C\left(\|u_1^\flat\|_{\mathcal{E}_{T_1}} + \|v_2^\flat\|_{\mathcal{E}_{T_1}} + \|y^\flat\|_{\mathcal{E}_{\mathcal{E}_1}}\right)\|y^\flat\|_{\mathcal{E}_{T_1}} + C\|u_1^\flat\|_{\mathcal{E}_{T_1}}\|v_2^\flat\|_{\mathcal{E}_{T_1}} \\
\leq & C\left(\frac{Q}{\sqrt{r}} + \sqrt{T_0}Q^2 + \sqrt{T_1 - T_0}Q^2 + \|y\|_{\mathcal{E}_{T_1}}\right)\|y\|_{\mathcal{E}_{T_1}} \\
& + C\frac{\sqrt{T_0} + \sqrt{T_1 - T_0}}{\sqrt{r}}Q^3.
\end{aligned}
$$

One may have some improvements by this method, repeating and repeating.

Acknowledgement. The author is grateful to Professor Matthias Hieber for letting him know many interesting literatures concerned with this paper, including [8]. He is also grateful to Professor Hisashi Okamoto, Professor Taku Yanagisawa, Professor Yasushi Taniuchi, Professor Hideyuki Miura, Professor Yasunori Maekawa and Professor Tsuyoshi Yoneda for discussing with him on the workshop at Nara Women's University. He also thanks Professor Tsukasa Iwabuchi who pointed out a gap of the previous works [48], and introduced the strategy of Bejenaru-Tao [3]. This work is partly supported by IRTG 1529 (Japanese-German International Research Training Group) from JSPS (Japan Society for the Promotion of Science) and DFG (Deutsche Forschungsgemeinschaft).

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