RIMS Kôkyûroku Bessatsu
B33 (2012), 111–121

A remark on multilinear Fourier multipliers satisfying Besov estimates

By

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Abstract

By using the L^r-based Sobolev space $H^r_s(\mathbb{R}^{Nn})$ with $1 < r \le 2$ and $s > Nn/r$, Grafakos and Si [6] proved the boundedness of multilinear Fourier multiplier operators. In this paper, we try to replace $H^r_s(\mathbb{R}^{Nn})$ by the Besov space $B^{r,1}_{Nn/r}(\mathbb{R}^{Nn})$ as the critical case for their result.

§ 1. Introduction

For $m \in L^\infty(\mathbb{R}^{Nn})$, the N-linear Fourier multiplier operator T_m is defined by

$$T_m(f_1, \dots, f_N)(x) = \frac{1}{(2\pi)^{Nn}} \int_{\mathbb{R}^{Nn}} e^{ix\cdot(\xi_1+\cdots+\xi_N)} m(\xi_1, \dots, \xi_N) \widehat{f_1}(\xi_1) \dots \widehat{f_N}(\xi_N)\, d\xi_1 \dots d\xi_N$$

for $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$. As the classical Coifman-Meyer theorem [1], it is well known that if $m \in C^L(\mathbb{R}^{Nn} \setminus \{0\})$ satisfies

$$|\partial^{\alpha_1}_{\xi_1} \dots \partial^{\alpha_N}_{\xi_N} m(\xi_1, \dots, \xi_N)| \le C_{\alpha_1,\dots,\alpha_N} (|\xi_1| + \cdots + |\xi_N|)^{-(|\alpha_1|+\cdots+|\alpha_N|)}$$

for all $|\alpha_1| + \cdots + |\alpha_N| \le L$, where L is a sufficiently large natural number, then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_N \le \infty$ and $1 < p < \infty$ satisfying $1/p_1 + \cdots + 1/p_N = 1/p$.

Let $\Psi \in \mathcal{S}(\mathbb{R}^d)$ be such that

$$\text{(1.1)} \qquad \operatorname{supp} \Psi \subset \{\xi \in \mathbb{R}^d : 1/2 \le |\xi| \le 2\}, \quad \sum_{k\in\mathbb{Z}} \Psi(\xi/2^k) = 1 \text{ for all } \xi \in \mathbb{R}^d \setminus \{0\}.$$

Received October 8, 2011. Revised December 11, 2011.
2000 Mathematics Subject Classification(s): 42B15, 42B20, 42B35
Key Words: *Besov spaces, Hörmander's multiplier theorem, multilinear operators*
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For $m \in L^\infty(\mathbb{R}^{Nn})$ and $j \in \mathbb{Z}$, we set

$$m_j(\xi) = m(2^j\xi_1, \dots, 2^j\xi_N)\Psi(\xi_1, \dots, \xi_N), \tag{1.2}$$

where $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ and Ψ is as in (1.1) with $d = Nn$. In order to weaken the regularity condition to assure the boundedness, Tomita [11] gave a Hörmander type theorem for multilinear Fourier multipliers. More precisely, he proved that if $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies

$$\sup_{j\in\mathbb{Z}} \|m_j\|_{H^2_s(\mathbb{R}^{Nn})} < \infty \qquad \text{with} \quad s > Nn/2,$$

where H^2_s is the L^2-based Sobolev space (see Section 2), then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_N, p < \infty$ satisfying $1/p_1 + \cdots + 1/p_N = 1/p$. Grafakos and Si [6] removed the condition $1 < p < \infty$, and extended this result as follows (see also Grafakos, Miyachi and Tomita [5], Miyachi and Tomita [7] for the cases where some indices p_j are equal to infinity, and $p_j \le 1$):

Theorem 1.1 ([6]). *Let $1 < r \le 2$, $r \le p_1, \dots, p_N < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. If $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies*

$$\sup_{j\in\mathbb{Z}} \|m_j\|_{H^r_s(\mathbb{R}^{Nn})} < \infty \qquad \textit{with} \quad s > Nn/r,$$

then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

The purpose of this paper is to consider the critical case $s = Nn/r$ for Theorem 1.1. We note that

$$H^r_s(\mathbb{R}^{Nn}) \hookrightarrow B^{r,1}_{Nn/r}(\mathbb{R}^{Nn}) \qquad \text{if} \quad s > Nn/r,$$

where $B^{r,1}_{Nn/r}(\mathbb{R}^{Nn})$ is the L^r-based Besov space (see Section 2), and try to replace $H^r_s(\mathbb{R}^{Nn})$ by $B^{r,1}_{Nn/r}(\mathbb{R}^{Nn})$. At least, by the slight modification of the arguments in [4, 6, 11], we have

Theorem 1.2. *Let $1 \le r < 2$, $r < p_1, \dots, p_N < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. If $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies*

$$\sup_{j\in\mathbb{Z}} \|m_j\|_{B^{r,1}_{Nn/r}(\mathbb{R}^{Nn})} < \infty,$$

then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

However, our argument given in this paper does not seem to work for the proof of Theorem 1.2 with $r = 2$, and its case will need a different method. It should be pointed out that we can replace $H^2_{(n/2)+\epsilon}(\mathbb{R}^n)$ by $B^{2,1}_{n/2}(\mathbb{R}^n)$ in the linear case (see Seeger [9]).

For the sake of simplicity, we only treat the (usual) Besov spaces in this paper. However, in Theorem 1.2, we can replace $B^{r,1}_{Nn/r}(\mathbb{R}^{Nn})$ by the Besov space of product type $B^{r,1}_{(n/r,\dots,n/r)}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n)$ (see Remark 4.1).

Our paper is organized as follows: In Section 2, we give definitions and preliminary lemmas. In Section 3, we give a key estimate used in the proof of Theorem 1.2. In Section 4, we prove Theorem 1.2.

§ 2. Preliminaries

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\,d\xi.$$

The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{r^n}\int_{|x-y|<r} |f(y)|\,dy$$

for locally integrable functions f on $\mathbb{R}^n$.

We recall the definitions of Sobolev and Besov spaces. For $1 < r < \infty$ and $s \in \mathbb{R}$, the Sobolev space $H^r_s(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{H^r_s} = \|(I-\Delta)^{s/2} f\|_{L^r} < \infty,$$

where $(I-\Delta)^{s/2} f = \mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\widehat{f}]$. Let $\Psi \in \mathcal{S}(\mathbb{R}^d)$ be as in (1.1), and set $\Psi_0(\xi) = 1 - \sum_{k=1}^{\infty} \Psi(\xi/2^k)$ and $\Psi_k(\xi) = \Psi(\xi/2^k)$ if $k \ge 1$. Note that $\operatorname{supp}\Psi_0 \subset \{|\xi| \le 2\}$, $\operatorname{supp}\Psi_k \subset \{2^{k-1} \le |\xi| \le 2^{k+1}\}$ if $k \ge 1$, and $\sum_{k=0}^{\infty} \Psi_k(\xi) = 1$. For $1 \le p, q \le \infty$ and $s \in \mathbb{R}$, the Besov space $B^{p,q}_s(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{B^{p,q}_s} = \left(\sum_{k=0}^{\infty} 2^{ksq}\|\mathcal{F}^{-1}[\Psi_k\widehat{f}]\|_{L^p}^q\right)^{1/q} = \left(\sum_{k=0}^{\infty} 2^{ksq}\|(\mathcal{F}^{-1}\Psi_k)*f\|_{L^p}^q\right)^{1/q} < \infty.$$

We refer to Triebel [12] and the references therein for details on Besov spaces.

The following lemmas will be used later on:

Lemma 2.1 ([3]). *Let $1 < p, q < \infty$. Then*

$$\left\| \left\{ \sum_{k\in\mathbb{Z}} (Mf_k)^q \right\}^{1/q} \right\|_{L^p} \lesssim \left\| \left\{ \sum_{k\in\mathbb{Z}} |f_k|^q \right\}^{1/q} \right\|_{L^p}$$

for all sequences $\{f_k\}_{k\in\mathbb{Z}}$ of locally integrable functions on $\mathbb{R}^n$.

Lemma 2.2 ([2, Theorem 8.6]). *Let $1 < p < \infty$, and let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that* $\operatorname{supp}\psi \subset \{\xi \in \mathbb{R}^n : 1/r \le |\xi| \le r\}$ *for some $r > 1$. Then*

$$\left\| \left\{ \sum_{k\in\mathbb{Z}} |\psi(D/2^k)f|^2 \right\}^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p} \qquad \text{for all } f \in L^p(\mathbb{R}^n),$$

where $\psi(D/2^k)f = \mathcal{F}^{-1}[\psi(\cdot/2^k)\widehat{f}]$.

Let N be a natural number, and let ϕ_0 be a C^∞-function on $[0,\infty)$ satisfying

$$\phi_0(t) = 1 \quad \text{on} \quad [0, 1/(4N)], \qquad \operatorname{supp}\phi_0 \subset [0, 1/(2N)].$$

We also set $\phi_1(t) = 1 - \phi_0(t)$. For $(i_1, i_2, \dots, i_N) \in \{0,1\}^N$, we define the function $\Phi_{(i_1,i_2,\dots,i_N)}$ on $\mathbb{R}^{Nn} \setminus \{0\}$ by

$$\Phi_{(i_1,i_2,\dots,i_N)}(\xi) = \phi_{i_1}(|\xi_1|/|\xi|)\phi_{i_2}(|\xi_2|/|\xi|)\dots\phi_{i_N}(|\xi_N|/|\xi|), \tag{2.1}$$

where $\xi = (\xi_1, \xi_2, \dots \xi_N) \in \mathbb{R}^n \times \mathbb{R}^n \times \dots \times \mathbb{R}^n$ and $|\xi| = \sqrt{|\xi_1|^2 + |\xi_2|^2 + \dots + |\xi_N|^2}$. Note that $\Phi_{(0,0,\dots,0)} = 0$. Then we have

Lemma 2.3. *Let $\Phi_{(i_1,\dots,i_N)}$ be the same as in* (2.1). *Then the following are true:*

(1) For $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n \setminus \{(0,\dots,0)\}$,

$$\sum_{\substack{(i_1,i_2,\dots,i_N)\in\{0,1\}^N \\ (i_1,i_2,\dots,i_N)\ne(0,0,\dots,0)}} \Phi_{(i_1,i_2,\dots,i_N)}(\xi) = 1.$$

(2) For $(i_1, \dots, i_N) \in \{0,1\}^N$ *and* $(\alpha_1, \dots, \alpha_N) \in \mathbb{Z}_+^n \times \dots \times \mathbb{Z}_+^n$,

$$|\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} \Phi_{(i_1,\dots,i_N)}(\xi)| \le C_{(i_1,\dots,i_N)}^{\alpha_1,\dots,\alpha_N} (|\xi_1| + \dots + |\xi_N|)^{-(|\alpha_1|+\dots+|\alpha_N|)}$$

for all $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n \setminus \{(0,\dots,0)\}$.

(3) If $i_j = 1$ for some $1 \le j \le N$ and $i_k = 0$ for all $1 \le k \le N$ with $k \ne j$, then $\operatorname{supp}\Phi_{(i_1,\dots,i_N)} \subset \{(\xi_1,\dots,\xi_N) : |\xi_k| \le |\xi_j|/N \text{ for } k \ne j\}$. *If $i_j = i_{j'} = 1$ for some $1 \le j, j' \le N$ with $j \ne j'$, then* $\operatorname{supp}\Phi_{(i_1,\dots,i_N)} \subset \{(\xi_1,\dots,\xi_N) : |\xi_j|/(4N) \le |\xi_{j'}| \le 4N|\xi_j|,\ |\xi_k| \le 4N|\xi_j| \text{ for } k \ne j, j'\}$.

See [11, Section 5], [4, Lemma 3.1] for the proof of Lemma 2.3.

§ 3. Key estimate

In this section, we prove the following lemma which plays an essential role in the proof of Theorem 1.2:

Lemma 3.1. *Let* $1 \le r \le 2$. *Then*

$$|T_{m(\cdot/2^j)}(f_1,\dots,f_N)(x)| \lesssim \|m\|_{B^{r,1}_{Nn/r}} M(|f_1|^r)(x)^{1/r} \dots M(|f_N|^r)(x)^{1/r}$$

for all $j \in \mathbb{Z}$, $m \in B^{r,1}_{Nn/r}(\mathbb{R}^{Nn})$ *and* $f_1,\dots,f_N \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Let $\{\Psi_k\}_{k=0}^{\infty} \subset \mathcal{S}(\mathbb{R}^{Nn})$ be a sequence of functions which appeared in the definition of Besov spaces. Then

$$\begin{aligned}
&T_{m(\cdot/2^j)}(f_1,\dots,f_N)(x)\\
&= \int_{\mathbb{R}^{Nn}} 2^{Njn}\mathcal{F}^{-1}m(2^j(x-y_1),\dots,2^j(x-y_N))f_1(y_1)\dots f_N(y_N)\,dy\\
&= (2\pi)^{-Nn}\sum_{k=0}^{\infty}\int_{\mathbb{R}^{Nn}} 2^{Njn}\Psi_k(2^j(y_1-x),\dots,2^j(y_N-x))\\
&\qquad\qquad \times \widehat{m}(2^j(y_1-x),\dots,2^j(y_N-x))f_1(y_1)\dots f_N(y_N)\,dy,
\end{aligned}$$

where $y=(y_1,\dots,y_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n$. Let r' be the conjugate exponent of r. Since

$$\operatorname{supp}\Psi_k \subset \{y \in \mathbb{R}^{Nn} \,:\, |y| \le 2^{k+1}\} \subset \{y \in \mathbb{R}^{Nn} \,:\, |y_j| \le 2^{k+1},\ j=1,\dots,N\},$$

we have by Hölder's inequality

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^{Nn}} 2^{Njn}\Psi_k(2^j(y_1-x),\dots,2^j(y_N-x))\widehat{m}(2^j(y_1-x),\dots,2^j(y_N-x))\\
&\qquad \times f_1(y_1)\dots f_N(y_N)dy\Big|\\
&\le 2^{Njn}\Big(\int_{\mathbb{R}^{Nn}}\Big|\Psi_k(2^j(y_1-x),\dots,2^j(y_N-x))\widehat{m}(2^j(y_1-x),\dots,2^j(y_N-x))\Big|^{r'}dy\Big)^{1/r'}\\
&\qquad \times \Big(\int_{|2^j(y_1-x)|\le 2^{k+1}}|f_1(y_1)|^r dy_1\Big)^{1/r}\dots\Big(\int_{|2^j(y_N-x)|\le 2^{k+1}}|f_N(y_N)|^r dy_N\Big)^{1/r}\\
&= 2^{N(k+1)n/r}\Big(\int_{\mathbb{R}^{Nn}}\Big|\Psi_k(y_1,\dots,y_N)\widehat{m}(y_1,\dots,y_N)\Big|^{r'}dy\Big)^{1/r'}\\
&\qquad \times \Big(\frac{1}{2^{(k-j+1)n}}\int_{|y_1-x|\le 2^{k-j+1}}|f_1(y_1)|^r dy_1\Big)^{1/r}\\
&\qquad\qquad \times\dots\times\Big(\frac{1}{2^{(k-j+1)n}}\int_{|y_N-x|\le 2^{k-j+1}}|f_N(y_N)|^r dy_N\Big)^{1/r}\\
&\le 2^{N(k+1)n/r}\|\Psi_k\widehat{m}\|_{L^{r'}}M(|f_1|^r)(x)^{1/r}\dots M(|f_N|^r)(x)^{1/r}.
\end{aligned}$$

It follows from the Hausdorff-Young inequality that $\|\Psi_k\widehat{m}\|_{L^{r'}} \lesssim \|\mathcal{F}^{-1}[\Psi_k\widehat{m}]\|_{L^r}$, and consequently

$$|T_{m(\cdot/2^j)}(f_1,\dots,f_N)(x)| \lesssim \left(\sum_{k=0}^{\infty} 2^{Nkn/r}\|\mathcal{F}^{-1}[\Psi_k\widehat{m}]\|_{L^r}\right) M(|f_1|^r)(x)^{1/r}\dots M(|f_N|^r)(x)^{1/r}.$$

This completes the proof. □

§ 4. Proof of Theorem 1.2

In this section, we use the following notation to distinguish linear and multilinear Fourier multiplier operators: For $\varphi \in L^\infty(\mathbb{R}^n)$, the (linear) Fourier multiplier operator $\varphi(D)$ is defined by $\varphi(D)f = \mathcal{F}^{-1}[\varphi\widehat{f}]$ for $f \in \mathcal{S}(\mathbb{R}^n)$. We also use the following notation: $\mathcal{A}_0$ denotes the set of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ for which $\operatorname{supp}\varphi$ is compact and $\varphi = 1$ on some neighborhood of the origin; $\mathcal{A}_1$ denotes the set of $\widetilde{\psi} \in \mathcal{S}(\mathbb{R}^n)$ for which $\operatorname{supp}\widetilde{\psi}$ is a compact subset of $\mathbb{R}^n \setminus \{0\}$.

Proof of Theorem 1.2. Let $1 \le r < 2$, $r < p_1,\dots,p_N < \infty$, $1/p_1+\cdots+1/p_N = 1/p$, and let $m \in L^\infty(\mathbb{R}^{Nn})$ satisfy $\sup_{j\in\mathbb{Z}}\|m_j\|_{B^{r,1}_{Nn/r}} < \infty$, where m_j is defined by (1.2). It follows from Lemma 2.3 (1) that

$$m(\xi) = \sum_{\substack{(i_1,i_2,\dots,i_N)\in\{0,1\}^N \\ (i_1,i_2,\dots,i_N)\neq(0,0,\dots,0)}} \Phi_{(i_1,i_2,\dots,i_N)}(\xi)m(\xi) = \sum_{\substack{(i_1,i_2,\dots,i_N)\in\{0,1\}^N \\ (i_1,i_2,\dots,i_N)\neq(0,0,\dots,0)}} m_{(i_1,i_2,\dots,i_N)}(\xi). \tag{4.1}$$

Estimate for $m_{(1,0,\dots,0)}$. We first consider the case where $(i_1,\dots,i_N)$ satisfies $\sharp\{j : i_j = 1\} = 1$, and may assume without loss of generality that $i_1 = 1$. This means $m_{(i_1,i_2,\dots,i_N)} = m_{(1,0,\dots,0)}$, and we simply write m instead of $m_{(1,0,\dots,0)}$. By Lemma 2.3 (3),

$$\operatorname{supp} m \subset \{\xi = (\xi_1,\dots,\xi_N) \in \mathbb{R}^n\times\cdots\times\mathbb{R}^n \,:\, |\xi_i| \le |\xi_1|/N,\ i = 2,\dots,N\}. \tag{4.2}$$

Let ψ be as in (1.1) with $d = n$. Note that

$$\|g\|_{L^p} \lesssim \|g\|_{\mathcal{H}^p} \approx \left\| \left(\sum_{j\in\mathbb{Z}} |\psi(D/2^j)g|^2\right)^{1/2} \right\|_{L^p}$$

for appropriate functions g, where $\mathcal{H}^p$ is the Hardy space (e.g. [6, Lemma 2.4]). Then

$$\|T_m(f_1,\dots,f_N)\|_{L^p} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |\psi(D/2^j)T_m(f_1,\dots,f_N)|^2 \right)^{1/2} \right\|_{L^p}. \tag{4.3}$$

It follows from (4.2) that if $(\xi_1,\dots,\xi_N) \in \operatorname{supp} m$ then $|\xi_1+\dots+\xi_N| \approx |\xi_1|$ and $|\xi_i| \lesssim |\xi_1|$ for $2 \le i \le N$, and we can find functions $\varphi \in \mathcal{A}_0$ and $\widetilde{\psi} \in \mathcal{A}_1$ independent of j such that

$$\begin{aligned} &m(\xi)\psi((\xi_1+\dots+\xi_N)/2^j) \\ &= m(\xi)\psi((\xi_1+\dots+\xi_N)/2^j)\widetilde{\psi}(\xi_1/2^j)^2\varphi(\xi_2/2^j)\dots\varphi(\xi_N/2^j), \end{aligned}$$

where we have used the fact $\operatorname{supp}\psi \subset \{\eta \in \mathbb{R}^n : 1/2 \le |\eta| \le 2\}$. Hence, setting

$$m_{(j)}(\xi) = m(2^j\xi)\psi(\xi_1+\dots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N),$$

we see that

$$\begin{aligned} &\psi(D/2^j)T_m(f_1,\dots,f_N)(x) \\ &= \frac{1}{(2\pi)^{Nn}} \int_{\mathbb{R}^{Nn}} e^{ix\cdot(\xi_1+\dots+\xi_N)} m(\xi)\psi((\xi_1+\dots+\xi_N)/2^j)\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\dots\widehat{f_N}(\xi_N)\,d\xi \\ &= \frac{1}{(2\pi)^{Nn}} \int_{\mathbb{R}^{Nn}} e^{ix\cdot(\xi_1+\dots+\xi_N)} m_{(j)}(\xi/2^j)(\widetilde{\psi}(\xi_1/2^j)\widehat{f_1}(\xi_1))\widehat{f_2}(\xi_2)\dots\widehat{f_N}(\xi_N)\,d\xi \\ &= T_{m_{(j)}(\cdot/2^j)}(\widetilde{\psi}(D/2^j)f_1,f_2,\dots,f_N)(x). \end{aligned}$$

By Lemma 3.1,

$$\begin{aligned} &|T_{m_{(j)}(\cdot/2^j)}(\widetilde{\psi}(D/2^j)f_1,f_2,\dots,f_N)(x)| \\ &\lesssim \|m_{(j)}\|_{B^{r,1}_{Nn/r}} M(|\widetilde{\psi}(D/2^j)f_1|^r)(x)^{1/r} M(|f_2|^r)(x)^{1/r}\dots M(|f_N|^r)(x)^{1/r}, \end{aligned}$$

and consequently

$$\begin{aligned} &\left(\sum_{j\in\mathbb{Z}} |\psi(D/2^j)T_m(f_1,\dots,f_N)(x)|^2 \right)^{1/2} \\ &\lesssim \left(\sup_{k\in\mathbb{Z}} \|m_{(k)}\|_{B^{r,1}_{Nn/r}} \right) \left(\sum_{j\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^j)f_1|^r)(x)^{2/r} \right)^{1/2} \\ &\qquad \times M(|f_2|^r)(x)^{1/r}\dots M(|f_N|^r)(x)^{1/r}. \end{aligned} \tag{4.4}$$

Since $1 \le r < 2$ and $r < p_1, \dots, p_N < \infty$, we see that $1 < 2/r, p_1/r, \dots, p_N/r < \infty$. Then, it follows from Hölder's inequality, Lemmas 2.1 and 2.2 that

$$
\begin{aligned}
(4.5)\quad & \left\| \left(\sum_{j\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^j)f_1|^r)^{2/r} \right)^{1/2} M(|f_2|^r)^{1/r} \dots M(|f_N|^r)^{1/r} \right\|_{L^p} \\
& \le \left\| \left(\sum_{j\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^j)f_1|^r)^{2/r} \right)^{1/2} \right\|_{L^{p_1}} \|M(|f_2|^r)^{1/r}\|_{L^{p_2}} \dots \|M(|f_N|^r)^{1/r}\|_{L^{p_N}} \\
& = \left\| \left(\sum_{j\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^j)f_1|^r)^{2/r} \right)^{r/2} \right\|_{L^{p_1/r}}^{1/r} \|M(|f_2|^r)\|_{L^{p_2/r}}^{1/r} \dots \|M(|f_N|^r)\|_{L^{p_N/r}}^{1/r} \\
& \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \dots \|f_N\|_{L^{p_N}}.
\end{aligned}
$$

Thus, by (4.3)-(4.5),

$$\|T_m(f_1, f_2, \dots, f_N)\|_{L^p} \lesssim \left(\sup_{j\in\mathbb{Z}} \|m_{(j)}\|_{B^{r,1}_{Nn/r}} \right) \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \dots \|f_N\|_{L^{p_N}}.$$

Recall that $m(\xi) = m_{(1,0,\dots,0)}(\xi)$, and

$$m_{(j)}(\xi) = m(2^j\xi)\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1 + \dots + \xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N).$$

Let us prove

$$(4.6)\qquad \sup_{j\in\mathbb{Z}} \|m_{(j)}\|_{B^{r,1}_{Nn/r}} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{B^{r,1}_{Nn/r}},$$

where m_j is defined by (1.2). Once this is proved, we have the desired estimate:

$$\|T_{m_{(1,0,\dots,0)}}(f_1, f_2, \dots, f_N)\|_{L^p} \lesssim \left(\sup_{j\in\mathbb{Z}} \|m_j\|_{B^{r,1}_{Nn/r}} \right) \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \dots \|f_N\|_{L^{p_N}}.$$

Let Ψ be as in (1.1) with $d = Nn$. Since $\operatorname{supp}\Psi(\cdot/2^\ell) \subset \{2^{\ell-1} \le |\xi| \le 2^{\ell+1}\}$, $\operatorname{supp}\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N) \subset \{2^{-j_0} \le |\xi| \le 2^{j_0}\}$ for some $j_0 \in \mathbb{N}$ and $B^{r,1}_{Nn/r}(\mathbb{R}^{Nn})$ is a multiplication algebra (Triebel [12, Theorem 2.8.3]), we have

$$
\begin{aligned}
\|m_{(j)}(\xi)\|_{B^{r,1}_{Nn/r}} & \le \sum_{\ell=-j_0}^{j_0} \|m_{(j)}(\xi)\Psi(\xi/2^\ell)\|_{B^{r,1}_{Nn/r}} \\
& \lesssim \sum_{\ell=-j_0}^{j_0} \|m(2^j\xi)\Psi(\xi/2^\ell)\|_{B^{r,1}_{Nn/r}} \\
& \qquad \times \|\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1 + \dots + \xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\|_{B^{r,1}_{Nn/r}}.
\end{aligned}
$$

By a change of variables,

$$\begin{aligned}\|m(2^j\xi)\Psi(\xi/2^\ell)\|_{B^{r,1}_{Nn/r}} &\lesssim (2^{-\ell})^{-Nn/r}(\max\{1,2^{-\ell}\})^{Nn/r}\|m(2^{j+\ell}\xi)\Psi(\xi)\|_{B^{r,1}_{Nn/r}}\\ &\lesssim \sup_{j\in\mathbb{Z}}\|m(2^j\xi)\Psi(\xi)\|_{B^{r,1}_{Nn/r}} = \sup_{j\in\mathbb{Z}}\|m_j\|_{B^{r,1}_{Nn/r}}\end{aligned}$$

for all $|\ell|\le j_0$ (see, for example, [8, Proposition 2.1.3/3], [10, Proposition 1.1]). On the other hand, by Lemma 2.3 (2),

$$\left|\partial_\xi^\alpha\left(\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\dots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\right)\right| \le C_\alpha\chi_{\{2^{-j_0}\le|\xi|\le 2^{j_0}\}}(\xi)$$

for all α and j, and consequently

$$\sup_{j\in\mathbb{Z}}\|\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\dots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\|_{B^{r,1}_{Nn/r}}<\infty.$$

Combining these estimates, we have (4.6).

Estimate for $m_{(1,1,i_3,\dots,i_N)}$. We next consider the case where $(i_1,\dots,i_N)$ satisfies $\sharp\{j : i_j=1\}\ge 2$, and may assume without loss of generality that $i_1=i_2=1$. This means $m_{(i_1,i_2,i_3,\dots,i_N)}=m_{(1,1,i_3,\dots,i_N)}$, where $i_3,\dots,i_N\in\{0,1\}$. We simply write m instead of $m_{(1,1,i_3,\dots,i_N)}$ as before. By Lemma 2.3 (3),

$$\operatorname{supp} m\subset\{|\xi_1|/(4N)\le|\xi_2|\le 4N|\xi_1|,\ |\xi_i|\le 4N|\xi_1|,\ i=3,\dots,N\}. \tag{4.7}$$

Let $\psi\in\mathcal{S}(\mathbb{R}^n)$ be as in (1.1) with $d=n$. By (4.7), we can find $\varphi\in\mathcal{A}_0$ and $\widetilde{\psi}\in\mathcal{A}_1$ independent of j such that

$$m(\xi)\psi(\xi_1/2^j)=m(\xi)\psi(\xi_1/2^j)\widetilde{\psi}(\xi_1/2^j)\widetilde{\psi}(\xi_2/2^j)^2\varphi(\xi_3/2^j)\dots\varphi(\xi_N/2^j).$$

Hence, setting

$$m_{(j)}(\xi)=m(2^j\xi)\psi(\xi_1)\widetilde{\psi}(\xi_2)\varphi(\xi_3)\dots\varphi(\xi_N),$$

we see that

$$\begin{aligned}&T_m(f_1,\dots,f_N)(x)\\ &=\sum_{j\in\mathbb{Z}}\frac{1}{(2\pi)^{Nn}}\int_{\mathbb{R}^{Nn}}e^{ix\cdot(\xi_1+\dots+\xi_N)}m(\xi)\psi(\xi_1/2^j)\widehat{f_1}(\xi_1)\dots\widehat{f_N}(\xi_N)\,d\xi\\ &=\sum_{j\in\mathbb{Z}}\frac{1}{(2\pi)^{Nn}}\int_{\mathbb{R}^{Nn}}e^{ix\cdot(\xi_1+\dots+\xi_N)}m_{(j)}(\xi/2^j)(\widetilde{\psi}(\xi_1/2^j)\widehat{f_1}(\xi_1))(\widetilde{\psi}(\xi_2/2^j)\widehat{f_2}(\xi_1))\\ &\qquad\times\widehat{f_3}(\xi_3)\dots\widehat{f_N}(\xi_N)\,d\xi\\ &=\sum_{j\in\mathbb{Z}}T_{m_{(j)}(\cdot/2^j)}(\widetilde{\psi}(D/2^j)f_1,\widetilde{\psi}(D/2^j)f_2,f_3,\dots,f_N)(x).\end{aligned}$$

It follows from Lemma 3.1 and Schwarz's inequality that

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} |T_{m_{(j)}}(\widetilde{\psi}(D/2^j)f_1, \widetilde{\psi}(D/2^j)f_2, f_3, \dots, f_N)(x)| \\
&\lesssim \sum_{j\in\mathbb{Z}} \|m_{(j)}\|_{B^{r,1}_{Nn/r}} M(|\widetilde{\psi}(D/2^j)f_1|^r)(x)^{1/r} M(|\widetilde{\psi}(D/2^j)f_2|^r)(x)^{1/r} \\
&\qquad\qquad \times M(|f_3|^r)(x)^{1/r} \dots M(|f_N|^r)(x)^{1/r} \\
&\le \left(\sup_{k\in\mathbb{Z}} \|m_{(k)}\|_{B^{r,1}_{Nn/r}}\right) \left(\sum_{j\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^j)f_1|^r)(x)^{2/r}\right)^{1/2} \\
&\qquad\qquad \times \left(\sum_{j\in\mathbb{Z}} M(|\widetilde{\psi}(D/2^j)f_2|^r)(x)^{2/r}\right)^{1/2} M(|f_3|^r)(x)^{1/r} \dots M(|f_N|^r)(x)^{1/r}.
\end{aligned}$$

The rest of the proof is similar to that of $m_{(1,0,\dots,0)}$, and we omit it.

We end this paper by giving the following remark:

Remark 4.1. Let $\{\Psi_k\}_{k=0}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ be a sequence of functions appearing in the definition of Besov spaces. For $1 \le r \le \infty$ and $s_1, \dots, s_N \in \mathbb{R}$, the Besov space of product type $B^{r,1}_{(s_1,\dots,s_N)}(\mathbb{R}^n \times \dots \times \mathbb{R}^n)$ is defined by the norm

$$\|f\|_{B^{r,1}_{(s_1,\dots,s_N)}} = \sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1 s_1+\dots+k_N s_N)} \|\mathcal{F}^{-1}[\Psi_{k_1}(\xi_1)\dots\Psi_{k_N}(\xi_N)\widehat{f}(\xi)]\|_{L^r(\mathbb{R}^{Nn})},$$

where $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n$. Note that if $s_1, \dots, s_N > 0$, then

$$B^{r,1}_{s_1+\dots+s_N}(\mathbb{R}^{Nn}) \hookrightarrow B^{r,1}_{(s_1,\dots,s_N)}(\mathbb{R}^n \times \dots \times \mathbb{R}^n).$$

In the same way as in the proof of Lemma 3.1, we can prove

$$\begin{aligned}
&|T_{m(\cdot/2^j)}(f_1, \dots, f_N)(x)| \\
&\lesssim \sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\dots+k_N)n/r} \left(\int_{\mathbb{R}^{Nn}} |\Psi_{k_1}(y_1)\dots\Psi_{k_N}(y_N)\widehat{m}(y)|^{r'} dy\right)^{1/r'} \\
&\qquad \times \left(\frac{1}{2^{(k_1-j+1)n}} \int_{|y_1-x|\le 2^{k_1-j+1}} |f_1(y_1)|^r dy_1\right)^{1/r} \\
&\qquad\qquad \times \dots \times \left(\frac{1}{2^{(k_N-j+1)n}} \int_{|y_N-x|\le 2^{k_N-j+1}} |f_N(y_N)|^r dy_N\right)^{1/r}.
\end{aligned}$$

As a result,

$$|T_{m(\cdot/2^j)}(f_1, \dots, f_N)(x)| \lesssim \|m\|_{B^{r,1}_{(n/r,\dots,n/r)}} M(|f_1|^r)(x)^{1/r} \dots M(|f_N|^r)(x)^{1/r}$$

for $1 \leq r \leq 2$. Then, in the case $1 \leq r < 2$, by using this estimate instead of Lemma 3.1, we can prove Theorem 1.2 with $B^{r,1}_{Nn/r}$ replaced by $B^{r,1}_{(n/r,\dots,n/r)}$. It should be mentioned that dilation and multiplication properties of Besov spaces were used in the proof of Theorem 1.2. See [10, Proposition 1.1, Theorem 1.4] for their properties of Besov spaces of product type.

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