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Remarks on value distributions of general Dirichlet series

By

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Abstract

In the paper, limit theorems for general Dirichlet series on the complex plane, in the space of analytic functions as well as in the space of meromorphic functions with explicit limit measures are proved without assumption of linear independence of the exponents.

§ 1. Introduction

A general Dirichlet series is a series of the form

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s}, \quad s = \sigma + it \in \mathbb{C}, \tag{1.1}$$

where $a_m \in \mathbb{C}$, and $\{\lambda_m\}$ is an increasing sequence of real numbers, $\lim_{m\to\infty} \lambda_m = +\infty$. Suppose that the series (1.1) converges absolutely for $\sigma > \sigma_a$ and has the sum $f(s)$. Then $f(s)$ is an analytic function in the half-plane $D := \{s \in \mathbb{C} : \sigma > \sigma_a\}$.

Limit theorems for general Dirichlet series on the complex plane, in the space of analytic functions as well as in the space of meromorphic functions have been studied relatively completely through papers [2, 3, 4, 5, 6, 7, 8]. Let us mention here the most recent results. For $T > 0$, let $\nu_T(\cdots)$ stand for the measure

$$\nu_T(\cdots) = \frac{1}{T}\mathrm{meas}\{t \in [0, T] : \cdots\},$$

where in place of dots a condition satisfied by t is to be given. Let $\mathcal{B}(S)$ be the class of Borel sets of the space S, and $H(D)$ be the space of analytic functions on D equipped

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with the topology of uniform convergence on compacta. Then the limit theorem for the absolutely convergent general Dirichlet series in the space of analytic functions was proved in [2].

Theorem A. *There exists a probability measure* P *on* $(H(D), \mathcal{B}(H(D)))$ *such that the sequence of probability measures*

$$\nu_T(f(s+it) \in A), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P *as* $T \to \infty$.

Suppose that $f(s)$ is meromorphically continuable to a wider half-plane $D_0 := \{s \in \mathbb{C} : \sigma > \sigma_0\}, \sigma_0 < \sigma_a$. Moreover, we require that all poles of $f(s)$ in D_0 are included in a compact set and that the following two conditions are satisfied.

(i) $f(s)$ is of finite order in any half-plane $\sigma \geq \sigma_1 (\sigma_1 > \sigma_0)$, that is, there exist constants $a > 0$ and $t_0 \geq 0$ such that the estimate

$$f(\sigma + it) = B(|t|^a), \quad |t| \geq t_0, \tag{1.2}$$

holds uniformly for $\sigma \geq \sigma_1$. Here and in the sequel, B is a quantity bounded by some constant.

(ii) For $\sigma > \sigma_0$ such that $\{\sigma + it : t \in \mathbb{R}\}$ does not contain any pole of $f(s)$,

$$\int_{-T}^{T} |f(\sigma + it)|^2 dt = B(T), \quad T \to \infty. \tag{1.3}$$

Let $\mathbb{C}_\infty$ be the Riemann sphere $\mathbb{C} \cup \{\infty\}$, and d be the sphere metric on $\mathbb{C}_\infty$ defined by

$$d(s_1, s_2) = \frac{2|s_1 - s_2|}{\sqrt{1+|s_1|^2}\sqrt{1+|s_2|^2}}, \quad d(s, \infty) = \frac{2}{\sqrt{1+|s|^2}}, \quad d(\infty, \infty) = 0,$$

where $s, s_1, s_2 \in \mathbb{C}$. This metric is compatible with the topology of $\mathbb{C}_\infty$. Let $M(D_0)$ denote the space of meromorphic functions $g \colon D_0 \to (\mathbb{C}_\infty, d)$ equipped with the topology of uniform convergence on compacta. Then the limit theorem in the space of meromorphic functions was obtained in [3].

Theorem B. *Suppose that conditions* (1.2) *and* (1.3) *are satisfied. Then there exists a probability measure* P *on* $(M(D_0), \mathcal{B}(M(D_0)))$ *such that the sequence of probability measures*

$$\nu_T(f(s+it) \in A), \quad A \in \mathcal{B}(M(D_0)),$$

converges weakly to P *as* $T \to \infty$.

The limit theorem on the complex plane was obtained in [4].

Theorem C. *Suppose that conditions* (1.2) *and* (1.3) *are satisfied. Then for each* $\sigma > \sigma_0$, *there exists a probability measure* P_σ *on* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *such that the sequence of probability measures*

$$\nu_T(f(\sigma + it) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ *as* $T \to \infty$.

To identify the limit probability measures in the three theorems above, some additional conditions are necessary. Suppose that the sequence of exponents $\{\lambda_m\}$ is linearly independent over the field of rational numbers. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ be the unit circle on the complex plane, and let

$$\Omega = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}$. With the product topology and point-wise multiplication the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the normalized Haar measure m_H exists, and we obtain a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(m)$ denote the projection of $\omega \in \Omega$ to the coordinate space γ_m. Assume further that, for $\sigma > \sigma_0$,

$$\sum_{m=1}^{\infty} |a_m|^2 e^{-2\lambda_m \sigma} (\log m)^2 < \infty. \tag{1.4}$$

Then it was proved in [7] that for $\sigma > \sigma_0$,

$$f(\sigma, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) e^{-\lambda_m \sigma}$$

is a complex-valued random variable defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Moreover, the limit probability measure P_σ in Theorem C coincides with the distribution of the random variable $f(\sigma, \omega)$. In addition, under conditions (1.2)–(1.4), it was proved in [8] that $f(s, \omega)$ defined by

$$f(s, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) e^{-\lambda_m s}$$

is an $H(D_0)$-valued random element and the limit probability measure in Theorem B coincides with the distribution of $f(s, \omega)$.

This paper is devoted to identify the limit probability measures without assumption of linear independence of $\{\lambda_m\}$. We will consider the probability space $(\Omega, \mathcal{B}(\Omega), \mathbf{P})$ with

P being a suitable probability measure. The limit probability measures in the three theorems above are shown to coincide with the distributions of appropriate random elements on $(\Omega, \mathcal{B}(\Omega), \mathbf{P})$. We point out that the condition (1.4) is not necessary, if $\{\lambda_m\}$ is linearly independent over the field of rational numbers.

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§ 2. General theory

The main aim of this section is to approximate the function $f(s)$ by a sequence of absolutely convergent Dirichlet series. If the function $f(s)$ is analytic in D_0, we can find this kind of result in [5, 6]. We begin with a result on the mean value of absolutely convergent Dirichlet series.

Theorem 2.1 (cf. [10, §9.5]). *For any $\sigma_1 > \sigma_a$, uniformly in $\sigma \geq \sigma_1$, we have*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^2 dt = \sum_{m=1}^{\infty} |a_m|^2 e^{-2\lambda_m \sigma}.$$

The following formula may be known as Perron's formula. We will use the Dirichlet series defined in that formula to approximate the function $f(s)$.

Lemma 2.2 (cf. [10, §9.43]). *For $\delta > 0, \lambda > 0$, and $c > 0, c > \sigma_a - \sigma$, we have*

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s} e^{-(e^{\lambda_m}\delta)^{\lambda}} = \frac{1}{2\pi i \lambda} \int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{w}{\lambda}\right) f(s+w) \delta^{-w} dw, \tag{2.1}$$

where Γ denotes the Gamma function.

Let

$$g_{\lambda,\delta}(s) := \sum_{m=1}^{\infty} a_m e^{-\lambda_m s} e^{-(e^{\lambda_m}\delta)^{\lambda}}, \quad (\lambda > 0, \delta > 0).$$

It is clear that the Dirichlet series $g_{\lambda,\delta}(s)$ is absolutely convergent for any $s \in \mathbb{C}$. The sequence $\{g_{\lambda,\delta}(s)\}_\delta$ approximates the function $f(s)$ in the following sense.

Corollary 2.3. *Let K be a compact subset in D. Then for fixed $\lambda > 0$,*

$$\lim_{\delta\to 0} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} |f(s+it) - g_{\lambda,\delta}(s+it)| dt = 0.$$

Proof. Let L be a simple closed contour lying in D and enclosing the set K and let δ_K denote the distance of L from the set K. It follows from Cauchy's integral formula that

$$\sup_{s\in K}|f(s+it)-g_{\lambda,\delta}(s+it)|\le\frac{1}{2\pi\delta_K}\int_L|f(z+it)-g_{\lambda,\delta}(z+it)||dz|,$$

then by the Cauchy-Schwarz inequality,

$$\left(\sup_{s\in K}|f(s+it)-g_{\lambda,\delta}(s+it)|\right)^2\le\frac{|L|}{(2\pi\delta_K)^2}\int_L|f(z+it)-g_{\lambda,\delta}(z+it)|^2|dz|.$$

Here $|L|$ denotes the length of the contour L. Thus when $T>\max_{z\in L}|\operatorname{Im} z|$,

$$\begin{aligned}
&\left(\frac{1}{T}\int_0^T\sup_{s\in K}|f(s+it)-g_{\lambda,\delta}(s+it)|dt\right)^2\\
&\le\frac{1}{T}\int_0^T\left(\sup_{s\in K}|f(s+it)-g_{\lambda,\delta}(s+it)|\right)^2dt\\
&\le\frac{1}{T}\int_0^T\left(\frac{|L|}{(2\pi\delta_K)^2}\int_L|f(z+it)-g_{\lambda,\delta}(z+it)|^2|dz|\right)dt\\
&=\frac{|L|}{(2\pi\delta_K)^2}\int_L\left(\frac{1}{T}\int_0^T|f(z+it)-g_{\lambda,\delta}(z+it)|^2dt\right)|dz|\\
&\le\frac{|L|}{(2\pi\delta_K)^2}\int_L\left(\frac{1}{T}\int_{-2T}^{2T}|f(\operatorname{Re} z+it)-g_{\lambda,\delta}(\operatorname{Re} z+it)|^2dt\right)|dz|\\
&\le\frac{4|L|^2}{(2\pi\delta_K)^2}\sup_{\sigma\ge\sigma_1}\frac{1}{4T}\int_{-2T}^{2T}|f(\sigma+it)-g_{\lambda,\delta}(\sigma+it)|^2dt,
\end{aligned}$$

where $\sigma_1=\min_{z\in L}\operatorname{Re} z>\sigma_a$. Now, uniformly in $\sigma\ge\sigma_1$,

$$\lim_{T\to\infty}\frac{1}{4T}\int_{-2T}^{2T}|f(\sigma+it)-g_{\lambda,\delta}(\sigma+it)|^2=\sum_{m=1}^{\infty}|a_m|^2e^{-2\lambda_m\sigma}(1-e^{-(e^{\lambda_m}\delta)^\lambda})^2,$$

by applying Theorem 2.1 to the function $f(s)-g_{\lambda,\delta}(s)$. Therefore,

$$\begin{aligned}
&\limsup_{T\to\infty}\left(\frac{1}{T}\int_0^T\sup_{s\in K}|f(s+it)-g_{\lambda,\delta}(s+it)|dt\right)^2\\
&\le\frac{4|L|^2}{(2\pi\delta_K)^2}\sum_{m=1}^{\infty}|a_m|^2e^{-2\lambda_m\sigma_1}(1-e^{-(e^{\lambda_m}\delta)^\lambda})^2.
\end{aligned}$$

The above series is dominated by $\sum_{m=1}^{\infty}|a_m|^2e^{-2\lambda_m\sigma_1}<\infty$ for any $\delta>0$, and each term converges to 0 as $\delta\to0$. Thus by the dominated convergence theorem, we arrive at

$$\lim_{\delta\to0}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\sup_{s\in K}|f(s+it)-g_{\lambda,\delta}(s+it)|dt=0.$$

The proof is complete. □

If there is no pole in D_0 or $f(s)$ is analytic in D_0, we have the following version of Theorem 2.1.

Theorem 2.4 (cf. [10, §9.51]). *Let $f(s)$ denote the analytic continuation of the function $f(s), \sigma > \sigma_a$ to the half-plane $\sigma \geq \alpha$. Assume that $f(s)$ is regular and of finite order for $\sigma \geq \alpha$, and that*

$$\int_{-T}^{T} |f(\alpha + it)|^2 dt = B(T), \quad T \to \infty. \tag{2.2}$$

Then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 dt = \sum_{m=1}^{\infty} |a_m|^2 e^{-2\lambda_m \sigma}, \tag{2.3}$$

for $\sigma > \alpha$, and uniformly in any strip $\alpha < \sigma_1 \leq \sigma \leq \sigma_2$.

Consequently, if $f(s)$ is analytic in D_0, the statement of Corollary 2.3 is still true for any compact subset K of D_0. We are now in a position to extend Corollary 2.3 to our considering case in which all poles of $f(s)$ in D_0 are included in a compact set. It then follows that the number of poles are finite. Denote the poles and their orders by $s_1, \dots, s_r$ and $n_1, \dots, n_r$, respectively.

Proposition 2.5. *Let K be a compact subset in D_0. Then for fixed $\lambda > \sigma_a - \sigma_0 + 1$,*

$$\lim_{\delta\to 0} \limsup_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} \sup_{s\in K} |f(s+it) - g_{\lambda,\delta}(s+it)| dt = 0,$$

where t_0 is a positive real number satisfying

$$\min_{s\in K}\{\operatorname{Im} s\} + t_0 > \max\{\operatorname{Im} s_1, \dots, \operatorname{Im} s_r\}.$$

Proof. From Corollary 2.3, we can assume without loss of generality that the compact subset K is included in the strip $\sigma_0 < \sigma < \sigma_a + 1$. Let L be a simple closed contour lying in the strip $\sigma_0 < \sigma < \sigma_a + 1$, enclosing the set K and

$$\min_{s\in L}\{\operatorname{Im} s\} + t_0 > \max\{\operatorname{Im} s_1, \dots, \operatorname{Im} s_r\}.$$

Then L lies in the strip $\sigma_1 \leq \sigma \leq \sigma_2$, where $\sigma_1 = \min_{s\in L} \operatorname{Re} s > \sigma_0$ and $\sigma_2 = \max_{s\in L} \operatorname{Re} s < \sigma_a + 1$. Choose $\alpha \in (\sigma_0, \sigma_1)$ such that all poles $s_1, \dots, s_r$ lie in the half-plane $\sigma > \alpha$.

For $s=\sigma+it$ with $\sigma\in[\sigma_1,\sigma_2]$ and $s\notin\{s_1,\dots,s_r\}$, by moving the contour in the formula (2.1) to $\operatorname{Re} w=\alpha-\sigma$, we pass a pole at $w=0$, with residue $\lambda f(s)$, poles at $w=s_1-s,\dots,w=s_r-s$. Since $\lambda>\sigma-\alpha$, no other pole is passed. Therefore, by the residue theorem, we obtain

$$\begin{aligned} g_{\lambda,\delta}(s)-f(s) &= \frac{1}{2\pi i\lambda}\int_{\alpha-\sigma-i\infty}^{\alpha-\sigma+i\infty}\Gamma(\frac{w}{\lambda})f(s+w)\delta^{-w}dw \\ &\quad+\frac{1}{\lambda}\sum_{j=1}^{r}\operatorname{Res}\Big(\Gamma(\frac{w}{\lambda})f(s+w)\delta^{-w},s_j-s\Big) \\ &=: I(s)+J(s). \end{aligned} \tag{2.4}$$

Observe that

$$\operatorname{Res}\Big(\Gamma(\frac{w}{\lambda})f(s+w)\delta^{-w},s_j-s\Big)=\sum_{k=0}^{n_j-1}\frac{a(k,s_j)}{k!}\Big(\Gamma(\frac{w}{\lambda})\delta^{-w}\Big)^{(k)}\Big|_{s_j-s},$$

where $^{(k)}$ denotes the kth derivative with respect to w and

$$a(k,s_j)=\frac{1}{(n_j-k-1)!}\lim_{w\to s_j-s}\left[\Big(f(s+w)(w-(s_j-s))^{n_j}\Big)^{(n_j-k-1)}(w)\right].$$

Thus, for fixed $\delta>0$,

$$J(s)=B\left(\sum_{k=0}^{n}\left|\Gamma^{(k)}(\frac{s_j-s}{\lambda})\right|\right),\qquad n=\max\{n_1,\dots,n_r\}-1. \tag{2.5}$$

Now, an argument similar to the one used in the proof of Corollary 2.3 shows that

$$\begin{aligned} &\left(\frac{1}{T}\int_{t_0}^{T}\sup_{s\in K}|f(s+it)-g_{\lambda,\delta}(s+it)|dt\right)^2 \\ &=B\left(\sup_{\sigma\in[\sigma_1,\sigma_2]}\frac{1}{4T}\int_{t_1}^{2T}|f(\sigma+it)-g_{\lambda,\delta}(\sigma+it)|^2dt\right), \end{aligned}$$

where $t_1=\min_{s\in L}\{\operatorname{Im} s\}+t_0>\max\{\operatorname{Im} s_1,\dots,\operatorname{Im} s_r\}$. For $\sigma\in[\sigma_1,\sigma_2]$ and $t\geq t_1$, the point $s=\sigma+it$ does not belong to the set $\{s_1,\dots,s_r\}$, thus the relation (2.4) implies

$$|g_{\lambda,\delta}(\sigma+it)-f(\sigma+it)|^2\leq 2\Big(|I(\sigma+it)|^2+|J(\sigma+it)|^2\Big).$$

Note that in the proof of Theorem 2.4 (see [10, §9.51]), we have the following

$$\frac{1}{2T}\int_{-T}^{T}|I(\sigma+it)|^2dt=B(\delta^{2\sigma-2\alpha})$$

uniformly with respect to T and $\sigma \in [\sigma_1, \sigma_2]$. It follows that

$$\begin{aligned}&\lim_{\delta\to 0}\limsup_{T\to\infty}\sup_{\sigma\in[\sigma_1,\sigma_2]}\frac{1}{4T}\int_{t_1}^{2T}|I(\sigma+it)|^2dt\\ &\le \lim_{\delta\to 0}\limsup_{T\to\infty}\sup_{\sigma\in[\sigma_1,\sigma_2]}\frac{1}{4T}\int_{-2T}^{2T}|I(\sigma+it)|^2dt \le \lim_{\delta\to 0}B(\delta^{2\sigma_1-2\alpha})=0.\end{aligned}$$

On the other hand, by Stirling's formula, there is a constant $A > 0$ such that uniformly in the strip $\sigma' \le \sigma \le \sigma''$, we have

$$|\Gamma(\sigma+it)| = B(e^{-A|t|}), \quad t\to\infty,$$

where $\sigma' < \min_j\{\mathrm{Re}((s_j-\sigma_2)/\lambda)\}$ and $\sigma'' > \max_j\{\mathrm{Re}((s_j-\sigma_1)/\lambda)\}$ are chosen first. This, together with (2.5), implies that

$$\limsup_{T\to\infty}\sup_{\sigma\in[\sigma_1,\sigma_2]}\frac{1}{4T}\int_{t_1}^{2T}|J(\sigma+it)|^2dt = B\left(\limsup_{T\to\infty}\frac{1}{4T}\int_{t_1}^{2T}e^{-At}dt\right)=0.$$

The proof is complete by combining the two estimates above. □

As a consequence of Proposition 2.5, we have the following.

Proposition 2.6. *Let K be a compact subset in D_0. Then for fixed $\lambda > \sigma_a - \sigma_0 + 1$*

$$\lim_{\delta\to 0}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\sup_{s\in K}d(f(s+it),g_{\lambda,\delta}(s+it))dt=0.$$

§ 3. Probability space associated with $\{\lambda_m\}$

For $T > 0$, we define a probability measure Q_T on $(\Omega, \mathcal{B}(\Omega))$ by

$$Q_T(A) := \nu_T\big((e^{-it\lambda_m})_{m\in\mathbb{N}}\in A\big), \quad A\in\mathcal{B}(\Omega).$$

If $\{\lambda_m\}$ is linearly independent over the field of rational numbers, then it is well known that $\{Q_T\}$ converges weakly the Haar measure m_H as $T\to\infty$, [5, 6, 7, 8]. Without assumption of linear independence of $\{\lambda_m\}$, the sequence $\{Q_T\}$ still converges weakly, but the limit probability measure, in general, is not m_H at all. The proof of this general case is not different so much.

Theorem 3.1. *There exists a probability measure $\mathbf{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that the sequence of probability measures $\{Q_T\}$ converges weakly to $\mathbf{P}$ as $T\to\infty$. If $\{\lambda_m\}$ is linearly independent over the field of rational numbers, then $\mathbf{P}$ coincides with the Haar measure m_H.*

Proof. The dual group of Ω is isomorphic to $\bigoplus_{m\in\mathbb{N}}\mathbb{Z}_m$, where $\mathbb{Z}_m=\mathbb{Z}$ for all $m\in\mathbb{N}$. $\underline{k}=\{k_m : m\in\mathbb{N}\}\in\bigoplus_{m\in\mathbb{N}}\mathbb{Z}_m$, where only a finite number of integers k_m are non-zero, acts on Ω by

$$\omega\mapsto\omega^{\underline{k}}=\prod_{m=1}^{\infty}\omega^{k_m}(m),\quad \omega\in\Omega.$$

Then, the Fourier transform $g_T(\underline{k})$ of the measure Q_T is of the form

$$\begin{aligned} g_T(\underline{k}) &= \int_\Omega\left(\prod_{m=1}^{\infty}\omega^{k_m}(m)\right)dQ_T=\frac{1}{T}\int_0^T\left(\prod_{m=1}^{\infty}e^{-it\lambda_m k_m}\right)dt \\ &= \begin{cases} 1, & \text{if } \sum_{m=1}^{\infty}\lambda_m k_m=0, \\ \dfrac{\exp\{-iT\sum_{m=1}^{\infty}\lambda_m k_m\}-1}{-iT\sum_{m=1}^{\infty}\lambda_m k_m}, & \text{if } \sum_{m=1}^{\infty}\lambda_m k_m\neq 0. \end{cases} \end{aligned}$$

Hence, the limit

$$g(\underline{k}):=\lim_{T\to\infty}g_T(\underline{k})=\begin{cases} 1, & \text{if } \sum_{m=1}^{\infty}\lambda_m k_m=0, \\ 0, & \text{if } \sum_{m=1}^{\infty}\lambda_m k_m\neq 0, \end{cases}$$

exists for every $\underline{k}$. A continuity theorem for probability measures on locally compact group implies that there exists a probability measure $\mathbf{P}$ on Ω such that $\{Q_T\}$ converges weakly to $\mathbf{P}$ as $T\to\infty$. Moreover, $g(\underline{k})$ is the Fourier transform of $\mathbf{P}$. Now, if $\{\lambda_m\}$ is linearly independent over the field of rational numbers, then $\sum_{m=1}^{\infty}\lambda_m k_m=0$ holds iff $k_m=0$ for all m. It then follows that $\mathbf{P}$ coincides with the Haar measure m_H. □

Lemma 3.2. *$\{\omega(m)\}_{m\in\mathbb{N}}$ is an orthonormal system in $L^2(\Omega,\mathbf{P})$, that is,*

$$\mathbf{E}^{(\mathbf{P})}\left[\omega(m_1)\overline{\omega(m_2)}\right]=\begin{cases} 1, & \text{if } m_1=m_2, \\ 0, & \text{if } m_1\neq m_2. \end{cases}$$

Here $\mathbf{E}^{(\mathbf{P})}$ denotes the expectation with respect to $\mathbf{P}$.

Proof. Let $m_1\neq m_2$. Take $\underline{k}=\{k_m : m\in\mathbb{N}\}$ such that $k_{m_1}=1, k_{m_2}=-1$ and the others are zero. We have $\sum_{m=1}^{\infty}\lambda_m k_m=\lambda_{m_1}-\lambda_{m_2}\neq 0$. Therefore

$$\mathbf{E}^{(\mathbf{P})}[\omega(m_1)\overline{\omega(m_2)}]=g(\underline{k})=0,$$

which completes the proof. □

The following lemma is fundamental in probability theory but useful.

Lemma 3.3. *Let $\{A_m\}_{m\in\mathbb{N}} \subset \mathbb{C}$ and $\sum_{m=1}^{\infty} |A_m|^2 < \infty$. Then there exists a random variable $F\colon \Omega \to \mathbb{C}$ such that*

$$F \overset{L^2(\Omega,\mathbf{P})}{=} \sum_{m=1}^{\infty} A_m \omega(m),$$

where "$\overset{L^2(\Omega,\mathbf{P})}{=}$" means that the equality holds in $L^2(\Omega, \mathbf{P})$. Assume further that

$$\sum_{m=1}^{\infty} A_m \omega(m)$$

converges $\mathbf{P}$-a.e. *Then*

$$F(\omega) = \sum_{m=1}^{\infty} A_m \omega(m), \quad \mathbf{P}\text{-a.e.}$$

§ 4. Limit theorems for general Dirichlet series on the complex plane

§ 4.1. Absolutely convergent case

For $\sigma > \sigma_a$, we define a random variable $f(\sigma, \cdot)\colon \Omega \to \mathbb{C}$ as

$$f(\sigma, \omega) := \sum_{m=1}^{\infty} a_m e^{-\lambda_m \sigma} \omega(m).$$

Then $f(\sigma, \omega)$ is continuous as a mapping from Ω to $\mathbb{C}$ because the series $f(\sigma, \omega)$ is dominated by

$$\sum_{m=1}^{\infty} |a_m| e^{-\lambda_m \sigma} < \infty.$$

Note that $f\big(\sigma, (e^{-i\lambda_m t})_{m\in\mathbb{N}}\big) = f(\sigma + it)$. In addition, it follows from the continuity of $f(\sigma, \omega)$ that the sequence of probability measures $\{Q_T f(\sigma, \cdot)^{-1}\}$ on $\mathbb{C}$ converges weakly to $\mathbf{P} f(\sigma, \cdot)^{-1}$, the distribution of $f(\sigma, \omega)$ under $\mathbf{P}$, as $T \to \infty$.

Theorem 4.1. *For $\sigma > \sigma_a$,*

$$\nu_T(f(\sigma + it) \in \cdot) \xrightarrow{d} \mathbf{P} f(\sigma, \cdot)^{-1} \quad \text{as} \quad T \to \infty,$$

where "$\xrightarrow{d}$" denotes the weak convergence of probability measures.

§ 4.2. General case

Lemma 4.2. *For $\sigma > \sigma_0$, we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} |f(\sigma + it)|^2 dt = \sum_{m=1}^{\infty} |a_m|^2 e^{-2\lambda_m \sigma} < \infty, \tag{4.1}$$

where t_0 is a number such that $\{\sigma + it : t \geq t_0\} \cap \{s_1, \ldots, s_r\} = \emptyset$.

Proof. This lemma is an easy consequence of Proposition 2.5 with $K = \{\sigma\}$. $\square$

Fix $\lambda > \sigma_a - \sigma_0 + 1$. For each $n \in \mathbb{N}$, we define

$$g_n(s) := g_{\lambda,e^{-\lambda_n}}(s) = \sum_{m=1}^{\infty} a_m \exp\{-e^{(\lambda_m-\lambda_n)\lambda}\}e^{-\lambda_m s} = \sum_{m=1}^{\infty} a_m v(m,n)e^{-\lambda_m s}.$$

The Dirichlet series g_n is absolutely convergent for $s \in \mathbb{C}$. Thus, given $\sigma > \sigma_0$, the infinite sum

$$g_n(\sigma,\omega) := \sum_{m=1}^{\infty} a_m v(m,n)e^{-\lambda_m\sigma}\omega(m)$$

is convergent for each $\omega \in \Omega$. By Lemma 3.3, we see

$$g_n(\sigma,\omega) \overset{L^2(\Omega,\mathbf{P})}{=} \sum_{m=1}^{\infty} a_m v(m,n)e^{-\lambda_m\sigma}\omega(m).$$

Besides, the relation (4.1) implies that there exists $f(\sigma,\cdot) \in L^2(\Omega,\mathbf{P})$ such that

$$f(\sigma,\omega) \overset{L^2(\Omega,\mathbf{P})}{=} \sum_{m=1}^{\infty} a_m e^{-\lambda_m\sigma}\omega(m).$$

Lemma 4.3. *For $\sigma > \sigma_0$, the sequence $\{g_n(\sigma,\omega)\}$ converges in $L^2(\Omega,\mathbf{P})$ to $f(\sigma,\omega)$ as $n \to \infty$.*

Proof. We have

$$\|g_n(\sigma,\omega) - f(\sigma,\omega)\|_{L^2}^2 = \sum_{m=1}^{\infty} |a_m|^2|1 - v(m,n)|^2 e^{-2\lambda_m\sigma}.$$

The above series is dominated by $\sum_{m=1}^{\infty} |a_m|^2 e^{-2\lambda_m\sigma} < \infty$. Moreover, for any fixed m, it is clear that $v(m,n) \to 1$ as $n \to \infty$. Therefore, $\|g_n(\sigma,\omega) - f(\sigma,\omega)\|_{L^2}^2 \to 0$ as $n \to \infty$ by using the dominated convergence theorem. $\square$

Theorem 4.4. *For $\sigma > \sigma_0$,*

$$\nu_T\big(f(\sigma+it) \in \cdot\big) \xrightarrow{d} \mathbf{P}f(\sigma,\omega)^{-1} \quad \textit{as} \quad T \to \infty.$$

Proof. Let $\theta_T \colon (\widehat{\Omega},\widehat{\mathbf{P}}) \to [0,T]$ be a random variable uniformly distributed on $[0,T]$. We put

$$X_{T,n}(\sigma) := g_n(\sigma + i\theta_T).$$

First, by Theorem 4.1,

$$X_{T,n} \xrightarrow{d} g_n(\sigma,\omega) \quad \text{as} \quad T \to \infty. \tag{4.2}$$

Secondly, Lemma 4.3 implies that

$$g_n(\sigma,\omega) \xrightarrow{d} f(\sigma,\omega) \quad \text{as} \quad n\to\infty. \tag{4.3}$$

Thirdly, take $K=\{\sigma\}$ in Proposition 2.5, we find that

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}|f(\sigma+it)-g_n(\sigma+it)|dt=0,$$

then it follows from Chebyshev's inequality that for any $\varepsilon>0$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\widehat{\mathbf{P}}(|X_{T,n}-Y_T|\ge\varepsilon)=0, \tag{4.4}$$

where $Y_T := f(\sigma+i\theta_T)$. Finally, by Theorem A.5, (4.2)–(4.4) are enough to deduce

$$Y_T \xrightarrow{d} f(\sigma,\omega) \quad \text{as} \quad T\to\infty.$$

□

Remark 1. Theorem 4.4 and Theorem 5.5 below are extensions of the main results in [7] and [8], respectively. Comparing with proofs in [7, 8], the basic idea does not change but a number of arguments are reduced. For instance, we use L^2-convergence instead of using the tightness of measures and ergodic theory.

§ 4.3. In the case of linearly independent $\{\lambda_m\}$

When $\{\lambda_m\}$ is linearly independent over the field of rational numbers, the probability measure $\mathbf{P}$ coincides with the Haar measure m_H. Under $\mathbf{P}$, the sequence $\{\omega(m)\}$ becomes independent.

Theorem 4.5. *For $\sigma>\sigma_0$, the series*

$$f(\sigma,\omega)=\sum_{m=1}^{\infty}a_m e^{-\lambda_m\sigma}\omega(m)$$

converges almost everywhere and converges in $L^2(\Omega,\mathbf{P})$.

Proof. For $M\in\mathbb{N}$, let

$$X_M(\omega):=\sum_{m=1}^{M}a_m e^{-\lambda_m\sigma}\omega(m).$$

Then $\{X_M\}_{M\in\mathbb{N}}$ is a martingale because $\{\omega(m)\}$ is a sequence of independent random variables with means 0. We have

$$\mathbf{E}^{(\mathbf{P})}[|X_M|^2]=\sum_{m=1}^{M}|a_m|^2e^{-2\lambda_m\sigma}\le\sum_{m=1}^{\infty}|a_m|^2e^{-2\lambda_m\sigma}<\infty.$$

Therefore, by Doob's martingale convergence theorem, the sequence $\{X_M\}_{M\in\mathbb{N}}$ converges almost everywhere and converges in $L^2(\Omega, \mathbf{P})$. $\square$

§ 5. Limit theorems for general Dirichlet series in functional spaces

§ 5.1. Absolutely convergent case

Recall that $D = \{s \in \mathbb{C} : \sigma > \sigma_a\}$ and $H(D)$ denotes the space of analytic functions on D equipped with the topology of uniform convergence on compacta. For $s \in D, \omega \in \Omega$, let

$$f(s,\omega) := \sum_{m=1}^{\infty} a_m e^{-\lambda_m s}\omega(m).$$

Then $f(s,\omega)$ is an $H(D)$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), \mathbf{P})$. In addition, we will prove that $f(s,\omega)$ is continuous as a mapping from Ω to $H(D)$. Indeed, let $\{\omega^{(n)}\}$ be a sequence converging to ω in Ω. We need to prove that $\{f(s,\omega^{(n)})\}$ converges to $f(s,\omega)$ in $H(D)$. Given a compact subset $K \subset D$, let $\sigma_1 = \min_{s\in K} \operatorname{Re} s > \sigma_a$. We have

$$\sup_{s\in K} |f(s,\omega^{(n)}) - f(s,\omega)| \leq \sum_{m=1}^{\infty} |a_m| e^{-\lambda_m \sigma_1} |\omega^{(n)}(m) - \omega(m)|.$$

Since $\sum_{m=1}^{\infty} |a_m| e^{-\lambda_m\sigma_1} < \infty$, it follows from the dominated convergence theorem that

$$\sup_{s\in K} |f(s,\omega^{(n)}) - f(s,\omega)| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, the mapping $f(s,\omega)$ is continuous. Consequently, the sequence of probability measures $\{Q_T f(s,\omega)^{-1}\}_T$ on $(H(D), \mathcal{B}(H(D)))$ converges weakly to $\mathbf{P}f(s,\omega)^{-1}$ as $T \to \infty$. Obviously, we have

$$\nu_T(f(s+it) \in A) = Q_T(\omega : f(s,\omega) \in A), \quad A \in \mathcal{B}(H(D)).$$

Therefore, we have just proved the following theorem.

Theorem 5.1. *The sequence of probability measures*

$$\nu_T(f(s+it) \in A), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $\mathbf{P}_f$ *as* $T \to \infty$, *where* $\mathbf{P}_f$ *denotes the distribution of the* $H(D)$*-valued random element* $f(s,\omega)$.

§ 5.2. General case

Recall that $D_0 = \{s \in \mathbb{C} : \sigma > \sigma_0\}$. There is a sequence $\{K_n\}$ of compact subsets of D_0 such that (i) $D_0 = \bigcup_{n=1}^{\infty} K_n$; (ii) $K_n \subset K_{n+1}$; (iii) if K is a compact set and $K \subset D_0$, then $K \subset K_n$ for some n. Then for $f, g \in H(D_0)$, let

$$\rho(f,g) := \sum_{n=1}^{\infty} 2^{-n} \frac{\sup_{s \in K_n} |f(s) - g(s)|}{1 + \sup_{s \in K_n} |f(s) - g(s)|}.$$

The topological space $H(D_0)$ becomes a complete separable metric space. Similarly, for $f, g \in M(D_0)$, let

$$\bar{\rho}(f,g) := \sum_{n=1}^{\infty} 2^{-n} \frac{\sup_{s \in K_n} d(f(s), g(s))}{1 + \sup_{s \in K_n} d(f(s), g(s))}.$$

Then $M(D_0)$ becomes a separable metric space.

For each $M \in \mathbb{N}$, we define an $H(D_0)$-valued random element $f_M(s, \omega)$ as

$$f_M(s,\omega) := \sum_{m=1}^{M} a_m e^{-\lambda_m s} \omega(m).$$

Lemma 5.2. *There is an $H(D_0)$-valued random element $f(s,\omega)$ such that*

$$\lim_{M \to \infty} \mathbf{E}^{(\mathbf{P})} \Big[\rho(f_M(\cdot, \omega), f(\cdot, \omega))^2 \Big] = 0.$$

In particular, for any fixed $s \in D_0$,

$$f(s,\omega) \overset{L^2(\Omega, \mathbf{P})}{=} \sum_{m=1}^{\infty} a_m e^{-\lambda_m s} \omega(m).$$

Proof. Let K be a compact subset in D_0. For $M < M'$, denote

$$h_{M,M'}(\omega) := \sup_{s \in K} |f_M(s,\omega) - f_{M'}(s,\omega)|.$$

Let L be a simple closed contour lying in D_0 and enclosing the set K and let δ denote the distance of L from the set K. For each $\omega \in \Omega$, Cauchy's integral formula implies that

$$h_{M,M'}(\omega) = \sup_{s \in K} |f_M(s,\omega) - f_{M'}(s,\omega)| \le \frac{1}{2\pi\delta} \int_L |f_M(z,\omega) - f_{M'}(z,\omega)| |dz|,$$

then by the Cauchy-Schwarz inequality,

$$|h_{M,M'}(\omega)|^2 \le \frac{|L|}{(2\pi\delta)^2} \int_L |f_M(z,\omega) - f_{M'}(z,\omega)|^2 |dz|.$$

Let $\sigma_1 = \min_{z\in L} \operatorname{Re} z > \sigma_0$. For $z = \sigma + it(\sigma \geq \sigma_1)$, it follows from the orthogonal property of the sequence $\{\omega(m)\}$ that

$$\mathbf{E}^{(\mathbf{P})}[|f_M(z,\omega) - f_{M'}(z,\omega)|^2] = \sum_{m=M+1}^{M'} |a_m|^2 e^{-2\lambda_m\sigma} \leq \sum_{m=M+1}^{M'} |a_m|^2 e^{-2\lambda_m\sigma_1}.$$

Consequently,

$$\begin{aligned}\mathbf{E}^{(\mathbf{P})}[|h_{M,M'}(\omega)|^2] &\leq \mathbf{E}^{(\mathbf{P})}\left[\frac{|L|}{(2\pi\delta)^2}\int_L |f_M(z,\omega) - f_{M'}(z,\omega)|^2 |dz|\right] \\ &= \frac{|L|}{(2\pi\delta)^2}\int_L \mathbf{E}^{(\mathbf{P})}[|f_M(z,\omega) - f_{M'}(z,\omega)|^2]|dz| \\ &\leq \frac{|L|^2}{(2\pi\delta)^2}\sum_{m=M+1}^{M'} |a_m|^2 e^{-2\lambda_m\sigma_1} \\ &\longrightarrow 0 \quad \text{as} \quad M, M' \to \infty.\end{aligned}$$

The above result holds for any compact subset K in D_0. Therefore, taking into account the definition of the metric ρ, we obtain

$$\lim_{M,M'\to\infty} \mathbf{E}^{(\mathbf{P})}\left[\rho(f_M(\cdot,\omega), f_{M'}(\cdot,\omega))^2\right] = 0.$$

Since $H(D_0)$ is a complete separable metric space, there is an $H(D_0)$-valued random element $f(s,\omega)$ such that

$$\lim_{M\to\infty} \mathbf{E}^{(\mathbf{P})}\left[\rho(f_M(\cdot,\omega), f(\cdot,\omega))^2\right] = 0.$$

It follows that there exists a subsequence $\{f_{M_k}(s,\omega)\}_k$ converging to $f(s,\omega)$ $\mathbf{P}$-a.e. as $k \to \infty$, that is,

$$\lim_{k\to\infty} \rho(f_{M_k}(\cdot,\omega), f(\cdot,\omega)) = 0, \quad \mathbf{P}\text{-a.e.}$$

In particular, for any fixed $s \in D_0$,

$$\lim_{k\to\infty} |f_{M_k}(s,\omega) - f(s,\omega)| = 0, \quad \mathbf{P}\text{-a.e.},$$

which implies

$$f(s,\omega) \overset{L^2(\Omega,\mathbf{P})}{=} \sum_{m=1}^{\infty} a_m e^{-\lambda_m s}\omega(m).$$

The lemma has been proved. □

For each $n \in \mathbb{N}$, we define a random element $g_n(s,\omega)\colon \Omega \to H(D_0)$ as

$$g_n(s,\omega) := \sum_{m=1}^{\infty} a_m v(m,n) e^{-\lambda_m s}\omega(m).$$

Lemma 5.3.
$$\lim_{n\to\infty} \mathbf{E}^{(\mathbf{P})}\left[\rho(g_n(\cdot,\omega), f(\cdot,\omega))^2\right] = 0.$$

Proof. This proof is similar to the proof of Lemma 5.2. Therefore, we only need to show that for a given compact subset K of D_0,

$$\lim_{n\to\infty} \mathbf{E}^{(\mathbf{P})}[|h_n(\omega)|^2] = 0, \tag{5.1}$$

where
$$h_n(\omega) := \sup_{s\in K} |g_n(s,\omega) - f(s,\omega)|.$$

To prove (5.1), let L be a simple closed contour lying in D_0 and enclosing the set K and let δ denote the distance of L from the set K. For $\omega \in \Omega$ for which $f(s,\omega) \in H(D_0)$, Cauchy's integral formula implies that

$$h_n(\omega) = \sup_{s\in K} |g_n(s,\omega) - f(s,\omega)| \le \frac{1}{2\pi\delta}\int_L |g_n(z,\omega) - f(z,\omega)||dz|,$$

then by the Cauchy-Schwarz inequality, we obtain

$$h_n^2(\omega) \le \frac{|L|}{(2\pi\delta)^2}\int_L |g_n(z,\omega) - f(z,\omega)|^2|dz|.$$

Let $\sigma_1 = \min_{z\in L} \operatorname{Re} z > \sigma_0$. For $z = \sigma + it(\sigma \ge \sigma_1)$, from Lemma 5.2, we have

$$f(z,\omega) \overset{L^2(\Omega,\mathbf{P})}{=} \sum_{m=1}^{\infty} a_m e^{-\lambda_m z}\omega(m),$$

which implies

$$\begin{aligned}\mathbf{E}^{(\mathbf{P})}[|g_n(z,\omega) - f(z,\omega)|^2] &= \sum_{m=1}^{\infty} |a_m|^2|1 - v(m,n)|^2 e^{-2\lambda_m\sigma}\\ &\le \sum_{m=1}^{\infty} |a_m|^2|1 - v(m,n)|^2 e^{-2\lambda_m\sigma_1} < \infty.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{E}^{(\mathbf{P})}[h_n^2(\omega)] &\le \mathbf{E}^{(\mathbf{P})}\left[\frac{|L|}{(2\pi\delta)^2}\int_L |g_n(z,\omega) - f(z,\omega)|^2|dz|\right]\\ &= \frac{|L|}{(2\pi\delta)^2}\int_L \mathbf{E}^{(\mathbf{P})}[|g_n(z,\omega) - f(z,\omega)|^2]|dz|\\ &\le \frac{|L|^2}{(2\pi\delta)^2}\sum_{m=1}^{\infty} |a_m|^2|1 - v(m,n)|^2 e^{-2\lambda_m\sigma_1}.\end{aligned}$$

Once again, our desired result (5.1) follows by using the dominated convergence theorem. The proof is complete. ☐

Corollary 5.4. (i) *For any* $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\Big(\rho(g_n(\cdot,\omega), f(\cdot,\omega)) \geq \varepsilon\Big) = 0.$$

(ii)

$$\mathbf{P}_{g_n} \xrightarrow{d} \mathbf{P}_f \quad as \quad n \to \infty, \tag{5.2}$$

where $\mathbf{P}_{g_n}$ *and* $\mathbf{P}_f$ *denote the distributions of the* $H(D_0)$*-valued or* $M(D_0)$*-valued random elements* g_n *and* f*, respectively.*

Proof. (i) follows from Lemma 5.3 by Chebyshev's inequality. (ii) follows from (i) by Theorem A.4. □

For every compact subset K of D_0, Proposition 2.6 claims that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K} d(g_n(s+it), f(s+it))dt = 0.$$

Thus, by Chebyshev's inequality, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \nu_T\Big(\bar{\rho}(g_n(\cdot+it), f(\cdot+it)) \geq \varepsilon\Big) = 0. \tag{5.3}$$

Since the Dirichlet series $g_n(s)$ is absolutely convergent in D_0, it follows from Theorem 5.1 that

$$\nu_T(g_n(s+it) \in \cdot) \xrightarrow{d} \mathbf{P}_{g_n} \quad \text{as} \quad T \to \infty, \tag{5.4}$$

where the weak convergence is still true in the space of meromorphic functions $M(D_0)$. Therefore, (5.2)–(5.4) imply the limit theorem for $f(s)$ in $M(D_0)$.

Theorem 5.5. *Suppose that conditions* (1.2) *and* (1.3) *are satisfied. Then there exists an* $H(D_0)$*-valued random element* $f(s,\omega)$ *such that*

$$f(s,\omega) \overset{L^2(\Omega,\mathbf{P})}{=} \sum_{m=1}^{\infty} a_m e^{-\lambda_m s}\omega(m), \quad s \in D_0.$$

Moreover, the sequence of probability measures

$$\nu_T(f(s+it) \in A), \quad A \in \mathcal{B}(M(D_0)),$$

converges weakly to $\mathbf{P}_f$ *as* $T \to \infty$*, where* $\mathbf{P}_f$ *denotes the distribution of* $f(s,\omega)$*.*

Remark 2. If the strip $D_1 = \{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2\}(\sigma_0 < \sigma_1 < \sigma_2 \le \infty)$ contains no pole of $f(s)$, then in view of the proof of Theorem 5.5 we can assert the following.

"The sequence of probability measures

$$\nu_T(f(s+it) \in A), \quad A \in \mathcal{B}(H(D_1)),$$

converges weakly to $\mathbf{P}_f$ *as* $T \to \infty$, *where* $\mathbf{P}_f$ *denotes the distribution of the* $H(D_1)$-*valued random element* $f(s, \omega)$*."*

Remark 3. Assume that the series

$$f(\sigma_1, \omega) := \sum_{m=1}^{\infty} a_m e^{-\lambda_m \sigma_1} \omega(m) \tag{5.5}$$

converges almost everywhere for any $\sigma_1 > \sigma_0$. Then for $\mathbf{P}$-a.e. $\omega \in \Omega$, the series

$$f(s, \omega) := \sum_{m=1}^{\infty} a_m e^{-\lambda_m s} \omega(m)$$

converges uniformly on each compact subset of the half-plane $\{s \in \mathbb{C} : \sigma > \sigma_1\}$. Let A_n denote the set of $\omega \in \Omega$ for which the series $f(s, \omega)$ converges uniformly on compact subsets of the half-plane $\{s \in \mathbb{C} : \sigma > \sigma_0 + 1/n\}$. Obviously, $\mathbf{P}(A_n) = 1$ for all $n \in \mathbb{N}$. Now if we take

$$A = \bigcap_{n=1}^{\infty} A_n,$$

then $\mathbf{P}(A) = 1$, and, for $\omega \in A$, the series $f(s, \omega)$ converges uniformly on compact subsets of D_0. It follows that $f(s, \omega)$ is an $H(D_0)$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), \mathbf{P})$. In other words, for $\mathbf{P}$-a.e. $\omega \in \Omega$, the sequence $\{f_M(s, \omega)\}$ converges to $f(s, \omega)$ in $H(D_0)$ as $M \to \infty$. This implies that the function $f(s, \omega)$ here coincides with the function defined in Lemma 5.2. The following two cases ensure the assumption (5.5).

(i) If $\{\lambda_m\}$ is linearly independent over the field of rational numbers, then the condition (5.5) is automatically satisfied (Theorem 4.5).

(ii) In general case, if the series

$$\sum_{m=1}^{\infty} |a_m|^2 e^{-2\lambda_m \sigma} (\log m)^2$$

converges for $\sigma > \sigma_0$, then the condition (5.5) is also satisfied. Indeed, let $X_m := a_m e^{-\lambda_m \sigma} \omega(m)$. Then $\{X_m\}$ becomes a sequence of orthogonal random variables and

$$\sum_{m=1}^{\infty} \mathbf{E}[|X_m|^2] (\log m)^2 < \infty.$$

Therefore, by [9], the series

$$\sum_{m=1}^{\infty} X_m$$

converges almost everywhere.

§ Appendix A. Convergence of probability measures on metric spaces

Let (S, ρ) be a separable metric space and let $\{X_n\}_{n\in\mathbb{N}}$ and X be S-valued random elements defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Definition A.1. The sequence $\{X_n\}_{n\in\mathbb{N}}$ converges in probability to X if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}(\rho(X_n, X) \geq \varepsilon) = 0.$$

Definition A.2. The sequence $\{X_n\}_{n\in\mathbb{N}}$ converges weakly to X if for every bounded continuous function $f\colon (S, \rho) \to \mathbb{R}$,

$$\lim_{n\to\infty} \mathbf{E}[f(X_n)] = \mathbf{E}[f(X)].$$

Lemma A.3 ([1, Theorem 2.1])**.** *The sequence $\{X_n\}_{n\in\mathbb{N}}$ converges weakly to X if and only if*

$$\lim_{n\to\infty} \mathbf{E}[f(X_n)] = \mathbf{E}[f(X)]$$

for every bounded uniformly continuous function $f\colon (S, \rho) \to \mathbb{R}$.

Theorem A.4. *The convergence in probability implies the weak convergence.*

Proof. Suppose that $\{X_n\}_{n\in\mathbb{N}}$ converges in probability to X. Let $f\colon (S, \rho) \to \mathbb{R}$ be a bounded uniformly continuous function. Then given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \quad \text{if } \rho(x, y) < \delta.$$

Now we have

$$\begin{aligned}
\mathbf{E}[|f(X_n) - f(X)|] &= \int_{\Omega} |f(X_n) - f(X)| d\mathbf{P} \\
&= \int_{\rho(X_n, X)<\delta} |f(X_n) - f(X)| d\mathbf{P} + \int_{\rho(X_n, X)\geq\delta} |f(X_n) - f(X)| d\mathbf{P} \\
&\leq \varepsilon + 2M\mathbf{P}(\rho(X_n, X) \geq \delta),
\end{aligned}$$

where $M := \sup_{x\in S}|f(x)| < \infty$. Therefore,

$$\limsup_{n\to\infty} \mathbf{E}[|f(X_n) - f(X)|] \le \varepsilon.$$

By letting $\varepsilon \to 0$, we obtain

$$\limsup_{n\to\infty} |\mathbf{E}[f(X_n)] - \mathbf{E}[f(X)]| \le \limsup_{n\to\infty} \mathbf{E}[|f(X_n) - f(X)|] = 0,$$

which completes the proof of Theorem A.4. □

Let $\{Y_n\}_n$ and $\{X_{k,n}\}_{k,n}$ be S-valued random elements.

Theorem A.5 ([1, Theorem 3.2]). *Assume that*

(i)
$$X_{k,n} \xrightarrow{d} X_k \quad as \quad n\to\infty;$$

(ii)
$$X_k \xrightarrow{d} X \quad as \quad k\to\infty;$$

(iii) *for every* $\varepsilon > 0$

$$\lim_{k\to\infty}\limsup_{n\to\infty} \mathbf{P}\{\rho(X_{k,n}, Y_n) \ge \varepsilon\} = 0.$$

Then $Y_n \xrightarrow{d} X$ *as* $n \to \infty$.

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