RIMS Kôkyûroku Bessatsu
B33 (2012), 87–96

A remark on the algebraic normal form method applied to the Dirac-Klein-Gordon system in two space dimensions

By

Masahiro IKEDA*, Akihiro SHIMOMURA** and Hideaki SUNAGAWA***

Abstract

We consider the massive Dirac-Klein-Gordon system in two space dimensions. Under the non-resonance mass condition, we show that the solution is asymptotically free if the initial data are sufficiently small in a suitable weighted Sobolev space. In particular, it turns out that the Dirac component of the DKG system tends to a solution of the free Dirac equation. Our proof is based on the algebraic normal form method.

§ 1. Introduction

This paper is intended to give a remark on applications of the algebraic normal form method developed by [8], [6], [7], [3], [10], [9], [4], etc. The model equation which we focus on is the two-dimensional massive Dirac-Klein-Gordon system

$$\begin{cases} \mathcal{D}_M\psi = ig\phi\beta\psi, \\ (\Box + m^2)\phi = g\langle\psi, \beta\psi\rangle_{\mathbb{C}^2}, \end{cases} \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^2 \tag{1.1}$$

with the initial condition

$$(\psi, \phi, \partial_t\phi)|_{t=0} = (\psi_0, \phi_0, \phi_1), \qquad x \in \mathbb{R}^2. \tag{1.2}$$

Received October 31, 2011. Revised January 17, 2012.
2000 Mathematics Subject Classification(s): 35L71, 35B40.
Key Words: Normal form method; Dirac-Klein-Gordon system.
*Department of Mathematics, Graduate School of Science, Osaka University. 1-1 Machikaneyama-cho, Toyonaka, Osaka 560-0043, Japan.
e-mail: `m-ikeda@cr.math.sci.osaka-u.ac.jp`
**Graduate School of Mathematical Sciences, The University of Tokyo. 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914 Japan.
e-mail: `simomura@ms.u-tokyo.ac.jp`
***Department of Mathematics, Graduate School of Science, Osaka University. 1-1 Machikaneyama-cho, Toyonaka, Osaka 560-0043, Japan.
e-mail: `sunagawa@math.sci.osaka-u.ac.jp`

Here (ψ, ϕ) is a $\mathbb{C}^2 \times \mathbb{R}$-valued unknown function of $(t,x) \in \mathbb{R} \times \mathbb{R}^2$. M, m are positive constants, g is a real constant, $\Box = \partial_t^2 - \Delta$, $\Delta = \partial_1^2 + \partial_2^2$, $\partial_j = \partial/\partial x_j$ $(j = 1, 2)$ and $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ denotes the standard scalar product in $\mathbb{C}^2$, i.e., $\langle u, v\rangle_{\mathbb{C}^2} = u^\dagger v$ for $u, v \in \mathbb{C}^2$ (regarded as column vectors), where $u^\dagger$ is the complex conjugate transpose of u. The Dirac operator $\mathcal{D}_M$ is defined by

$$\mathcal{D}_M = \partial_t + \alpha_1\partial_1 + \alpha_2\partial_2 + iM\beta = \partial_t + \alpha \cdot \nabla_x + iM\beta$$

with 2×2 hermitian matrices α_1, α_2, β satisfying

$$\begin{aligned} &\alpha_1^2 = \alpha_2^2 = \beta^2 = I, \\ &\alpha_1\alpha_2 + \alpha_2\alpha_1 = \alpha_1\beta + \beta\alpha_1 = \alpha_2\beta + \beta\alpha_2 = O. \end{aligned}$$

We also set

$$\widetilde{\mathcal{D}}_M = \partial_t - (\alpha \cdot \nabla_x + iM\beta),$$

then we can easily check that the following relations hold:

$$\mathcal{D}_M\widetilde{\mathcal{D}}_M = \widetilde{\mathcal{D}}_M\mathcal{D}_M = (\Box + M^2)I. \tag{1.3}$$

This implies that the solution (ψ, ϕ) of (1.1)–(1.2) also solves

$$\begin{cases} (\Box + M^2)\psi = ig\widetilde{\mathcal{D}}_M(\phi\beta\psi), \\ (\Box + m^2)\phi = g\langle \psi, \beta\psi\rangle_{\mathbb{C}^2}, \end{cases} \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2 \tag{1.4}$$

with the initial condition

$$(\psi, \partial_t\psi, \phi, \partial_t\phi)|_{t=0} = (\psi_0, \psi_1, \phi_0, \phi_1), \qquad x \in \mathbb{R}^2, \tag{1.5}$$

where $\psi_1 = -(\alpha \cdot \nabla_x + iM\beta)\psi_0 + ig\phi_0\beta\psi_0$. According to Theorem 6.1 of [9] (see also [10], [4]), the solution of (1.4)-(1.5) exists globally in time if $m \neq 2M$ and the data are sufficiently small, smooth and decay fast as $|x| \to \infty$. Moreover there exists a solution $(\psi^\pm, \phi^\pm)$ of the free Klein-Gorodon equation

$$\begin{cases} (\Box + M^2)\psi^\pm = 0, \\ (\Box + m^2)\phi^\pm = 0, \end{cases}$$

such that

$$\lim_{t\to\pm\infty} \sum_{j=0}^{1} \Big(\|\partial_t^j(\psi(t,\cdot) - \psi^\pm(t,\cdot))\|_{H^{1-j}} + \|\partial_t^j(\phi(t,\cdot) - \phi^\pm(t,\cdot))\|_{H^{1-j}} \Big) = 0.$$

In this sense, the solution of (1.1)–(1.2) behaves like a solution of the free Klein-Gordon equations in the large time if $m \neq 2M$. However, this does not directly imply that the solution is asymptotically free. What we emphasize here is that a solution u of the free Klein-Gordon equation $(\Box + M^2)u = 0$ is not necessarily a solution of the free Dirac equation $\mathcal{D}_M u = 0$ in general. So the following question arises: *Does the Dirac componet $\psi(t)$ of (1.1) tend to a solution of the free Dirac quation as $t \to \pm\infty$?* As far as the authors know, there are no previous papers which address this question in the case of two space dimensions. There are several results in 3D case (see e.g., [1] and the referances therein), however, those methods do not work well in 2D case because of the insufficiency of expected decay rate with respect to t of the nonlinear terms. We will give an affirmative answer to this question by using the algebraic normal form method.

To state the main result, let us introduce the weighted Sobolev space

$$H^{s,k}(\mathbb{R}^2) = \{u \in L^2(\mathbb{R}^2) \,:\, (1 + |\cdot|^2)^{k/2}(1-\Delta)^{s/2}u \in L^2(\mathbb{R}^2)\}$$

equipped with the norm

$$\|u\|_{H^{s,k}(\mathbb{R}^2)} = \|(1 + |\cdot|^2)^{k/2}(1-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^2)}.$$

As usual, we write $H^s = H^{s,0}$ and $\|u\|_{H^s} = \|u\|_{H^{s,0}}$. Our main result is as follows.

Theorem 1.1. *Let $m \neq 2M$. Assume that $(\psi_0, \phi_0, \phi_1) \in H^{s+1,s} \times H^{s+1,s} \times H^{s,s}(\mathbb{R}^2)$ with $s \geq 18$. There exists a positive constant ε such that if*

$$\|\psi_0\|_{H^{s+1,s}(\mathbb{R}^2)} + \|\phi_0\|_{H^{s+1,s}(\mathbb{R}^2)} + \|\phi_1\|_{H^{s,s}(\mathbb{R}^2)} \leq \varepsilon, \tag{1.6}$$

the Cauchy problem (1.1)–(1.2) *admits a unique global solution (ψ, ϕ) satisfying*

$$\psi \in C(\mathbb{R}; H^{s+1}(\mathbb{R}^2)), \quad \phi \in \bigcap_{k=0}^{1} C^k([0,\infty); H^{s+1-k}(\mathbb{R}^2)).$$

Furthermore, there exist $\psi_0^\pm \in H^{s-1}(\mathbb{R}^2)$ and $(\phi_0^\pm, \phi_1^\pm) \in H^{s-1} \times H^{s-2}(\mathbb{R}^2)$ such that

$$\lim_{t\to\pm\infty} \|\psi(t,\cdot) - \psi^\pm(t,\cdot)\|_{H^{s-1}} = 0 \quad \textit{and} \quad \lim_{t\to\pm\infty} \sum_{j=0}^{1} \|\partial_t^j(\phi(t,\cdot) - \phi^\pm(t,\cdot))\|_{H^{s-1-j}} = 0,$$

where $\psi^\pm$ and $\phi^\pm$ are the solutions to

$$\begin{cases} \mathcal{D}_M \psi^\pm = 0 \\ \psi^\pm|_{t=0} = \psi_0^\pm \end{cases} \quad \textit{and} \quad \begin{cases} (\Box + m^2)\phi^\pm = 0 \\ (\phi^\pm, \partial_t\phi^\pm)|_{t=0} = (\phi_0^\pm, \phi_1^\pm), \end{cases}$$

respectively.

Remark 1. The condition $m \neq 2M$ is often called the non-resonance mass condition. Difficulties appearing in the resonant case $(m = 2M)$ are explained in [5].

Remark 2. Recently, the first author considered the final value problem for (1.1) in two space dimensions and succeeded in showing the existence of wave operators for (1.1) under the non-resonance mass condition. See [2] for the detail.

The rest of this paper is organized as follows: In the next section, we give some preliminaries mainly on the commuting vector fields and the null forms. In Section 3, we recall and develop an algebraic normal form transformation. We will get an a priori esimate of the solution in Section 4. After that, Theorem 1.1 will be proved in Section 5. A few remarks will be given in the final section. Throughout this paper, we will frequently use the following conventions on implicit constants:

- $A \lesssim B$ (resp. $A \gtrsim B$) stands for $A \le CB$ (resp. $A \ge CB$) with a positive constant C.
- The expression $f = \sum'_{\kappa \in K} g_\kappa$ means that there exists a family $\{C_\kappa\}_{\kappa \in K}$ of constants such that $f = \sum_{\kappa \in K} C_\kappa g_\kappa$.

Also, the notation $\langle y \rangle = (1 + |y|^2)^{1/2}$ will be used for $y \in \mathbb{R}^N$ with a positive integer N.

§ 2. Commuting vector fields and the null forms

In this section, we summarize basic properties of some vector fields associated with the Klein-Gordon operators. We put $x_0 = -t$, $x = (x_1, x_2)$, $\Omega_{ab} = x_a \partial_b - x_b \partial_a$, $0 \le a, b \le 2$, $\partial = (\partial_0, \partial_1, \partial_2) = (\partial_t, \partial_{x_1}, \partial_{x_2})$ and

$$Z = (Z_1, \dots, Z_6) = \big(\partial_0, \partial_1, \partial_2, \Omega_{01}, \Omega_{02}, \Omega_{12}\big).$$

Note that the following commutation relations hold:

$$\begin{aligned} &[\Box + m^2, Z_j] = 0, \\ &[\Omega_{ab}, \partial_c] = \eta_{bc}\partial_a - \eta_{ca}\partial_b, \\ &[\Omega_{ab}, \Omega_{cd}] = \eta_{ad}\Omega_{bc} + \eta_{bc}\Omega_{ad} - \eta_{ac}\Omega_{bd} - \eta_{bd}\Omega_{ac} \end{aligned} \tag{2.1}$$

for $m \in \mathbb{R}$, $1 \le j \le 6$, $0 \le a, b \le 2$. Here $[\cdot, \cdot]$ denotes the commutator of linear operators, and $(\eta_{ab})_{0 \le a, b \le 2} = \mathrm{diag}(-1, 1, 1)$. Note that $\Box = -\sum_{a,b=0}^{2} \eta_{ab}\partial_a\partial_b$. For a smooth function u of $(t, x) \in \mathbb{R}^{1+2}$ and for a non-negative integer s, we define

$$|u(t,x)|_s := \sum_{|\nu| \le s} |Z^\nu u(t,x)|$$

and

$$\|u(t)\|_s := \sum_{|\nu| \le s} \|Z^\nu u(t, \cdot)\|_{L^2(\mathbb{R}^2)},$$

where $\nu = (\nu_1, \ldots, \nu_6)$ is a multi-index, $Z^\nu = Z_1^{\nu_1} \cdots Z_6^{\nu_6}$ and $|\nu| = \nu_1 + \cdots + \nu_6$. Next we introduce the null form Q_0 and the strong null forms Q_{ab} as follows:

$$Q_0(u,v) = -\sum_{a,b=0}^{2} \eta_{ab}(\partial_a u)(\partial_b v), \tag{2.2}$$

$$Q_{ab}(u,v) = (\partial_a u)(\partial_b v) - (\partial_b u)(\partial_a v), \quad 0 \le a, b \le 2. \tag{2.3}$$

We summarize well known properties on the strong null forms.

Lemma 2.1. *Let u, v be smooth functions of $(t,x) \in \mathbb{R}^{1+2}$. We have*

$$|Q_{ab}(u,v)| \lesssim \frac{1}{\langle |t| + |x| \rangle}\big(|u|_1|\partial v| + |\partial u||v|_1\big)$$

for $0 \le a, b \le 2$, and

$$Z^\nu Q_{ab}(u,v) = \sum_{c,d=0}^{2} \sideset{}{'}\sum_{|\lambda|+|\mu| \le |\nu|} Q_{cd}(Z^\lambda u, Z^\mu v)$$

for any multi-index ν.

§ 3. Algebraic normal form transformation

This section is devoted to some decomposition of the nonlinear terms in (1.1).

Let v_j and $\widetilde{v}_j$ be smooth functions of $(t,x) \in \mathbb{R}^{1+2}$ (not necessarily scalar-valued), and let m_1, m_2 be real constants. We set $h_j = (\Box + m_j^2)v_j$ and $\widetilde{h}_j = (\Box + m_j^2)\widetilde{v}_j$ for $j = 1, 2$. We write

$$F \sim G$$

if $F - G$ can be written as a linear combination of $Q_{ab}(\partial^\mu v_k, \partial^\nu \widetilde{v}_l)$, $(\partial^\mu v_k)(\partial^\nu \widetilde{h}_l)$, $(\partial^\mu h_k)(\partial^\nu \widetilde{v}_l)$ or $h_k \widetilde{h}_l$ with $|\mu|$, $|\nu| \le 1$, $0 \le a, b \le 2$ and $1 \le k, l \le 2$. The following lemma is important for our main purpose.

Lemma 3.1 ([9], [4]). *Put $\mathbf{e}_{kl} = v_k \widetilde{v}_l$, $\tilde{\mathbf{e}}_{kl} = Q_0(v_k, \widetilde{v}_l)$ and $\mathcal{L}_j = \Box + m_j^2$, where Q_0 is given by (2.2). We have*

$$\big(\mathcal{L}_j(\mathbf{e}_{kl}) \quad \mathcal{L}_j(\tilde{\mathbf{e}}_{kl})\big) \sim (\mathbf{e}_{kl} \quad \tilde{\mathbf{e}}_{kl}) A_{jkl},$$

where

$$A_{jkl} = \begin{pmatrix} m_j^2 - m_k^2 - m_l^2 & 2m_k^2 m_l^2 \\ 2 & m_j^2 - m_k^2 - m_l^2 \end{pmatrix}.$$

See Proposition 4.1 of [4] or Lemma 6.1 of [9] for the proof of this lemma. Remark that the proof remains valid in the vector-valued case.

Now we focus our attention to the structure of the matrix A_{jkl}. Since

$$\det A_{jkl} = \prod_{\sigma_1,\sigma_2\in\{\pm 1\}} (m_j + \sigma_1 m_k + \sigma_2 m_l),$$

we see that A_{121} and A_{211} are invertible if $m_2 \neq 2m_1$. Moreover we have

$$\begin{aligned} v_k\widetilde{v}_l &= (\mathbf{e}_{kl} \quad \tilde{\mathbf{e}}_{kl}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= (\mathbf{e}_{kl} \quad \tilde{\mathbf{e}}_{kl}) A_{jkl} \begin{pmatrix} p_{jkl} \\ \tilde{p}_{jkl} \end{pmatrix} \\ &\sim (\mathcal{L}_j(\mathbf{e}_{kl}) \quad \mathcal{L}_j(\tilde{\mathbf{e}}_{kl})) \begin{pmatrix} p_{jkl} \\ \tilde{p}_{jkl} \end{pmatrix} \\ &= (\Box + m_j^2)\Big(p_{jkl}v_k\widetilde{v}_l + \tilde{p}_{jkl}Q_0(v_k,\widetilde{v}_l)\Big) \end{aligned}$$

with

$$\begin{pmatrix} p_{jkl} \\ \tilde{p}_{jkl} \end{pmatrix} = A_{jkl}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for $(j,k,l) = (1,2,1)$ or $(2,1,1)$. By using the above formula with $(m_1, m_2, v_2, \widetilde{v}_1) = (M, m, \phi, \beta\psi)$ or $(m_1, m_2, v_1, \widetilde{v}_1) = (M, m, \psi^\dagger, \beta\psi)$, we arrive at the following decompositions for the nonlinear terms in (1.1) :

Corollary 3.2. *Let (ψ, ϕ) be a solution for (1.1) with $m \neq 2M$. We have*

$$\begin{cases} ig\phi\beta\psi = \mathcal{D}_M(\widetilde{\mathcal{D}}_M\Lambda_D) + N_D + R_D, \\ g\langle\psi, \beta\psi\rangle_{\mathbb{C}^2} = (\Box + m^2)\Lambda_{KG} + N_{KG} + R_{KG}, \end{cases}$$

where

$$\begin{aligned} \Lambda_D &= {\sum_{|\mu|,|\nu|\le 1}}' (\partial^\mu\phi)\beta\partial^\nu\psi, \\ \Lambda_{KG} &= {\sum_{|\mu|,|\nu|\le 1}}' \langle\partial^\mu\psi, \beta\partial^\nu\psi\rangle_{\mathbb{C}^2}, \\ N_D &= \sum_{a,b=0}^{2} {\sum_{|\mu|,|\nu|\le 1}}' Q_{ab}(\partial^\mu\phi, \beta\partial^\nu\psi), \\ N_{KG} &= \sum_{a,b=0}^{2} {\sum_{|\mu|,|\nu|\le 1}}' Q_{ab}(\partial^\mu\psi^\dagger, \beta\partial^\nu\psi), \end{aligned}$$

and R_D, R_{KG} are smooth functions of $((\partial^\mu\psi)_{|\mu|\le 2}, (\partial^\nu\phi)_{|\nu|\le 2})$ which vanish of cubic order at $(0,0)$.

Remark 3. Roughly saying, the above assertion tells us that the right-hand side in (1.1) are splitted into two parts: The first one is the image of the corresponding linear operator, and the second one consists of faster decaying terms which can be regarded as harmless remainder when $t \gg 1$. By pushing the first part into the left-hand side, we can rewrite (1.1) as

$$\begin{cases} \mathcal{D}_M(\psi - \widetilde{\mathcal{D}}_M \Lambda_D) = N_D + R_D, \\ (\Box + m^2)(\phi - \Lambda_{KG}) = N_{KG} + R_{KG}. \end{cases} \tag{3.1}$$

This is what we call the normal form transformation.

§ 4. A priori estimate

The goal of this section is to get some a priori estimate. From now on, we consider only the forward Cauchy problem (i.e., $t > 0$) since the backward problem can be treated in the same way. Let (ψ, ϕ) be a solution of (1.1)–(1.2) for $t \in [0, T)$. We define

$$\begin{aligned} E(T) = \sup_{0 \le t < T} \Big[&\langle t \rangle^{-\delta} \big(\|\psi(t)\|_s + \|\partial\psi(t)\|_s + \|\phi(t)\|_s + \|\partial\phi(t)\|_s \big) \\ &+ \|\psi(t)\|_{s-2} + \|\partial\psi(t)\|_{s-2} + \|\phi(t)\|_{s-2} + \|\partial\phi(t)\|_{s-2} \\ &+ \sup_{x \in \mathbb{R}^2} \big\{ \langle t + |x| \rangle (|\psi(t,x)|_{s-8} + |\phi(t,x)|_{s-8}) \big\} \Big], \end{aligned}$$

where $s \ge 18$ and $0 < \delta < 1$. Then we have the following.

Proposition 4.1. *Let $m \neq 2M$. Assume that* (1.6) *is satisfied. Suppose that $E(T) \le 1$. There exists a positive constant C_0, which is independent of ε and T, such that*

$$E(T) \le C_0(\varepsilon + E(T)^2). \tag{4.1}$$

We omit the proof of this proposition because it is exactly the same as that of the previous works ([3], [9], [4], etc.). The point is that Corollary 3.2 and the commutation relation (2.1) imply

$$\begin{cases} (\Box + M^2) Z^\nu (\psi - \widetilde{\mathcal{D}}_M \Lambda_D) = Z^\nu \widetilde{\mathcal{D}}_M (N_D + R_D), \\ (\Box + m^2) Z^\nu (\phi - \Lambda_{KG}) = Z^\nu (N_{KG} + R_{KG}) \end{cases}$$

with

$$\begin{aligned} |Z^\nu \Lambda_*(t,x)| &\lesssim |u|_{[|\nu|/2]+1} (|u|_{|\nu|} + |\partial u|_{|\nu|}), \\ |Z^\nu R_*(t,x)| &\lesssim |u|^2_{[|\nu|/2]+2} \big(|u|_{|\nu|+1} + |\partial u|_{|\nu|+1} \big), \end{aligned}$$

and

$$|Z^\nu N_*(t,x)| \lesssim \frac{1}{\langle t+|x|\rangle}|u|_{[|\nu|/2]+2}\big(|u|_{|\nu|+1}+|\partial u|_{|\nu|+1}\big),$$

where $u=(\psi,\phi)$, and $*$ stands for "D" or "KG". Remark that the restriction $s \geq 18$ comes from the relation $[(s+1)-2]/2+2 \leq s-8$.

§ 5. Proof of Theorem 1.1

Now we are ready to prove Theorem 1.1. First we examine the global existence part of the theorem. The inequality (4.1) implies that there exists a constant $\rho > 0$, which does not depend on T, such that

$$E(T) \leq \rho$$

if we choose ε sufficiently small. The unique global existence of the solution for (1.1)–(1.2) is an immediate consequence of this a priori bound and the classical local existence theorem.

Next we turn to the proof of the existence of the scattering state. Remember that

$$\mathcal{D}_M(\psi - \widetilde{\mathcal{D}}_M\Lambda_D) = N_D + R_D$$

with

$$\begin{aligned}&\|(N_D+R_D)(t,\cdot)\|_{H^{s-1}} \lesssim \langle t\rangle^{-2+\delta},\\ &\|\widetilde{\mathcal{D}}_M\Lambda_D(t,\cdot)\|_{H^{s-1}} \lesssim \langle t\rangle^{-1+\delta}.\end{aligned}$$

Now we set

$$\psi_0^+ := \psi_0 - (\widetilde{\mathcal{D}}_M\Lambda_D)|_{t=0} + \int_0^\infty U_D(-\tau)(N_D+R_D)(\tau)d\tau$$

and $\psi^+(t) = U_D(t)\psi_0^+$, where $U_D(t) = \exp(-t(\alpha\cdot\nabla_x + iM\beta))$. Since the Duhamel formula yields

$$\begin{aligned}\psi(t) - \widetilde{\mathcal{D}}_M\Lambda_D(t) &= U_D(t)(\psi_0 - (\widetilde{\mathcal{D}}_M\Lambda_D)|_{t=0}) + \int_0^t U_D(t-\tau)(N_D+R_D)(\tau)d\tau\\ &= \psi^+(t) - \int_t^\infty U_D(t-\tau)(N_D+R_D)(\tau)d\tau,\end{aligned}$$

we have

$$\begin{aligned}\|\psi(t)-\psi^+(t)\|_{H^{s-1}} &\leq \|\widetilde{\mathcal{D}}_M\Lambda_D(t)\|_{H^{s-1}} + \int_t^\infty \|(N_D+R_D)(\tau)\|_{H^{s-1}}d\tau\\ &\lesssim \langle t\rangle^{-1+\delta} + \int_t^\infty \langle\tau\rangle^{-2+\delta}d\tau\\ &\lesssim \langle t\rangle^{-1+\delta}.\end{aligned}$$

As for the Klein-Gordon component, we just have to set

$$\phi_0^+ = \phi_0 - \Lambda_{KG}\big|_{t=0} + \int_0^\infty \frac{\sin(-\tau\mathbf{\Omega}_m)}{\mathbf{\Omega}_m}(N_{KG}+R_{KG})(\tau,\cdot)d\tau,$$
$$\phi_1^+ = \phi_1 - \partial_t\Lambda_{KG}\big|_{t=0} + \int_0^\infty (\cos(-\tau\mathbf{\Omega}_m))\,(N_{KG}+R_{KG})(\tau,\cdot)d\tau$$

with $\mathbf{\Omega}_m = (m^2-\Delta)^{1/2}$. □

§ 6. Additional Remarks

We add a few remarks concerning the arguments in the preceding sections. After submitting the first version of this paper, the authors are informed by the referee that the following argument gives an alternative proof of Theorem 1.1: "Suppose that we already know

$$\lim_{t\to\infty}\sum_{j=0}^{1}\|\partial_t^j(\psi(t)-\psi^+(t))\|_{H^{1-j}} = 0 \tag{6.1}$$

with

$$(\Box + M^2)\psi^+ = 0. \tag{6.2}$$

Then, since

$$\|\mathcal{D}_M(\psi^+(\tau)-\psi(\tau))\|_{L^2} \lesssim \sum_{j=0}^{1}\|\partial_\tau^j(\psi(\tau)-\psi^+(\tau))\|_{H^{1-j}}$$

and

$$\|\mathcal{D}_M\psi(\tau)\|_{L^2} \lesssim \|\phi(\tau)\|_{L^\infty}\|\psi(\tau)\|_{L^2} \lesssim \rho^2\langle\tau\rangle^{-1},$$

we have

$$\|\mathcal{D}_M\psi^+(\tau)\|_{L^2} \le \|\mathcal{D}_M(\psi^+(\tau)-\psi(\tau))\|_{L^2} + \|\mathcal{D}_M\psi(\tau)\|_{L^2} \to 0$$

as $\tau\to\infty$. On the other hand, because of the L^2-conservation law for the equation $\widetilde{\mathcal{D}}_M u = 0$, it follows from (6.2) and (1.3) that

$$\|\mathcal{D}_M\psi^+(t)\|_{L^2} = \|\mathcal{D}_M\psi^+(\tau)\|_{L^2}$$

for any t, $\tau\in\mathbb{R}$. This implies $\mathcal{D}_M\psi^+ = 0$." The above argument does not require (3.1) explicitly. However, it should be remembered that the proof of (6.1)–(6.2) relies heavily on the normal form transformation for the reduced Klein-Gordon system in the previous paper [9]. In contrast, our proof of Theorem 1.1 avoids the use of (6.1)–(6.2); we only use (3.1) and show directly that $\psi(t)$ tends to a solution to the free Dirac equation. What is important in our approach is to apply the normal form transformation for the original system (1.1), not for the reduced system (1.4).

Acknowledgments

The authors would like to thank the referee for valuable comments and suggestions. The first author (M.I.) is partially supported by Grant-in-Aid for JSPS Fellows 23-2083. The second author (A.S.) is partially supported by JSPS, Grant-in-Aid for Young Scientists (B) 21740102. The third author (H.S.) is partially supported by JSPS, Grant-in-Aid for Young Scientists (B) 22740089.

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