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Rauzy fractals induced from automorphisms on the free group of rank 2 related to continued fractions

By

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Abstract

For a substitution satisfying the Pisot, irreducible, unimodular condition, a tiling substitution plays a key role in the construction of Rauzy fractals (see [15, 3]). To try extending techniques developed for substitutions to automorphisms, [6] gives the way to construct Rauzy fractals by using tiling substitutions for automorphisms related to hyperbolic companion matrices of quadratic polynomials. This paper shows application to another class of automorphisms related to continued fraction expansions.

§ 0. Introduction

After Rauzy introduced fractal, called Rauzy fractal, as a geometric representation of substitutive dynamical system for a special Pisot, irreducible unimodular substitution in [15], it has been extensively studied (e.g., [3, 11, 14]). In especial, Arnoux and Ito [3] gave a method to construct Rauzy fractals by using tiling substitutions. After that, Arnoux, Berthé, Hilion and Siegel [2] initiated the investigation into automorphisms on free groups. They actually forcused on automorphisms whose iterations, applied to letters, do not give rise to any cancellations of letters. An incidence matrix A_σ is a linear map associated to an automorphism σ by abelianization (for details, see the next section). [6] is devoted to construct stepped surfaces and Rauzy fractals induced from automorphisms on the free group F_2 of rank 2 whose incidence matrices are the companion matrices of quadratic polynomials $x^2 - ax \mp 1$ such that

$$A_\pm = \begin{pmatrix} 0 & \pm 1 \\ 1 & a \end{pmatrix}, \tag{0.1}$$

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and assume "hyperbolicity" instead of the Pisot condition. Then there are four cases corresponding to configurations of eigenvalues of the matrix (0.1). Let us consider one case related to the matrix A_- whose eigenvalues satisfy $\lambda > 1,\ 0 < \lambda' < 1$, concretely the automorphism σ with $a = 3$:

$$\sigma : \begin{cases} 1 \to 2 \\ 2 \to 21^{-1}22 \end{cases}, \quad A_\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}.$$

Then we encounter the following cancellation problem under the iteration of σ as follows:

$$\sigma^2(2) = \sigma(21^{-1}22) = 21^{-1}22\ \underline{2^{-1}\ 2}1^{-1}22\ 21^{-1}22.$$

Such a cancellation never occurs for a substitution. The idea to solve such a problem is to find a substitution τ and an automorphism δ such that

$$\sigma = \delta^{-1} \circ \tau \circ \delta,$$

in other words, σ is conjugate to a substitution τ. In fact, for the above example, we can find

$$\tau : \begin{cases} 1 \to 12 \\ 2 \to 212 \end{cases}, \quad \delta : \begin{cases} 1 \to 21^{-1} \\ 2 \to 2 \end{cases} \quad \left(\delta^{-1} : \begin{cases} 1 \to 1^{-1}2 \\ 2 \to 2 \end{cases} \right).$$

Through the discussion on the classes of automorphisms related to the matrix (0.1), the possibility to apply the method in [6] to another class of automorphisms was found. To do it, it is necessary for σ to be decomposed as $\sigma = \delta^{-1} \circ \tau \circ \delta$ with δ and τ satisfying the following conditions:

1. τ is a substitution or an alternating substitution, where an alternating substitution is an endomorphism on the free group F_2 such that $\tau(i)$ $(i = 1, 2)$ are words with the alphabet $\{1^{-1}, 2^{-1}\}$.

2. δ is an automorphism on F_2 such that both words $\delta^{-1}(i)$ $(i = 1, 2)$ are words with one of the alphabets $\{1, 2\}$, $\{1^{-1}, 2\}$, $\{1, 2^{-1}\}$ or $\{1^{-1}, 2^{-1}\}$.

Ito and Yasutomi [12] gives automorphisms which fix characteristic sequences $C(x)$ of quadratic irrational numbers x (see the next section). Namely, $C(x)$ is a fixed point of the following automorphism σ_x:

$$\sigma_x = (\sigma_{1,a_1} \circ \sigma_{2,a_2} \circ \cdots \circ \sigma_{t_N,a_N}) \circ (\sigma_{t_{N+1},a_{N+1}} \circ \cdots \circ \sigma_{t_{N+2K},a_{N+2K}}) \circ (\sigma^{-1}_{t_N,a_N} \circ \cdots \circ \sigma^{-1}_{1,a_1}),$$

where $[0, a_1, a_2, \cdots, a_N, \overline{a_{N+1}, \cdots, a_{N+2K}}]$ is the continued fraction expansion of a quadratic irrational number $x \in [0, 1)$, and $\sigma_{1,a},\ \sigma_{2,a}$ are substitutions defined by

$$\sigma_{1,a} : \begin{cases} 1 \to 1^a2 \\ 2 \to 1^{a-1}2 \end{cases}, \quad \sigma_{2,a} : \begin{cases} 1 \to 12^{a-1} \\ 2 \to 12^a \end{cases},$$

$$t_k = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 2 & \text{if } k \text{ is even} \end{cases}.$$

Note that a substitution is naturally extended to an endomorphism on the free group, and $\sigma_{1,a},\ \sigma_{2,a}$ are invertible in this sense. Put

$$B_{a_1,a_2,\ldots,a_n} := \begin{pmatrix} a_1 - 1 & a_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_2 - 1 & a_2 \\ 1 & 1 \end{pmatrix} \cdots \begin{pmatrix} a_n - 1 & a_n \\ 1 & 1 \end{pmatrix}.$$

It is easy to check that the incidence matrices of automorphisms σ_x related to $x = [0, a_1, a_2, \cdots, a_N, \overline{a_{N+1}, \cdots, a_{N+2K}}]$ are given by

$$A_{\sigma_x} = B_{a_1,a_2,\ldots,a_{N+2K}} B^{-1}_{a_1,a_2,\ldots,a_N},$$

which are different from (0.1). Since the automorphisms σ_x related to continued fraction expansions satisfy the two conditions 1, 2 by taking $\tau = \sigma_{t_{N+1},a_{N+1}} \circ \cdots \circ \sigma_{t_{N+2K},a_{N+2K}}$ and $\delta = \sigma^{-1}_{t_N,a_N} \circ \cdots \circ \sigma^{-1}_{1,a_1}$, we apply the method in [6] for this new class of automorphisms.

In the next section, we recall results associated to characteristic sequences. In the last section, we construct Rauzy fractals induced from automorphisms σ_x, and define domain exchange transformations. The purpose of this paper is to find set equations for Rauzy fractals by using a conjugacy δ of an automorphism σ_x in Theorem 2.6, and show that the orbit of the origin point by the domain exchange transformation gives the sequence $C(x)$ in Corollary 2.10.

§ 1. Characteristic sequences and continued fraction expansions

For an endomorphism σ on the free group of rank 2 denoted by F_2 (resp. a substitution σ over the alphabet $\mathcal{A} = \{1, 2\}$), the incidence matrix A_σ defined by $(\mathbf{f}(\sigma(1)), \mathbf{f}(\sigma(2)))$ satisfies the following commutative diagram:

$$\begin{array}{ccc} F_2 & \xrightarrow{\sigma} & F_2 \\ \mathbf{f}\downarrow & & \downarrow\mathbf{f}, \\ \mathbb{Z}^2 & \xrightarrow[A_\sigma]{} & \mathbb{Z}^2 \end{array}$$

where $\mathbf{f} : F_2 \to \mathbb{Z}^2$ is a canonical homomorphism defined by $\mathbf{f}(\epsilon) = \boldsymbol{o}$ and $\mathbf{f}(i^{\pm 1}) = \pm \boldsymbol{e}_i,\ i \in \mathcal{A}$, and $\boldsymbol{e}_j$ are fundamental vectors. The characteristic polynomial of A_σ is denoted by $\Phi_\sigma(x)$. For an endomorphism on F_2 or a substitution, there are the following four conditions:

- (Pisot condition) The maximum root of $\Phi_\sigma(x)$ is Pisot number, that is, the dominant eigenvalue of A_σ is greater than one and the other has modulus less than one,

- (Irreducible condition) $\Phi_\sigma(x)$ is irreducible over $\mathbb{Q}$,
- (Unimodular condition) $|\det A_\sigma| = 1$,
- (Primitive condition) the matrix A_σ is primitive.

For a number $x \in I := [0,1)$, define the map C from I to $\mathcal{A}^{\mathbb{N}}$ by

$$C(x) := (C_1(x), C_2(x), \cdots, C_n(x), \cdots),$$

where $C_n(x) := \lfloor nx \rfloor - \lfloor (n-1)x \rfloor + 1$, and $\lfloor \cdot \rfloor$ means the floor function. Note that $C_n(x) := \lfloor nx \rfloor - \lfloor (n-1)x \rfloor$ when we use the alphabet $\mathcal{A} = \{0,1\}$. Put $c(x) = (C_2(x), C_3(x), \cdots, C_n(x), \cdots)$. Since $C_1(x) = 1$, $C(x) = 1c(x)$; and $C(x)$ or $c(x)$ are called the *characteristic sequence* or the *characteristic word* of x (cf. [13]).

Is there a non-trivial substitution σ for which $C(x)$ (resp. $c(x)$) is invariant, that is, $C(x)$ (resp. $c(x)$) is a fixed point of σ such that $\sigma(C(x)) = C(x)$ (resp. $\sigma(c(x)) = c(x)$)? Ito and Yasutomi [12] and Crisp, Moran, Pollington and Shiue [5] give the answer to this question by using the continued fraction expansion of x.

Let us define the functions $S: I \to I$ and $a: I \to \mathbb{Z}$ by

$$S(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$
$$a(x) := \left\lfloor \frac{1}{x} \right\rfloor,$$

and the sequence $\{a_n\}_{n=1}^{\infty}$ by $a_n := a(S^{n-1}(x))$, then we get the continued fraction expansion of x denoted by $[0, a_1, a_2, \cdots, a_n, \cdots]$.

Remark 1. (1) A quadratic irrational number x is reduced, that is, $0 < x < 1$ and $\overline{x} < -1$, where $\overline{x}$ means the algebraic conjugate of x, if and only if the continued fraction expansion of x is purely periodic (see [17]).
(2) A number x is quadratic if and only if the continued fraction expansion of x is eventually periodic.

Ito and Yasutomi give a substitution over $\mathcal{A}$ or an automorphism on F_2 for which $C(x)$ is invariant for a quadratic irrational number $x \in I$ in the following theorem.

Theorem 1.1 (Theorem 2.4 in [12]). *(1) If $x \in I$ is a quadratic irrational and reduced number, then $\gamma_x(C(x)) = C(x)$, where $[0, \overline{a_1, a_2, \cdots, a_{2K}}]$ is the continued fraction expansion of x, and γ_x is a substitution given by*

$$\gamma_x = \sigma_{t_1, a_1} \circ \cdots \circ \sigma_{t_{2K}, a_{2K}}.$$

(2) If $x \in I$ is a quadratic irrational number, then $\sigma_x(C(x)) = C(x)$, where $[0, a_1, a_2, \cdots, a_N, \overline{a_{N+1}, \cdots, a_{N+2K}}]$ is the continued fraction expansion of x, and σ_x is an automorphism given by

$$\sigma_x = \delta_x^{-1} \circ \tau_x \circ \delta_x,$$

where

$$\begin{aligned} \tau_x &= \sigma_{t_{N+1}, a_{N+1}} \circ \cdots \circ \sigma_{t_{N+2K}, a_{N+2K}} \\ \delta_x &= \sigma_{t_N, a_N}^{-1} \circ \cdots \circ \sigma_{1, a_1}^{-1}. \end{aligned}$$

In the case (1), the sequence $C(x)$ is a fixed point of the substitution γ_x, and the case boils down to the substitution case (cf. [3]). In the case (2), automorphisms σ_x are conjugate to invertible substitutions τ_x satisfying the Pisot, irreducible, unimodular, primitive conditions.

§ 2. Rauzy fractals and domain exchange transformations

In this section, we consider only the automorphisms σ_x in Theorem 1.1 (2) for quadratic irrational numbers $x \in I$ which are not reduced, and for simplicity, we use notations σ, τ, δ instead of σ_x, τ_x, δ_x, unless otherwise noted. Note that the substitution τ satisfies the Pisot, irreducible, unimodular, primitive conditions and it is invertible. Results in this section are derived by applying the previous paper [6], so brief proofs or ideas of proofs are explained.

Let us define the tiling substitution for a unimodular endomorphism on F_2 (see [7, 16]).

Definition 2.2. The free $\mathbb{Z}$-module $\mathcal{G}^*$ is defined by

$$\mathcal{G}^* := \left\{ \sum_{k=1}^{l} n_k(\boldsymbol{x}_k, i_k^*) \mid n_k \in \mathbb{Z},\ \boldsymbol{x}_k \in \mathbb{Z}^2,\ i_k \in \mathcal{A} \text{ for any } k,\ l < \infty \right\}.$$

An arbitrary endomorphism σ on F_2 is written by

$$\sigma(i) = w_1^{(i)} w_2^{(i)} \cdots w_{l^{(i)}}^{(i)},\ i \in \mathcal{A},$$

in reduced form, and define the k-prefix $P_k^{(i)}$ and k-suffix $S_k^{(i)} \in F_2$ for $0 \le k \le l^{(i)}$ by

$$P_k^{(i)} := w_1^{(i)} w_2^{(i)} \cdots w_{k-1}^{(i)},\ S_k^{(i)} := w_{k+1}^{(i)} w_{k+2}^{(i)} \cdots w_{l^{(i)}}^{(i)}.$$

The tiling substitution σ^* on $\mathcal{G}^*$ for a unimodular endomorphism σ on F_2 is defined by $\sigma^*(\boldsymbol{x}, i^*) :=$

$$\sum_{j \in \mathcal{A}} \left\{ \sum_{w_k^{(j)} = i} \left(A_\sigma^{-1}(\boldsymbol{x} + \mathbf{f}(S_k^{(j)})), j^* \right) + \sum_{w_k^{(j)} = i^{-1}} - \left(A_\sigma^{-1}(\boldsymbol{x} + \mathbf{f}(w_k^{(j)} S_k^{(j)})), j^* \right) \right\}.$$

Identify $(\boldsymbol{x}, i^*) \in \mathbb{Z}^2 \times \{1^*, 2^*\}$ with the positive oriented unit segment spanned by the fundamental vector $\boldsymbol{e}_j$ translated by $\boldsymbol{x}$, where $\{i, j\} = \{1, 2\}$, that is,

$$(\boldsymbol{x}, 1^*) := \{\boldsymbol{x} + t\boldsymbol{e}_2 \mid 0 \le t \le 1\},\ (\boldsymbol{x}, 2^*) := \{\boldsymbol{x} + t\boldsymbol{e}_1 \mid 0 \le t \le 1\}.$$

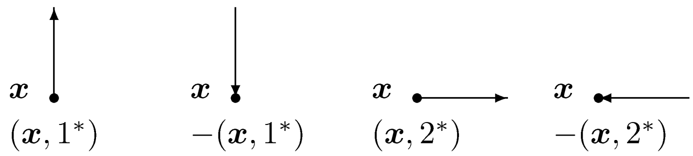

Figure 1. The segments $(\boldsymbol{x}, 1^*), (\boldsymbol{x}, 2^*)$ with orientation

The element of $\mathcal{G}^*$ is also identified with a union of oriented unit segments with their multiplicity.

Since τ is primitive and Pisot, the incidence matrix A_τ has a positive column eigenvector $\boldsymbol{u}_\tau$ and a positive row eigenvector $\boldsymbol{v}_\tau$ corresponding to the maximum eigenvalue $\lambda > 1$ by Perron-Frobenius Theorem. The matrix A_σ has the same eigenvalue λ, and a column eigenvector $\boldsymbol{u}_\sigma$ and a row eigenvector $\boldsymbol{v}_\sigma$ of A_σ corresponding to the eigenvalue λ are respectively given by $\boldsymbol{u}_\sigma = A_\delta^{-1}\boldsymbol{u}_\tau$ and $\boldsymbol{v}_\sigma = \boldsymbol{v}_\tau A_\delta$ by $A_\sigma = A_\delta^{-1} A_\tau A_\delta$; and the contractive eigenspaces P_τ of A_τ and P_σ of A_σ are given by $P_\tau = \{\boldsymbol{x} \in \mathbb{R}^2 \mid < \boldsymbol{x}, {}^t\boldsymbol{v}_\tau >= 0\}$ and $P_\sigma = \{\boldsymbol{x} \in \mathbb{R}^2 \mid < \boldsymbol{x}, {}^t\boldsymbol{v}_\sigma >= 0\}$, where $< \cdot, \cdot >$ means an inner product. The projection from $\mathbb{R}^2$ to P_τ (resp. P_σ) along the expanding column eigenvector $\boldsymbol{u}_\tau$ (resp. $\boldsymbol{u}_\sigma$) is denoted by π_τ (resp. π_σ). Note $\boldsymbol{u}_\sigma$ is positive since the matrix A_δ^{-1} is positive, but it is possible for $\boldsymbol{v}_\sigma$ not to be positive (see Figure 2).

Example 2.3. For the quadratic number x with the continued fraction expansion $x = [0, 4, \overline{4, 2}]$, the automorphism σ is given by $\sigma(1) = 1(1112)^4$, $\sigma(2) = (2^{-1}1^{-1}1^{-1}1^{-1})^3 2^{-1}1^{-1}2$. And it is conjugate to the substitution τ given by $\tau(1) = (1222)^3 2$, $\tau(2) = (1222)^2 2$ with the conjugacy δ given by $\delta(1) = 12^{-1}$, $\delta(2) = (21^{-1})^3 2$. The direction of P_σ is not the same as in the substitutive case (see Figure 2).

Let us introduce Rauzy fractals induced from τ and σ by using tiling substitutions.

Proposition 2.4. *The following limit sets exist in the sense of Hausdorff metric (cf. [3, 6]):*

$$X_\tau := \lim_{n\to\infty} A_\tau^n \pi_\tau \tau^{*\ n}(\mathcal{U}),$$
$$X_\tau^{(i)} := \lim_{n\to\infty} A_\tau^n \pi_\tau \tau^{*\ n}(\boldsymbol{e}_i, i^*),\ i \in \mathcal{A},$$
$$X_\sigma := \lim_{n\to\infty} A_\sigma^n \pi_\sigma \sigma^{*\ n}(\mathcal{U}),$$
$$X_\sigma^{(i)} := \lim_{n\to\infty} A_\sigma^n \pi_\sigma \sigma^{*\ n}(\boldsymbol{e}_i, i^*),\ i \in \mathcal{A},$$

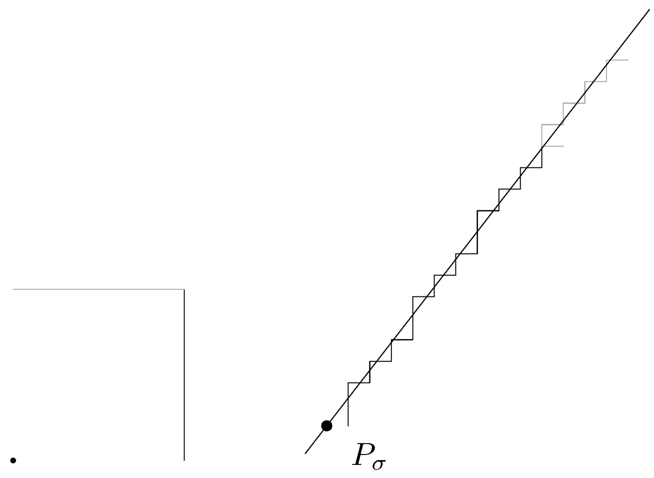

Figure 2. $(\boldsymbol{e}_1, 1^*) + (\boldsymbol{e}_2, 2^*)$ and $\sigma^*((\boldsymbol{e}_1, 1^*) + (\boldsymbol{e}_2, 2^*))$ for $x = [0, 4, \overline{4, 2}]$

where $\mathcal{U} = (\boldsymbol{e}_1, 1^*) + (\boldsymbol{e}_2, 2^*) \in \mathcal{G}^*$.

These sets $X_\tau, X_\tau^{(i)}$ (resp. $X_\sigma, X_\sigma^{(i)}$) are called Rauzy fractals induced from τ (resp. σ) (see Figure 2).

Proof. The existence of limit sets for a substitution is known (see [3]), so we consider it for the automorphism σ. Put

$$\mathcal{C}_i := \left\{ \sum_{k=1}^{l} (\boldsymbol{x}_k, j_k^*) \in \mathcal{G}^* \mid \sigma^*(\boldsymbol{o}, i^*) \cap \sigma^*(\boldsymbol{x}_k, j_k^*) \neq \emptyset \ for\ any\ k \right\},$$

where $\sum_{k=1}^{l} (\boldsymbol{x}_k, i_k^*) \cap \sum_{t=1}^{m} (\boldsymbol{y}_t, j_t^*) \neq \emptyset$ means there exist k, t such that $(\boldsymbol{x}_k, i_k^*) = (\boldsymbol{y}_t, j_t^*)$. For $\gamma \in \mathcal{C}_i$, put

$$\pi_\sigma \sigma^*(\boldsymbol{o}, i^*)_\gamma := \pi_\sigma \sigma^*(\boldsymbol{o}, i^*) \setminus (\pi_\sigma \sigma^*(\boldsymbol{o}, i^*) \cap \pi_\sigma \sigma^*(\gamma)).$$

Let positive numbers c_1, c_2 be

$$c_1 = \max_{i \in \mathcal{A}} d_H(\pi_\sigma(\boldsymbol{o}, i^*), A_\sigma \pi_\sigma \sigma^*(\boldsymbol{o}, i^*)),$$
$$c_2 = \max_{i \in \mathcal{A}} \max_{\gamma \in \mathcal{C}_i} \{ d_H(A_\sigma \pi_\sigma \sigma^*(\boldsymbol{o}, i^*), A_\sigma \pi_\sigma \sigma^*(\boldsymbol{o}, i^*)_\gamma) \},$$

where d_H is the Hausdorff metric. From the fact that for any compact sets A, B, C, D,

$$d_H(A \cup B, C \cup D) \leq \max(d_H(A, C), d_H(B, D)),$$

and the triangle inequality,

$$d_H(\pi_\sigma \sigma^{*\,n}(\mathcal{U}), A_\sigma \pi_\sigma \sigma^{*\,n+1}(\mathcal{U})) \leq c_1 + c_2.$$

So

$$d_H(A_\sigma^n \pi_\sigma \sigma^{*\ n}(\mathcal{U}), A_\sigma^{n+1}\pi_\sigma \sigma^{*\ n+1}(\mathcal{U})) \leq (c_1+c_2)\lambda'^n,$$

where λ' is the other eigenvalue given by $\lambda' = \frac{1}{\lambda} < 1$. Therefore the sequence $A_\sigma^n \pi_\sigma \sigma^{*\ n}(\mathcal{U})$ is a Cauchy sequence in the space of all compact subsets on P_σ, and it is convergent. The existence of the limit set $X_\sigma^{(i)}$ is proved by the same way. □

Remark 2. The substitution τ is Pisot, unimodular, irreducible, primitive and invertible, thus Rauzy fractals $X_\tau,\ X_\tau^{(i)},\ i \in \mathcal{A}$ are intervals, and moreover, we have

$$X_\tau^{(i)} = \pi_\tau(\boldsymbol{e}_i, i^*) + \boldsymbol{h}$$

for some $\boldsymbol{h} \in P_\tau$ (see [4]).

The following proposition and theorem show the set equation of $X_\tau^{(i)},\ i \in \mathcal{A}$ and the relations between Rauzy fractals $X_\tau^{(i)}$ and $X_\sigma^{(i)}$ induced from τ and σ.

Proposition 2.5 ([3]). *The following equations hold:*

$$A_\tau^{-1} X_\tau^{(i)} = \cup_{j\in\mathcal{A}} \cup_{w_k^{(j)}=i} (-A_\tau^{-1}\pi_\tau \mathbf{f}(P_k^{(j)}) + X_\tau^{(j)}),\ i \in \mathcal{A},$$

where $\tau(i)$ is written as $\tau(i) = w_1^{(i)} \cdots w_k^{(i)} \cdots w_{l^{(i)}}^{(i)} = P_k^{(i)} w_k^{(i)} S_k^{(i)}$.

The sets $(-A_\tau^{-1}\pi_\tau \mathbf{f}(P_k^{(j)}) + X_\tau^{(j)}),\ j \in \mathcal{A}$ such that $w_k^{(j)} = i$ are pairwise disjoint up to a set of Lebesgue measure 0.

By noticing that $\delta^{-1} = \sigma_{1,a_1} \circ \sigma_{2,a_2} \circ \cdots \circ \sigma_{t_N,a_N}$ is an invertible substitution, we have the following theorem:

Theorem 2.6 ([6]).

$$X_\sigma^{(i)} = \cup_{j\in\mathcal{A}} \cup_{w_k^{(j)}=i} (-\pi_\sigma \mathbf{f}(P_k^{(j)}) + A_\delta^{-1} X_\tau^{(j)}),\ i \in \mathcal{A},$$

where $\delta^{-1}(i)$ is written as $\delta^{-1}(i) = w_1^{(i)} \cdots w_k^{(i)} \cdots w_{l^{(i)}}^{(i)} = P_k^{(i)} w_k^{(i)} S_k^{(i)}$.

The sets $(-\pi_\sigma \mathbf{f}(P_k^{(j)}) + A_\delta^{-1} X_\tau^{(j)}),\ j \in \mathcal{A}$ such that $w_k^{(j)} = i$ are pairwise disjoint up to a set of Lebesgue measure 0. Moreover,

$$A_\delta X_\sigma^{(i)} = \pi_\tau((\delta^{-1})^*(\boldsymbol{e}_i, i^*)) + \boldsymbol{h}$$

for some $\boldsymbol{h} \in P_\tau$. Thus, $X_\sigma^{(i)}$, X_σ are intervals.

Proof. The idea of the proof is mentioned here (for details, see [6]). By $\sigma = \delta^{-1} \circ \tau \circ \delta$ and according to properties of a tiling substitution (see [7]), we have

$$\sigma^* = \delta^* \circ \tau^* \circ (\delta^{-1})^*,$$
$$A_\delta^{-1}\pi_\tau \boldsymbol{x} = \pi_\sigma A_\delta^{-1} \boldsymbol{x} \quad \text{for } \boldsymbol{x} \in \mathbb{R}^2.$$

Therefore,

$$\begin{aligned}
X_\sigma^{(i)} &= \lim_{n\to\infty} A_\sigma^n \pi_\sigma \sigma^{*\ n}(\boldsymbol{e}_i, i^*)\\
&= \lim_{n\to\infty} A_\sigma^n \pi_\sigma \delta^* \circ \tau^{*\ n} \circ (\delta^{-1})^*(\boldsymbol{e}_i, i^*)\\
&= \cup_{j\in\mathcal{A}} \cup_{w_k^{(j)}=i} \lim_{n\to\infty} A_\sigma^n \pi_\sigma \delta^* \tau^{*\ n}(\boldsymbol{e}_j - A_\delta \mathbf{f}(P_k^{(j)}), j^*)\\
&= \cup_{j\in\mathcal{A}} \cup_{w_k^{(j)}=i} \lim_{n\to\infty} A_\sigma^n \pi_\sigma A_\delta^{-1} \tau^{*\ n}(\boldsymbol{e}_j - A_\delta \mathbf{f}(P_k^{(j)}), j^*)\\
&= \cup_{j\in\mathcal{A}} \cup_{w_k^{(j)}=i} \lim_{n\to\infty} A_\delta^{-1} A_\tau^n \pi_\tau \tau^{*\ n}(\boldsymbol{e}_j - A_\delta \mathbf{f}(P_k^{(j)}), j^*)\\
&= \cup_{j\in\mathcal{A}} \cup_{w_k^{(j)}=i} (-\pi_\sigma \mathbf{f}(P_k^{(j)}) + A_\delta^{-1} X_\tau^{(j)}),
\end{aligned}$$

for the equality of the fourth line, see [6].

Disjointness and the property X_σ and $X_\sigma^{(i)}$ are intervals derive from Remark 3 as follows.

$$\begin{aligned}
A_\delta X_\sigma^{(i)} &= \cup_{j\in\mathcal{A}} \cup_{w_k^{(j)}=i} (-\pi_\tau A_\delta \mathbf{f}(P_k^{(j)}) + X_\tau^{(j)})\\
&= \cup_{j\in\mathcal{A}} \cup_{w_k^{(j)}=i} \pi_\tau(-A_\delta \mathbf{f}(P_k^{(j)}) + \boldsymbol{e}_j, j^*) + \boldsymbol{h}\\
&= \pi_\tau (\delta^{-1})^*(\boldsymbol{e}_i, i^*) + \boldsymbol{h}
\end{aligned}$$

for some $\boldsymbol{h} \in P_\tau$. Recall that if a substitution β is invertible, then $\beta^*(\boldsymbol{o}, i^*)$, $i \in \mathcal{A}$ are connected (see [8]). Since δ^{-1} is invertible substitution, the remaining part is proved. □

Remark 3. Theorem 2.6 shows the role played by δ^{-1}: the tiling substitution $(\delta^{-1})^*$ is used to get Rauzy fractals $X_\sigma^{(i)}$ $(i \in \mathcal{A})$ from $X_\tau^{(j)}$ $(j \in \mathcal{A})$. On the other hand, in the paper [6], δ is used to get the stepped surface of P_σ, which is a discrete approximation of P_σ, from the one of P_τ.

Definition 2.7. The domain exchange transformations T_τ, T_σ on X_τ, X_σ for a quadratic irrational number $x \in X$, are defined by

$$\begin{aligned}
&T_\tau : X_\tau \to X_\tau\\
&T_\tau(\boldsymbol{x}) = \boldsymbol{x} - \pi_\tau \mathbf{f}(i)\ if\ \boldsymbol{x} \in X_\tau^{(i)},
\end{aligned}$$

and

$$\begin{aligned}
&T_\sigma : X_\sigma \to X_\sigma\\
&T_\sigma(\boldsymbol{x}) = \boldsymbol{x} - \pi_\sigma \mathbf{f}(i)\ if\ \boldsymbol{x} \in X_\sigma^{(i)}.
\end{aligned}$$

The domain exchange transformations T_τ, T_σ are well-defined from the definitions of X_τ and X_σ (see Figure 3) (cf. [3, 6]).

Definition 2.8. Let (X, T, μ) be a measure-theoretical dynamical system, σ an arbitrary substitution over the alphabet $\mathcal{A}$ such that

$$\sigma(i) = w_1^{(i)} w_2^{(i)} \cdots w_{l^{(i)}}^{(i)}\ ,$$

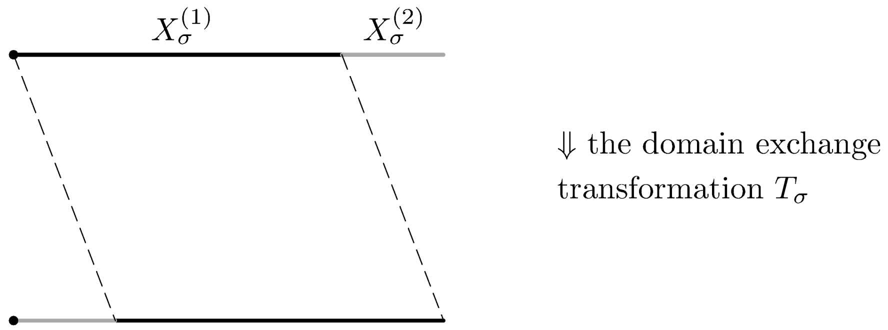

Figure 3. The domain exchange transformation T_σ for $x = [0, 4, \overline{4, 2}]$

$\{X^{(i)} \mid i \in \mathcal{A}\}$ a measurable partition of X, and $\{A^{(i)} \mid i \in \mathcal{A}\}$ a measurable partition of a subset A of X. We say that the transformation T has σ-structure with respect to the pair of partitions $\{X^{(i)}\}$, $\{A^{(i)}\}$ if the following conditions hold up to a set of measure 0:

$$\begin{array}{ll} T^k A^{(i)} \subset X^{(w_{k+1}^{(i)})} & \textit{for all} \ \ i \in \mathcal{A}\ , \ k = 0, 1, \cdots, l^{(i)} - 1 \\ T^k A^{(i)} \cap A = \emptyset & \textit{for all} \ \ i \in \mathcal{A}, \ 0 < k < l^{(i)} \\ T^{l^{(i)}} A^{(i)} \subset A & \textit{for all } i \in \mathcal{A} \\ X = \bigcup_{i \in \mathcal{A}} \bigcup_{0 \le k \le l^{(i)} - 1} T^k A^{(i)} & (\textit{non}-\textit{overlapping}) \end{array} .$$

From Proposition 2.5 and Theorem 2.6, we have the following theorem.

Theorem 2.9. *For a quadratic irrational number $x \in X$, the measure-theoretical dynamical system $(X_\sigma, T_\sigma, \mu)$ with Lebesgue measure μ has δ^{-1}-structure with respect to the pair of partitions $\{X_\sigma^{(i)} |\ i \in \mathcal{A}\}$, $\{A_\delta^{-1} X_\tau^{(i)} |\ i \in \mathcal{A}\}$. Moreover, $(X_\sigma, T_\sigma, \mu)$ has $\delta^{-1}\tau^n$-structure with respect to the pair of partitions $\{X_\sigma^{(i)} |\ i \in \mathcal{A}\}$, $\{A_\delta^{-1} A_\tau^n X_\tau^{(i)} |\ i \in \mathcal{A}\}$ for any positive integer n.*

Recall that the sequence $C(x) = (c_1(x), c_2(x), \cdots, c_n(x), \cdots)$ is a fixed point of σ, thus it is given by $\lim_{n \to \infty} \delta^{-1}\tau^n(1)$ with the product topology. We have $\boldsymbol{o} \in \pi_\tau(\mathcal{U}) \subset \pi_\tau \tau^*(\boldsymbol{e}_1, 1^*)$ and $\boldsymbol{o} \in X_\tau^{(1)}$, since $\tau(1) = \tau(2) = 1$. From the theorem we obtain the following corollary:

Corollary 2.10. *For a quadratic irrational number x,*

$$T_\sigma^{(k-1)}(\boldsymbol{o}) \in X_\sigma^{(c_k(x))}, \ k = 1, 2, \cdots .$$

Since $X_\sigma, X_\sigma^{(i)}$, $i \in \mathcal{A}$ are intervals by Theorem 2.6, the domain exchange transformation T_σ is just a two interval exchange transformation; and the orbit of the origin point by T_σ gives the characteristic sequence $C(x)$.

There are algorithms corresponding to the continued fraction algorithm in the higher dimensional case (cf. [10]), and it is expected for the strategy developed in the paper to work well. We consider automorphisms of rank 2 in the paper, and some of results may seem obvious. But as Rauzy fractals have fractal boundaries in the higher dimensional case, the discussion is more complicated in general, and the generalization of the strategy should be important.

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