A remark on the algebraic normal form method applied to the Dirac-Klein-Gordon system in two space dimensions

By

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Abstract

We consider the massive Dirac-Klein-Gordon system in two space dimensions. Under the non-resonance mass condition, we show that the solution is asymptotically free if the initial data are sufficiently small in a suitable weighted Sobolev space. In particular, it turns out that the Dirac component of the DKG system tends to a solution of the free Dirac equation. Our proof is based on the algebraic normal form method.

§1. Introduction

This paper is intended to give a remark on applications of the algebraic normal form method developed by [8], [6], [7], [3], [10], [9], [4], etc. The model equation which we focus on is the two-dimensional massive Dirac-Klein-Gordon system

\[
\begin{aligned}
\mathcal{D}_M \psi &= ig \phi \beta \psi, \\
(\Box + m^2) \phi &= g \langle \psi, \beta \psi \rangle_{\mathbb{C}^2},
\end{aligned}
\]

\((t, x) \in \mathbb{R} \times \mathbb{R}^2\)

with the initial condition

\[
(\psi, \phi, \partial_t \phi)|_{t=0} = (\psi_0, \phi_0, \phi_1), \quad x \in \mathbb{R}^2.
\]
Here $(\psi, \phi)$ is a $\mathbb{C}^2 \times \mathbb{R}$-valued unknown function of $(t, x) \in \mathbb{R} \times \mathbb{R}^2$. $M, m$ are positive constants, $g$ is a real constant, $\Box = \partial_t^2 - \Delta$, $\Delta = \partial_1^2 + \partial_2^2$, $\partial_j = \partial/\partial x_j$ ($j = 1, 2$) and $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ denotes the standard scalar product in $\mathbb{C}^2$, i.e., $\langle u, v \rangle_{\mathbb{C}^2} = u^\dagger v$ for $u, v \in \mathbb{C}^2$ (regarded as column vectors), where $u^\dagger$ is the complex conjugate transpose of $u$. The Dirac operator $\mathcal{D}_M$ is defined by

$$\mathcal{D}_M = \partial_t + \alpha_1 \partial_1 + \alpha_2 \partial_2 + iM \beta = \partial_t + \alpha \cdot \nabla_x + iM \beta$$

with $2 \times 2$ hermitian matrices $\alpha_1, \alpha_2, \beta$ satisfying

$$\alpha_1^2 = \alpha_2^2 = \beta^2 = I,$$

$$\alpha_1 \alpha_2 + \alpha_2 \alpha_1 = \alpha_1 \beta + \beta \alpha_1 = \alpha_2 \beta + \beta \alpha_2 = O.$$

We also set

$$\widetilde{\mathcal{D}}_M = \partial_t - (\alpha \cdot \nabla_x + iM \beta),$$

then we can easily check that the following relations hold:

$$\mathcal{D}_M \mathcal{D}_M = \mathcal{D}_M \mathcal{D}_M = (\Box + M^2)I.\tag{1.3}$$

This implies that the solution $(\psi, \phi)$ of (1.1)-(1.2) also solves

$$\begin{cases}
(\Box + M^2)\psi = ig \mathcal{D}_M (\phi \beta \psi), \\
(\Box + m^2)\phi = g \langle \psi, \beta \psi \rangle_{\mathbb{C}^2},
\end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2\tag{1.4}$$

with the initial condition

$$(\psi, \partial_t \psi, \phi, \partial_t \phi)|_{t=0} = (\psi_0, \psi_1, \phi_0, \phi_1), \quad x \in \mathbb{R}^2,\tag{1.5}$$

where $\psi_1 = - (\alpha \cdot \nabla_x + iM \beta) \psi_0 + ig \phi_0 \beta \psi_0$. According to Theorem 6.1 of [9] (see also [10], [4]), the solution of (1.4)-(1.5) exists globally in time if $m \neq 2M$ and the data are sufficiently small, smooth and decay fast as $|x| \rightarrow \infty$. Moreover there exists a solution $(\psi^\pm, \phi^\pm)$ of the free Klein-Gordon equation

$$\begin{cases}
(\Box + M^2) \psi^\pm = 0, \\
(\Box + m^2) \phi^\pm = 0,
\end{cases}$$

such that

$$\lim_{t \rightarrow \pm \infty} \sum_{j=0}^1 \left( \| \partial_t^j (\psi(t, \cdot) - \psi^\pm(t, \cdot)) \|_{H^{1-j}} + \| \partial_t^j (\phi(t, \cdot) - \phi^\pm(t, \cdot)) \|_{H^{1-j}} \right) = 0.$$

In this sense, the solution of (1.1)-(1.2) behaves like a solution of the free Klein-Gordon equations in the large time if $m \neq 2M$. However, this does not directly imply that
the solution is asymptotically free. What we emphasize here is that a solution \( u \) of the free Klein-Gordon equation \( (\Box + M^2)u = 0 \) is not necessarily a solution of the free Dirac equation \( \mathcal{D}_M u = 0 \) in general. So the following question arises: Does the Dirac component \( \psi(t) \) of (1.1) tend to a solution of the free Dirac equation as \( t \to \pm \infty ? \) As far as the authors know, there are no previous papers which address this question in the case of two space dimensions. There are several results in 3D case (see e.g., [1] and the references therein), however, those methods do not work well in 2D case because of the insufficiency of expected decay rate with respect to \( t \) of the nonlinear terms. We will give an affirmative answer to this question by using the algebraic normal form method.

To state the main result, let us introduce the weighted Sobolev space

\[
H^{s,k}(\mathbb{R}^2) = \{u \in L^2(\mathbb{R}^2) : (1 + |x|^2)^{k/2}(1 - \Delta)^{s/2}u \in L^2(\mathbb{R}^2)\}
\]
equipped with the norm

\[
\|u\|_{H^{s,k}(\mathbb{R}^2)} = \|(1 + |x|^2)^{k/2}(1 - \Delta)^{s/2}u\|_{L^2(\mathbb{R}^2)}.
\]

As usual, we write \( H^s = H^{s,0} \) and \( \|u\|_{H^s} = \|u\|_{H^{s,0}} \). Our main result is as follows.

**Theorem 1.1.** Let \( m \neq 2M \). Assume that \((\psi_0, \phi_0, \phi_1) \in H^{s+1,s} \times H^{s+1,s} \times H^{s,s}(\mathbb{R}^2)\) with \( s \geq 18 \). There exists a positive constant \( \epsilon \) such that if

\[
(1.6) \quad \|\psi_0\|_{H^{s+1,s}(\mathbb{R}^2)} + \|\phi_0\|_{H^{s+1,s}(\mathbb{R}^2)} + \|\phi_1\|_{H^{s,s}(\mathbb{R}^2)} \leq \epsilon,
\]

the Cauchy problem (1.1)–(1.2) admits a unique global solution \((\psi, \phi)\) satisfying

\[
\psi \in C(\mathbb{R}; H^{s+1}(\mathbb{R}^2)), \quad \phi \in \bigcap_{k=0}^{1} C^k([0, \infty); H^{s+1-k}(\mathbb{R}^2)).
\]

Furthermore, there exist \( \psi_0^\pm \in H^{s-1}(\mathbb{R}^2) \) and \( (\phi_0^\pm, \phi_1^\pm) \in H^{s-1} \times H^{s-2}(\mathbb{R}^2) \) such that

\[
\lim_{t \to \pm \infty} \|\psi(t, \cdot) - \psi^\pm(t, \cdot)\|_{H^{s-1}} = 0 \quad \text{and} \quad \lim_{t \to \pm \infty} \sum_{j=0}^{1} \|\partial_t^j(\phi(t, \cdot) - \phi^\pm(t, \cdot))\|_{H^{s-1-j}} = 0,
\]

where \( \psi^\pm \) and \( \phi^\pm \) are the solutions to

\[
\begin{aligned}
&\mathcal{D}_M \psi^\pm = 0 \\
&\psi^\pm|_{t=0} = \psi_0^\pm
\end{aligned}
\quad \text{and} \quad
\begin{aligned}
&(\Box + m^2)\phi^\pm = 0 \\
&(\phi^\pm, \partial_t \phi^\pm)|_{t=0} = (\phi_0^\pm, \phi_1^\pm),
\end{aligned}
\]

respectively.

**Remark 1.** The condition \( m \neq 2M \) is often called the non-resonance mass condition. Difficulties appearing in the resonant case \( (m = 2M) \) are explained in [5].
Recent work by the first author explores the final value problem for equation (1.1) in two space dimensions, successfully demonstrating the existence of wave operators under the non-resonance mass condition. See [2] for more details.

The paper is organized as follows: In the next section, we examine preliminary properties of vector fields associated with Klein-Gordon operators. Section 3 recaps and develops an algebraic normal form transformation. In Section 4, we derive an a priori estimate for the solution. Section 5 will prove Theorem 1.1. A few remarks will be included in the final section. Throughout, we use conventions on implicit constants:

- $A \lesssim B$ (resp. $A \gtrsim B$) indicates $A \leq CB$ (resp. $A \geq CB$) with a constant $C$.
- The expression $f = \sum'_{\kappa \in K} g_\kappa$ means there exists a family $\{C_\kappa\}_{\kappa \in K}$ such that $f = \sum_{\kappa \in K} C_\kappa g_\kappa$.

Also, the notation $\langle y \rangle = (1 + |y|^2)^{1/2}$ is used for $y \in \mathbb{R}^N$ with $N$ a positive integer.

§ 2. Commuting vector fields and the null forms

In this section, we summarize basic properties of some vector fields related to Klein-Gordon operators. We define $x_0 = -t$, $x = (x_1, x_2)$, $\Omega_{ab} = x_a \partial_b - x_b \partial_a$, $0 \leq a, b \leq 2$, $\partial = (\partial_t, \partial_1, \partial_2) = (\partial_t, \partial_{x_1}, \partial_{x_2})$, and $Z = (Z_1, \ldots, Z_6) = (\partial_0, \partial_1, \partial_2, \Omega_{01}, \Omega_{02}, \Omega_{12})$.

Note that the following commutation relations hold:

\begin{equation}
[\Box + m^2, Z_j] = 0,
\end{equation}

\begin{align*}
[\Omega_{ab}, \partial_c] &= \eta_{bc} \partial_a - \eta_{ca} \partial_b, \\
[\Omega_{ab}, \Omega_{cd}] &= \eta_{ad} \Omega_{bc} + \eta_{bc} \Omega_{ad} - \eta_{ac} \Omega_{bd} - \eta_{bd} \Omega_{ac}
\end{align*}

for $m \in \mathbb{R}, 1 \leq j \leq 6, 0 \leq a, b \leq 2$. Here $[\cdot, \cdot]$ denotes the commutator of linear operators, and $\eta_{ab} = \text{diag}(-1, 1, 1)$. Note that $\Box = -\sum_{a,b=0}^{2} \eta_{ab} \partial_a \partial_b$. For a smooth function $u$ of $(t, x) \in \mathbb{R}^{1+2}$ and for a non-negative integer $s$, we define

$$|u(t, x)|_s := \sum_{|\nu| \leq s} |Z^\nu u(t, x)|$$

and

$$\|u(t)\|_s := \sum_{|\nu| \leq s} \|Z^\nu u(t, \cdot)\|_{L^2(\mathbb{R}^2)},$$
where $\nu = (\nu_1, \ldots, \nu_6)$ is a multi-index, $Z^\nu = Z_1^{\nu_1} \cdots Z_6^{\nu_6}$ and $|\nu| = \nu_1 + \cdots + \nu_6$. Next we introduce the null form $Q_0$ and the strong null forms $Q_{ab}$ as follows:

\begin{align}
Q_0(u, v) &= -\sum_{a,b=0}^{2} \eta_{ab}(\partial_a u)(\partial_b v), \\
Q_{ab}(u, v) &= (\partial_a u)(\partial_b v) - (\partial_b u)(\partial_a v), \quad 0 \leq a, b \leq 2.
\end{align}

We summarize well known properties on the strong null forms.

**Lemma 2.1.** Let $u, v$ be smooth functions of $(t, x) \in \mathbb{R}^{1+2}$. We have

$$|Q_{ab}(u, v)| \lesssim \frac{1}{|t| + |x|} (|u|_{1}|\partial v| + |\partial u||v|_{1})$$

for $0 \leq a, b \leq 2$, and

$$Z^\nu Q_{ab}(u, v) = \sum_{c,d=0}^{2} \sum' Q_{cd}(Z^\lambda u, Z^\mu v)$$

for any multi-index $\nu$.

§3. Algebraic normal form transformation

This section is devoted to some decomposition of the nonlinear terms in (1.1).

Let $v_j$ and $\tilde{v}_j$ be smooth functions of $(t, x) \in \mathbb{R}^{1+2}$ (not necessarily scalar-valued), and let $m_1, m_2$ be real constants. We set $h_j = (\Box + m_j^2)v_j$ and $\tilde{h}_j = (\Box + m_j^2)\tilde{v}_j$ for $j = 1, 2$. We write

$$F \sim G$$

if $F - G$ can be written as a linear combination of $Q_{ab}(\partial^\mu v_k, \partial^\nu \tilde{v}_l)$, $(\partial^\mu v_k)(\partial^\nu \tilde{h}_l)$, $(\partial^\mu h_k)(\partial^\nu \tilde{v}_l)$ or $h_k \tilde{h}_l$ with $|\mu|, |\nu| \leq 1$, $0 \leq a, b \leq 2$ and $1 \leq k, l \leq 2$. The following lemma is important for our main purpose.

**Lemma 3.1** ([9], [4]). Put $e_{kl} = v_k \tilde{v}_l$, $\tilde{e}_{kl} = Q_0(v_k, \tilde{v}_l)$ and $L_j = \Box + m_j^2$, where $Q_0$ is given by (2.2). We have

$$(L_j(e_{kl}) L_j(\tilde{e}_{kl})) \sim (e_{kl} \cdot \tilde{e}_{kl})A_{jkl},$$

where

$$A_{jkl} = \begin{pmatrix}
m_j^2 - m_k^2 - m_l^2 & \frac{2m_j^2m_k^2}{2m_j^2 - m_k^2 - m_l^2} \\
\frac{2m_j^2m_k^2}{2m_j^2 - m_k^2 - m_l^2} & m_j^2 - m_k^2 - m_l^2
\end{pmatrix}.$$

Now we focus our attention to the structure of the matrix $A_{jkl}$. Since
\[
\det A_{jkl} = \prod_{\sigma_1, \sigma_2 \in \{\pm 1\}} (m_j + \sigma_1 m_k + \sigma_2 m_l),
\]
we see that $A_{121}$ and $A_{211}$ are invertible if $m_2 \neq 2m_1$. Moreover we have
\[
v_k \bar{v}_l = (e_{kl} \, \bar{e}_{kl}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}
= (e_{kl} \, \bar{e}_{kl}) A_{jkl} \begin{pmatrix} p_{jkl} \\ \bar{p}_{jkl} \end{pmatrix}
\sim (\mathcal{L}_j(e_{kl}) \mathcal{L}_j(\bar{e}_{kl})) \begin{pmatrix} p_{jkl} \\ \bar{p}_{jkl} \end{pmatrix}
= (\Box + m_j^2)(p_{jkl} v_k \bar{v}_l + \bar{p}_{jkl} Q_0(v_k, \bar{v}_l))
\]
with
\[
\begin{pmatrix} p_{jkl} \\ \bar{p}_{jkl} \end{pmatrix} = A_{jkl}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
for $(j, k, l) = (1, 2, 1)$ or $(2, 1, 1)$. By using the above formula with $(m_1, m_2, v_2, \bar{v}_1) = (M, m, \phi, \beta \psi)$ or $(m_1, m_2, v_1, \bar{v}_1) = (M, m, \psi^\dagger, \beta \psi)$, we arrive at the following decompositions for the nonlinear terms in (1.1):

**Corollary 3.2.** Let $(\psi, \phi)$ be a solution for (1.1) with $m \neq 2M$. We have
\[
\begin{cases}
ig \phi \beta \psi = \mathcal{D}_M(\mathcal{D}_M \Lambda_D) + N_D + R_D, \\
\langle \psi, \beta \psi \rangle_{\mathbb{C}^2} = (\Box + m^2)\Lambda_{KG} + N_{KG} + R_{KG},
\end{cases}
\]
where
\[
\Lambda_D = \sum'_{|\mu|, |\nu| \leq 1} (\partial^\mu \phi) \beta \partial^\nu \psi,
\]
\[
\Lambda_{KG} = \sum'_{|\mu|, |\nu| \leq 1} \langle \partial^\mu \psi, \beta \partial^\nu \psi \rangle_{\mathbb{C}^2},
\]
\[
N_D = \sum_{a,b=0}^2 \sum'_{|\mu|, |\nu| \leq 1} Q_{ab}(\partial^\mu \phi, \beta \partial^\nu \psi),
\]
\[
N_{KG} = \sum_{a,b=0}^2 \sum'_{|\mu|, |\nu| \leq 1} Q_{ab}(\partial^\mu \psi^\dagger, \beta \partial^\nu \psi),
\]
and $R_D$, $R_{KG}$ are smooth functions of $((\partial^\mu \psi)_{|\mu| \leq 2}, (\partial^\nu \phi)_{|\nu| \leq 2})$ which vanish of cubic order at $(0, 0)$. 
**Remark 3.** Roughly saying, the above assertion tells us that the right-hand side in (1.1) are splitted into two parts: The first one is the image of the corresponding linear operator, and the second one consists of faster decaying terms which can be regarded as harmless remainder when \( t \gg 1 \). By pushing the first part into the left-hand side, we can rewrite (1.1) as

\[
\begin{cases}
\mathcal{D}_M(\psi - \tilde{D}_M \Lambda_D) = N_D + R_D, \\
(\Box + m^2)(\phi - \Lambda_{KG}) = N_{KG} + R_{KG}.
\end{cases}
\]

This is what we call the normal form transformation.

\section*{§4. A priori estimate}

The goal of this section is to get some a priori estimate. From now on, we consider only the forward Cauchy problem (i.e., \( t > 0 \)) since the backward problem can be treated in the same way. Let \((\psi, \phi)\) be a solution of (1.1) for \( t \in [0, T) \). We define

\[
E(T) = \sup_{0 \leq t < T} \left[ t^{-\delta} \left( \|\psi(t)\|_s + \|\partial \psi(t)\|_s + \|\phi(t)\|_s + \|\partial \phi(t)\|_s + \|\psi(t)\|_{s-2} + \|\partial \psi(t)\|_{s-2} + \|\phi(t)\|_{s-2} + \|\partial \phi(t)\|_{s-2} + \sup_{x \in \mathbb{R}^2} \{ (t + |x|)(|\psi(t, x)|_{s-8} + |\phi(t, x)|_{s-8}) \} \right) \right],
\]

where \( s \geq 18 \) and \( 0 < \delta < 1 \). Then we have the following.

**Proposition 4.1.** Let \( m \neq 2M \). Assume that (1.6) is satisfied. Suppose that \( E(T) \leq 1 \). There exists a positive constant \( C_0 \), which is independent of \( \varepsilon \) and \( T \), such that

\[
E(T) \leq C_0(\varepsilon + E(T)^2).
\]

We omit the proof of this proposition because it is exactly the same as that of the previous works ([3], [9], [4], etc.). The point is that Corollary 3.2 and the commutation relation (2.1) imply

\[
\begin{cases}
(\Box + M^2)Z^\nu(\psi - \tilde{D}_M \Lambda_D) = Z^\nu \tilde{D}_M(N_D + R_D), \\
(\Box + m^2)Z^\nu(\phi - \Lambda_{KG}) = Z^\nu(N_{KG} + R_{KG})
\end{cases}
\]

with

\[
|Z^\nu \Lambda_*(t, x)| \lesssim |u||\nu|/2 + 1(|u|\nu| + |\partial u|\nu|),
\]

\[
|Z^\nu R_*(t, x)| \lesssim |u|^2||\nu|/2 + 2(|u|\nu| + 1 + |\partial u|\nu| + 1),
\]

where
and
\[ |Z\nu N_{*}(t, x)| \lesssim \frac{1}{\langle t + |x| \rangle} |u|_{|\nu|/2} + 2 (|u|_{|\nu|+1} + |\partial u|_{|\nu|+1}), \]
where \( u = (\psi, \phi) \), and \(* \) stands for “\( D \)” or “\( KG \)”. Remark that the restriction \( s \geq 18 \) comes from the relation \( [(s+1) - 2]/2 + 2 \leq s - 8 \).

§ 5. Proof of Theorem 1.1

Now we are ready to prove Theorem 1.1. First we examine the global existence part of the theorem. The inequality (4.1) implies that there exists a constant \( \rho > 0 \), which does not depend on \( T \), such that
\[ E(T) \leq \rho \]
if we choose \( \varepsilon \) sufficiently small. The unique global existence of the solution for (1.1)–(1.2) is an immediate consequence of this a priori bound and the classical local existence theorem.

Next we turn to the proof of the existence of the scattering state. Remember that
\[ \mathcal{D}_{M}(\psi - \mathcal{D}_{M}\Lambda_{D}) = N_{D} + R_{D} \]
with
\[ \|(N_{D} + R_{D})(t, \cdot)\|_{H^{s-1}} \lesssim \langle t \rangle^{-2+\delta}, \]
\[ \|\mathcal{D}_{M}\Lambda_{D}(t, \cdot)\|_{H^{s-1}} \lesssim \langle t \rangle^{-1+\delta}. \]
Now we set
\[ \psi_{0}^{+} := \psi_{0} - (\mathcal{D}_{M}\Lambda_{D})|_{t=0} + \int_{0}^{\infty} U_{D}(-\tau)(N_{D} + R_{D})(\tau) d\tau \]
and \( \psi^{+}(t) = U_{D}(t)\psi_{0}^{+} \), where \( U_{D}(t) = \exp(-t(\alpha \cdot \nabla_{x} + iM \beta)) \). Since the Duhamel formula yields
\[ \psi(t) - \mathcal{D}_{M}\Lambda_{D}(t) = U_{D}(t)(\psi_{0} - (\mathcal{D}_{M}\Lambda_{D})|_{t=0}) + \int_{0}^{t} U_{D}(t - \tau)(N_{D} + R_{D})(\tau) d\tau \]
\[ = \psi^{+}(t) - \int_{t}^{\infty} U_{D}(t - \tau)(N_{D} + R_{D})(\tau) d\tau, \]
we have
\[ \|\psi(t) - \psi^{+}(t)\|_{H^{s-1}} \leq \|\mathcal{D}_{M}\Lambda_{D}(t)\|_{H^{s-1}} + \int_{t}^{\infty} \|(N_{D} + R_{D})(\tau)\|_{H^{s-1}} d\tau \]
\[ \lesssim \langle t \rangle^{-1+\delta} + \int_{t}^{\infty} \langle \tau \rangle^{-2+\delta} d\tau \]
\[ \lesssim \langle t \rangle^{-1+\delta}. \]
As for the Klein-Gordon component, we just have to set
\[
\phi_0^+ = \phi_0 - \Lambda_{KG}\big|_{t=0} + \int_0^\infty \frac{\sin(-\tau\Omega_m)}{\Omega_m} (N_{KG} + R_{KG})(\tau, \cdot) d\tau,
\]
\[
\phi_1^+ = \phi_1 - \partial_t \Lambda_{KG}\big|_{t=0} + \int_0^\infty \cos(-\tau\Omega_m) (N_{KG} + R_{KG})(\tau, \cdot) d\tau
\]
with \(\Omega_m = (m^2 - \Delta)^{1/2}\).

\[
\square
\]

§ 6. Additional Remarks

We add a few remarks concerning the arguments in the preceding sections. After submitting the first version of this paper, the authors are informed by the referee that the following argument gives an alternative proof of Theorem 1.1: “Suppose that we already know
\[
\lim_{t \to \infty} \sum_{j=0}^1 \|\partial_t^j(\psi(t) - \psi^+(t))\|_{H^{1-j}} = 0
\]
with
\[
(\Box + M^2)\psi^+ = 0.
\]
Then, since
\[
\|\mathcal{D}_M(\psi^+(\tau) - \psi(\tau))\|_{L^2} \lesssim \sum_{j=0}^1 \|\partial^j_\tau(\psi(\tau) - \psi^+(\tau))\|_{H^{1-j}}
\]
and
\[
\|\mathcal{D}_M\psi(\tau)\|_{L^2} \lesssim \|\phi(\tau)\|_{L^\infty} \|\psi(\tau)\|_{L^2} \lesssim \rho^2(\tau)^{-1},
\]
we have
\[
\|\mathcal{D}_M\psi^+(\tau)\|_{L^2} \leq \|\mathcal{D}_M(\psi^+(\tau) - \psi(\tau))\|_{L^2} + \|\mathcal{D}_M\psi(\tau)\|_{L^2} \to 0
\]
as \(\tau \to \infty\). On the other hand, because of the \(L^2\)-conservation law for the equation \(\mathcal{D}_M u = 0\), it follows from (6.2) and (1.3) that
\[
\|\mathcal{D}_M\psi^+(t)\|_{L^2} = \|\mathcal{D}_M\psi^+(\tau)\|_{L^2}
\]
for any \(t, \tau \in \mathbb{R}\). This implies \(\mathcal{D}_M\psi^+ = 0\).” The above argument does not require (3.1) explicitly. However, it should be remembered that the proof of (6.1)–(6.2) relies heavily on the normal form transformation for the reduced Klein-Gordon system in the previous paper [9]. In contrast, our proof of Theorem 1.1 avoids the use of (6.1)–(6.2); we only use (3.1) and show directly that \(\psi(t)\) tends to a solution to the free Dirac equation. What is important in our approach is to apply the normal form transformation for the original system (1.1), not for the reduced system (1.4).
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