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An analytic function in 3 variables related to the value-distribution of $\log L$, and the "Plancherel volume"

By

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Introduction At the conference, the author spoke on his recent article [7] and on related joint work with Kohji Matsumoto [9, 10]. These papers which contain full details have meanwhile been published. On the other hand, his survey article [12] on more or less the same subject written about a year ago remains formally unpublished. So, here, we shall present a slightly revised version of [12]. In [7, 10], we worked over general global base fields K, and treated both the "$\log L$-" and the "$d\log L$-versions" simultaneously. But here, as in [9], we restrict our attention to the $\log L$-version over $K = \mathbf{Q}$. For a more recent work related to the $d\log L$-analogue of [9], cf. [13].

The first subject is of general and elementary nature. For a continuous density measure $M(x)|dx|$ on $\mathbf{R}^d$, let $\mu = \mu_M$ denote the variance and

$$\nu = \nu_M = \int M(x)^2|dx| = \int |\hat{M}(y)|^2|dy|$$

the "Plancherel volume", where $|dx|$ is the self-dual Haar measure of $\mathbf{R}^d$ and $M \to \hat{M}$ denotes the Fourier transform. We pay our attention to this basic integral invariant ν of the measure, giving a basic elementary inequality which is slightly more general than the one given in [7].

The main subject is a complex analytic function in 3 variables s, z_1, z_2 defined by the Euler product

$$\tilde{M}(s; z_1, z_2) = \prod_p F(iz_1/2, iz_2/2; 1; p^{-2s}) \qquad (\Re(s) > 1/2). \tag{0.1}$$

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Here, $i = \sqrt{-1}$, and $F(a, b; c; t)$ is the Gauss hypergeometric series. This function arose in connection with the two "mean-values"

$$\{\overline{\zeta(s)}^{iz_1/2}\zeta(s)^{iz_2/2}\}_{\mathrm{Re}(s)=\sigma}, \qquad \{\overline{L(s,\chi)}^{iz_1/2}L(s,\chi)^{iz_2/2}\}_{\chi}.$$

The average on the left probably requires no explanation. As for the one on the right, s is fixed with $\mathrm{Re}(s) = \sigma$, and χ runs over all Dirichlet characters with prime conductors. By [9, 10] (to be reviewed in §3), when σ is real $> 1/2$ and at least when $z_2 = \bar{z}_1$, $\tilde{M}(\sigma, z_1, z_2)$ can be interpreted as the function giving the above two mean-values (which are, as expected, equal). In other words, $\tilde{M}(\sigma, z, \bar{z})$ is the Fourier dual of the density function $M_\sigma(w)$ for the distribution of values of $\{\log\zeta(\sigma+ti)\}_{t\in\mathbf{R}}$ and of $\{\log L(s,\chi)\}_\chi$.

But we consider $\tilde{M}(s; z_1, z_2)$ as an analytic function also of the complex variable s. We shall briefly review the main results of [7] on analytic continuation of $\tilde{M}(s; z_1, z_2)$ to the left of $\Re(s) > 1/2$, two other infinite product expansions, and also its limit behaviours at $s \to 1/2$, which will be applied to the determination of the corresponding limits of the invariants related to $M_\sigma(w)$ such as $\mu_M\nu_M$ for $M = M_\sigma$.

§ 1. The Plancherel volume and a basic inequality (cf. [7],§1.1)

The readers mainly interested in the analytic function $\tilde{M}(s; z_1, z_2)$ might skip this section. Let $\mathbf{R}^d$ be the d-dimensional Euclidean space, with points denoted as $x = (x_1, \cdots, x_d)$, and with the Haar measure $|dx| = (dx_1...dx_d)/(2\pi)^{d/2}$, which is self-dual with respect to the dual pairing $e^{i\langle x,x'\rangle}$ of $\mathbf{R}^d$, where $\langle x, x'\rangle = \sum_{i=1}^d x_i x_i'$. Write, as usual, $|x| = \langle x, x\rangle^{1/2}$. Let $M(x)|dx|$ be any density measure on $\mathbf{R}^d$ with center $\mathbf{0}$; in other words, $M(x)$ is a non-negative real-valued measurable function on $\mathbf{R}^d$ such that

$$\int M(x)|dx| = 1; \qquad \int M(x)x_i|dx| = 0 \qquad (1 \le i \le d), \tag{1.1}$$

where the integrals are over $\mathbf{R}^d$. For any such $M(x)$, put

$$\nu := \nu_M = \int M(x)^2|dx| \tag{1.2}$$

and call it the *Plancherel volume* of $M(x)$ (or of $M(x)|dx|$). A reason for this naming is that if $M(x)$ is analytically "good enough" (which we shall not need as assumption in §1), the following standard formulas in Fourier analysis hold;

$$\hat{M}(y) := \int M(x)e^{i\langle x,y\rangle}|dx|, \qquad M(x) = \int \hat{M}(y)e^{-i\langle x,y\rangle}|dy|; \tag{1.3}$$

$$\nu_M = \int M(x)^2|dx| = \int |\hat{M}(y)|^2|dy| \quad \text{(the Plancherel formula).} \tag{1.4}$$

Now, if we write $M^-(x) := M(-x)$, then

$$\nu_M = (M * M^-)\,|_{x=0}, \tag{1.5}$$

where $*$ denotes the convolution product with respect to $|dx|$. Thus, ν_M is the density at the origin, of the differences of two random points in the measure space $(\mathbf{R}^d, M(x)|dx|)$. On the other hand, for each $k > 0$, put

$$\mu^{(k)} := \int M(x)|x|^k|dx|; \qquad \mu = \mu^{(2)} : \textit{the variance}. \tag{1.6}$$

Observe that the quantity

$$(\mu^{(k)})^{d/k}\nu \tag{1.7}$$

is invariant under scalar transforms

$$M(x) \longmapsto c^d M(cx) \tag{1.8}$$

$(c > 0)$, and observe (intuitively) that *not both* of ν and $\mu^{(k)}$ can be small at the same time. It is thus natural to ask: "does there exist a positive universal lower bound for the quantity (1.7)?" In [7] Theorem 1, we proved the existence of such a bound for $k = 2$;

$$\mu^{d/2}\nu \geq \left(\frac{2d}{d+4}\right)^{d/2} \frac{4\,\Gamma(\frac{d+4}{2})}{d+4}, \tag{1.9}$$

and described precisely when the equality holds. Here, we just add that this can be generalized, with almost the same proof, to the case of any $k > 0$:

Theorem 1 *For any $k > 0$ we have*

$$(\mu^{(k)})^{d/k}\nu \geq 2^{\frac{d}{2}+1}\Gamma\left(\frac{d}{2}+1\right)\frac{d+k}{d+2k}\left(\frac{d}{d+2k}\right)^{d/k}, \tag{1.10}$$

with the equality if and only if $M(x)$ coincides almost everywhere with a scalar transform of the function $\mathrm{Max}(0, 1 - |x|^k)$.

Proof (I) We may assume that $M(x)$ is rotation-invariant; $M(x) = f(|x|)$ with some non-negative valued measurable function $f(r)$ on $r \geq 0$. As in *loc.cit*, put $\gamma_d := (2\pi)^{d/2}/\mathrm{Vol}(S_{d-1}) = 2^{(d/2)-1}\Gamma(d/2)$, where $\mathrm{Vol}(S_{d-1})$ denotes the (ordinary) Euclidean volume of the $(d-1)$-dimensional sphere. Then by definitions,

$$\int_0^\infty f(r)r^{d-1}dr = \gamma_d, \quad \int_0^\infty f(r)r^{d-1+k}dr = \gamma_d\,\mu^{(k)}, \quad \int_0^\infty f(r)^2 r^{d-1}dr = \gamma_d\,\nu. \tag{1.11}$$

(II) The case $f(r) = c\mathrm{Max}(0, 1 - r^k)$. By the first formula of (1.11) we obtain $c = (d(d+k)/k)\gamma_d$, and by direct calculations, we also obtain the following formulas for the invariants for this case, which will be distinguished from the invariants for the general case by the subscript $*$:

$$\mu_*^{(k)} = \frac{d}{d+2k}, \qquad \nu_* = \frac{2d(d+k)}{d+2k}\gamma_d, \tag{1.12}$$

$$(\mu_*^{(k)})^{d/k}\nu_* = 2^{\frac{d}{2}+1}\Gamma\left(\frac{d}{2}+1\right)\frac{d+k}{d+2k}\left(\frac{d}{d+2k}\right)^{d/k}. \tag{1.13}$$

Note that the last value is equal to the quantity on the right hand side of (1.10).

(III) The general case. By a suitable scalar transform, we may assume $\mu^{(k)} = \mu_*^{(k)}$. The Schwarz inequality gives $AB \geq C^2$, for

$$A = \int_0^1 (1-r^k)^2 r^{d-1}dr, \; B = \int_0^1 f(r)^2 r^{d-1}dr; \; C = \int_0^1 (1-r^k)f(r)r^{d-1}dr. \tag{1.14}$$

By (1.12) we have

$$A = \frac{2k^2}{d(d+k)(d+2k)}, \tag{1.15}$$

while, obviously,

$$B \leq \int_0^\infty f(r)^2 r^{d-1}dr = \gamma_d\,\nu. \tag{1.16}$$

As for C, we have

$$\begin{aligned} C &\geq \int_0^\infty (1-r^k)f(r)r^{d-1}dr = \gamma_d(1-\mu^{(k)}) \\ &= \gamma_d(1-\mu_*^{(k)}) = \gamma_d\frac{2k}{d+2k} > 0 \end{aligned} \tag{1.17}$$

(note the positivity of the right hand side of (1.17)). Therefore,

$$\left(\frac{2k^2}{d(d+k)(d+2k)}\gamma_d\,\nu\right)^{1/2} \geq (AB)^{1/2} \geq C \geq \gamma_d\frac{2k}{d+2k}, \tag{1.18}$$

which gives

$$\nu \geq \frac{2d(d+k)}{d+2k}\gamma_d = \nu_*; \tag{1.19}$$

hence $(\mu^{(k)})^{d/k}\nu = (\mu_*^{(k)})^{d/k}\nu \geq (\mu_*^{(k)})^{d/k}\nu_*$, as desired. The second statement of the theorem is clear from this proof.

Examples (i) $M(x) = \exp(-|x|^2/2)$ (Gaussian). Then

$$(\mu^{(k)})^{d/k}\nu = \left(\Gamma\left(\frac{d+k}{2}\right)/\Gamma\left(\frac{d}{2}\right)\right)^{d/k}.$$

(ii) Let $M(x)$ be the defining function of a compact set $S \subset \mathbf{R}^d$ with center of gravity $\mathbf{0}$ and the volume $\int_S |dx| = 1$. Then

$$(\mu^{(k)})^{d/k}\nu \geq 2^{d/2}\Gamma\left(\frac{d}{2}+1\right)\left(\frac{d}{d+k}\right)^{d/k}, \tag{1.20}$$

with the equality if and only if S is a ball with center $\mathbf{0}$ (modulo a set of measure 0). For curious readers, the right hand side of (1.10) divided by that of (1.20) is $2(y/(1+y))^y < 1$, $y = (d+k)/k > 1$.

When $M(x)$ has analytic parameters, the quantity $(\mu^{(k)})^{d/k}\nu$ is often expressible as the product of powers of Gamma functions whose arguments are simple functions of the parameters. (This is in fact so e.g. when $M(x) = |x|^{p-1}\exp(-|x|^q)$ $(p, q > 0)$, or $M(x) = (|x|^\alpha + 1)^{-\beta}$ $(\alpha, \beta > 0, \alpha\beta \gg 1)$, but the formulas do not seem illuminating.) Thus, although the quantity itself takes positive real values, it can often be continued analytically as a function of complex parameters. I have not understood the reason even in the present case of interest.

§ 2. The analytic function $\tilde{M}(s; z_1, z_2)$; introduction

First, recall that the Riemann zeta function $\zeta(s)$ has the Euler product expansion

$$\zeta(s) = \prod_p \zeta_p(s) \tag{2.1}$$

on $\mathrm{Re}(s) > 1$, where

$$\zeta_p(s) = (1 - p^{-s})^{-1}, \tag{2.2}$$

and also the Riemann-Hadamard decomposition

$$\zeta(s) = \epsilon(s)^{-1}\prod_\rho \left(1 - \frac{s}{\rho}\right)e^{\frac{s}{\rho}}, \tag{2.3}$$

where $\epsilon(s)$ is of the form $s(s-1)e^{Bs}\Gamma(s/2)$ and ρ runs over all non-trivial zeros of $\zeta(s)$. As is well-known, comparison of the two decompositions (2.1) and (2.3) leads to various identities connecting "$\{p\}$" with "$\{\rho\}$".

The function in the title, called $\tilde{M}(s; z_1, z_2)$, in which complex powers of $\zeta(2s)$ are comprised, also has two types of infinite product decompositions, each of which having some common features with *both* (2.1) and (2.3) (see §7). Let us recall the definition.

First, the local factor $\tilde{M}_p(s;z_1,z_2)$ for each prime p. Consider the power series expansion in p^{-s} of the complex x-th power of $\zeta_p(s)$:

$$\zeta_p(s)^x = (1-p^{-s})^{-x} = 1 + \sum_{n=1}^{\infty} a_n(x)p^{-ns}, \tag{2.4}$$

$$a_n(x) = (x)_n = \frac{x(x+1)\cdots(x+n-1)}{n!}. \tag{2.5}$$

It is convenient to use complex variables (x_1,x_2) and (z_1,z_2) related to each other by

$$x_\nu = iz_\nu/2 \qquad (\nu = 1,2), \tag{2.6}$$

where $i = \sqrt{-1}$. Then $\tilde{M}_p(s;z_1,z_2)$ is defined by

$$\tilde{M}_p(s;z_1,z_2) = 1 + \sum_{n=1}^{\infty} a_n(x_1)a_n(x_2)p^{-2ns} = F(x_1,x_2;1;\, p^{-2s}), \tag{2.7}$$

where

$$F(a,b;c;\,t) = 1 + \frac{a.b}{1.c}\,t + \frac{a(a+1)b(b+1)}{1.2c(c+1)}\,t^2 + \cdots \tag{2.8}$$

($|t| < 1$) denotes the Gauss hypergeometric series. It is clear that $\tilde{M}_p(s;z_1,z_2)$ is a holomorphic function of s, z_1, z_2 on $\mathrm{Re}(s) > 0$, symmetric in z_1, z_2. The zero divisor of $\tilde{M}_p(s;z_1,z_2)$ is non-trivial (see below §6). The global holomorphic function $\tilde{M}(s;z_1,z_2)$ of s, z_1, z_2 on the domain $\mathrm{Re}(s) > 1/2$ is defined by

$$\tilde{M}(s;z_1,z_2) = \prod_p \tilde{M}_p(s;z_1,z_2) \tag{2.9}$$

which is absolutely convergent in the following sense. Fix any $\sigma_0 > 1/2$ and $R > 0$. Then $|\tilde{M}_p(s;z_1,z_2) - 1| < 1$ holds on $\mathrm{Re}(s) \geq \sigma_0$ and $|z_1|, |z_2| \leq R$ for almost all p (depending on σ_0, R), and the sum of $\log \tilde{M}_p(s;z_1,z_2)$ (the principal branch) over these p is absolutely convergent; thus $\tilde{M}(s;z_1,z_2)$ is defined as the product of finitely many local factors and the exponential of a holomorphic function on this domain. In particular, the zero divisor of $\tilde{M}(s;z_1,z_2)$ is the sum of those of local factors. Note that $\tilde{M}(s;-2i,-2ix) = \zeta(2s)^x$ ($x \in \mathbf{C}$).

This function $\tilde{M}(s;z_1,z_2)$ has a Dirichlet series expansion on $\mathrm{Re}(s) > 1/2$ whose coefficients are polynomials of z_1, z_2, formally arising from the Euler product expansion (2.9). It is absolutely convergent also as Dirichlet series on the same domain. We recall [10]§4 that (again for $\mathrm{Re}(s) > 1/2$) it has an everywhere absolutely convergent power series expansion in z_1, z_2:

$$\tilde{M}(s;z_1,z_2) = 1 + \sum_{a,b\geq 1} \mu^{(a,b)}(s)\frac{x_1^a x_2^b}{a!b!} = 1 - \frac{1}{4}\mu(s)z_1z_2 + \cdots, \tag{2.10}$$

where each $\mu^{(a,b)}(s)$ is a certain Dirichlet series, and

$$\mu(s) = \mu^{(1,1)}(s) = \sum_p (\sum_{n=1}^{\infty} \frac{1}{n^2 p^{2ns}}). \tag{2.11}$$

Now, in the joint work with Matsumoto [9](cf. also [10]§4.1), we have constructed for each $\sigma > 1/2$ a density measure $M_\sigma(w)|dw|$ satisfying and determined by the following mutually inverse Fourier-transform equalities:

$$\tilde{M}(\sigma, z_1, z_2) = \int_{\mathbf{C}} M_\sigma(w) \exp\left(\frac{i}{2}(z_1\bar{w} + z_2 w)\right) |dw| \tag{2.12}$$

$$M_\sigma(w) = \int_{\mathbf{C}} \tilde{M}(\sigma; z, \bar{z}) e^{-i\mathrm{Re}(\bar{z}w)} |dz| \tag{2.13}$$

($|dz| = dxdy/2\pi$ for $z = x + iy$). The first equality holds for any $z_1, z_2 \in \mathbf{C}$, and the second, for any $w \in \mathbf{C}$. This $M_\sigma(w)$ is a non-negative real valued continuous (in fact, C^∞-) function on $\mathbf{C}$, satisfying

$$\int_{\mathbf{C}} M_\sigma(w)|dw| = 1, \qquad \int_{\mathbf{C}} M_\sigma(w) w |dw| = 0. \tag{2.14}$$

It is thus a density function with the center of gravity $\mathbf{0}$. Its variance μ_σ is

$$\begin{aligned} \mu_\sigma = \int_{\mathbf{C}} M_\sigma(w)|w|^2|dw| &= \frac{\partial^2}{\partial x_1 \partial x_2} \tilde{M}(\sigma; z_1, z_2) \mid_{(0,0)} \\ = \mu(\sigma) > 0, & \end{aligned} \tag{2.15}$$

$\mu(s)$ being the Dirichlet series (2.11). This is real analytic in σ. On the other hand, its Plancherel volume

$$\nu_\sigma = \int M_\sigma(w)^2 |dw| = \int |\tilde{M}(\sigma; z, \bar{z})|^2 |dz| \tag{2.16}$$

is at least continuous on $\sigma > 1/2$, but I do not know whether ν_σ is real analytic, and even if so, whether it has an analytic continuation to the left of 1/2. We know by (1.9) that $\mu_\sigma \nu_\sigma \geq 8/9$, and for a numerical example, $\mu_1 = 0.474...$, $\nu_1 = 1.967...$; $\mu_1 \nu_1 = 0.93...$.

§ 3. Connection with the value-distribution of the logarithm of Dirichlet L-functions; review of joint work with K. Matsumoto [9, 10]

The value-distribution theory related to ζ and L-functions has a long history since Bohr-Jessen [1]. Our work is closest in spirit with Bohr-Jessen and Jessen-Wintner [14], but some of the works of Elliott [2, 3], Stankus [18, 19], Granville-Soundararajan [4], Lamzouri [15] are more or less related; cf. also Laurinčikas [16], Steuding [20]. We leave precise descriptions of relations with past works to the introductory parts in [9, 11].

In [9, 10], we have established the Bohr-Jessen type equalities:

Theorem 2 (with Kohji Matsumoto [9, 10]) *Let* $\sigma > 1/2$. *Then:*
(i) *The equalities*

$$\int_{\mathbf{C}} M_\sigma(w)\Phi(w)|dw| = \mathrm{Avg}_{\mathrm{Re}(s)=\sigma}\Phi(\log\zeta(s)) \tag{3.1}$$

$$= \mathrm{Avg}_\chi \Phi(\log L(s,\chi)) \tag{3.2}$$

hold for any bounded continuous function Φ *on* $\mathbf{C}$. *Here,* $\mathrm{Avg}_{\mathrm{Re}(s)=\sigma}$ *denotes the limit of the average over the segment defined by* $\mathrm{Re}(s)=\sigma$ *and* $|\mathrm{Im}(s)| \le T$ $(T \to \infty)$, χ *runs over a "density* $\to 1$*" subset of the set of all Dirichlet characters with prime conductors,* $L(s,\chi)$ *is the Dirichlet L-function,* Avg_χ *is an average over* χ *in a suitable sense. Under GRH, the Generalized Riemann Hypothesis,* Φ *(at least) in (3.2) can be any continuous function with at most exponential growth.* (ii) *In particular, for* $\Phi(w) = \exp(x_1\bar{w} + x_2 w)$ *with fixed pair of complex numbers* $x_\nu = iz_\nu/2$ $(\nu = 1,2)$,

$$\tilde{M}(\sigma; z_1, z_2) = \mathrm{Avg}_{\mathrm{Re}(s)=\sigma}\left(\overline{\zeta(s)}^{\,x_1}\zeta(s)^{x_2}\right) \tag{3.3}$$

$$= \mathrm{Avg}_\chi\left(\overline{L(s,\chi)}^{\,x_1} L(s,\chi)^{x_2}\right) \tag{3.4}$$

holds unconditionally as long as $z_2 = \bar{z}_1$. *Under GRH, (3.4) holds for any* $z_1, z_2 \in \mathbf{C}$.

The equality (3.1) for bounded continuous test functions Φ, and (3.3) for $z_2 = \bar{z}_1$, are, at least essentially, due to Bohr-Jessen [1](cf. [9] Remark 9.1). Over $\mathbf{C}$, the two types of averages, the"vertical", i.e., $\log\zeta(s)$ over $\mathrm{Re}(s) = \sigma$, and the "character-type", i.e., $\log L(s,\chi)$ over χ, correspond to the same density function, and we find it meaningful to present this explicitly. However,

(Warning) This is not at all a general phenomenon. In fact, these two types of averages (distributions) possess the same density *only when* the base field is $\mathbf{Q}$. The main reason is that vertical type distribution corresponds to consideration of characters that depend only on the norm of primes. The vertical type for a number field case is interesting and offers deep problems cf. [17], while for function fields over finite fields, the zeta functions are vertically periodic and the average on this direction is not interesting. In contrast to these, the character type average has a common feature for all global fields. Many people consider that the vertical type distribution is the main thing and the character type results are something secondary and easily predictable. Is it really so ?

Finally, in the character type case, there are stronger and weaker averages. Results on stronger averages can be obtained for function fields over finite fields, or for some number fields under GRH. For these details, cf. [9, 10] (or a survey [11]).

The following equalities are expected to hold in general:

$$\mu_\sigma = \mathrm{Avg}_{\mathrm{Re}(s)=\sigma} |\log \zeta(s)|^2 = \mathrm{Avg}_\chi |\log L(s,\chi)|^2; \tag{3.5}$$

$$\nu_\sigma = \textit{the density at } 0 \textit{ of the distribution of} \tag{3.6}$$

$$\{\log(\zeta(\sigma+ti)/\zeta(\sigma+t'i))\}_{t,t'\in\mathbf{R}},\ \{\log(L(s,\chi)/L(s,\chi'))\}_{\chi,\chi'}.$$

§ 4. Limit behaviors at $s = 1/2$ (review of [7])

It is natural to pay attention to the "variance-normalized" function

$$M_\sigma^\star(w) = \mu_\sigma M_\sigma(\mu_\sigma^{1/2} w) \tag{4.1}$$

which has the variance $= 1$ and the Fourier transform

$$\tilde{M}_\sigma^\star(z) = \tilde{M}(\sigma; \mu_\sigma^{-1/2} z, \mu_\sigma^{-1/2}\bar{z}). \tag{4.2}$$

As in §1, consider the Plancherel volume

$$\nu_\sigma := \int_{\mathbf{C}} M_\sigma(w)^2 |dw| = \int_{\mathbf{C}} |\tilde{M}(\sigma; z, \bar{z})|^2 |dz|. \tag{4.3}$$

The product $\mu_\sigma \nu_\sigma$, which may be expressed as

$$\mu_\sigma \nu_\sigma = \int_{\mathbf{C}} M_\sigma^\star(w)^2 |dw| = \int_{\mathbf{C}} |\tilde{M}_\sigma^\star(z)|^2 |dz|, \tag{4.4}$$

is an interesting object of study. Recall that $\mu_\sigma \nu_\sigma \geq 8/9$.

Theorem 3 ([7]§2) *As* $s \to 1/2 + 0$,

$$\mu(s)/\log \frac{1}{2s-1} \to 1, \tag{4.5}$$

$$\tilde{M}(s; \mu(s)^{-1/2} z_1, \mu(s)^{-1/2} z_2) \to \exp(-z_1 z_2/4). \tag{4.6}$$

In particular,

$$\tilde{M}_\sigma^\star(z) \to \exp(-|z|^2/4). \tag{4.7}$$

The convergences in (4.6)(4.7) are uniform in the wider sense. These follow from the special case $N = 1$ of Theorem 4 below. We have also proved the following rapid decay property of $|\tilde{M}_\sigma(z)|$: Take any $0 < \epsilon < 1$, and let $(2\sigma - 1)^{-1} \gg_\epsilon 1$. Then the inequality

$$|\tilde{M}(\sigma; z, \bar{z})|^2 \leq \exp\left(-\frac{1-\epsilon}{2}\mu_\sigma |z|^{2(1-\epsilon')}\right) \tag{4.8}$$

holds for all $z \in \mathbf{C}$, where $\epsilon' = \epsilon$ (resp. 0) for $|z| \geq 1$ (resp. $|z| < 1$) ;[7]§4 Theorem 7C.

These provide enough ingredients for the proof of:

Theorem 3'([7]§2) *As* $\sigma \to 1/2 + 0$,

$$M_\sigma^\star(w) \to 2\exp(-|w|^2), \tag{4.9}$$

$$\mu_\sigma \nu_\sigma \to 1. \tag{4.10}$$

(As for the equality (4.9), the author first worked on its $d\log$-analogue and computed the corresponding values for two special "central" points $w = \mathbf{0}$ and $w = -(d\log\zeta)(2\sigma)$, for comparison. Professor S.Takanobu who attended the author's talk in a workshop (July, 2008) kindly pointed out how this can immedidately be generalized (without further ingredients) to a formula for any w. An analogous method works for the present log-case.)

§ 5. Analytic continuation (cf. [7]§3)

Put

$$\mathcal{D} = \{\mathrm{Re}(s) > 0;\ s \neq \frac{1}{2n}, \frac{\rho}{2n};\ \rho : \textit{nontrivial zeros of}\ \zeta(s),\ n \in \mathbf{N}\}. \tag{5.1}$$

Theorem 4 ([7]§3) $\tilde{M}(s; z_1, z_2)$ *extends to a multivalent analytic function on* $\mathcal{D} \times \mathbf{C}^2$.

This means that $\tilde{M}(s; z_1, z_2)$ extends to an analytic function on $\tilde{\mathcal{D}} \times \mathbf{C}^2$, where $\tilde{\mathcal{D}}$ is the universal covering of $\mathcal{D}$. Actually, $\tilde{\mathcal{D}}$ can be replaced by the maximal unramified *abelian* covering of $\mathcal{D}$. Let

$$\ell(t) = -\log(1-t) = t + \frac{1}{2}t^2 + \cdots, \tag{5.2}$$

and $P_n(x_1, x_2)$ $(n = 1, 2, \cdots)$ be the polynomial of degree $\leq n$ in each variable defined by the formal power series equality

$$\log F(x_1, x_2; 1;\ t) = \sum_{n=1}^{\infty} P_n(x_1, x_2)\ell(t^n). \tag{5.3}$$

Then a more descriptive account of Theorem 4 reads as follows.

Theorem 4'([7]§3)

$$\tilde{M}(s; z_1, z_2) = \prod_{n=1}^{\infty} \zeta(2ns)^{P_n(x_1, x_2)} \tag{5.4}$$

holds in the following sense; (i) for any $N \geq 0$*, the quotient of* $\tilde{M}(s; z_1, z_2)$ *by the partial product over* $n \leq N$ *on the right hand side extends to a holomorphic function on* $\mathrm{Re}(s) > 1/(2N+2)$*; (ii) the equality (5.4) holds on* $|z_1|, |z_2| \leq R$ *and* $\mathrm{Re}(s) \geq \sigma_0 > 1/2$*, provided that either* R *is fixed and* σ_0 *is sufficiently large, or* σ_0 *is fixed and* R *is sufficiently small.*

We have $P_1(x_1, x_2) = x_1 x_2$ and $P_2(x_1, x_2) = -x_1 x_2 (x_1 - 1)(x_2 - 1)/4$. Note that (5.3) already gives the "formal local version"

$$\log \tilde{M}_p(s; z_1, z_2) = \sum_{n=1}^{\infty} P_n(x_1, x_2) \log \zeta_p(2ns) \tag{5.5}$$

of (5.4). To prove the global analytic equality (5.4), we need to justify the commutativity of summations over p and those over the exponents of p^{-2s}, x_1, x_2. This follows from suitable estimations of various summands. If z_1, z_2 are fixed and s encircles a punctured point $s_0 \in \{\mathrm{Re}(s) > 0\} \setminus \mathcal{D}$ in the positive direction, and if, say, s_0 can be expressed in just one way as $s_0 = \rho/2n$ with some $n \geq 1$ and with a simple zero ρ of $\zeta(s)$, then the function $\tilde{M}(s; z_1, z_2)$ is multiplied by

$$\exp(2\pi i P_n(x_1, x_2)).$$

§ 6. Zeros of $\tilde{M}(s; z_1, z_2)$ ([7]§0.4)

One can prove that the zero divisor of the analytic continuation of $\tilde{M}(s; z_1, z_2)$ on $\tilde{\mathcal{D}} \times \mathbf{C}^2$ is well-defined as a divisor on $\mathcal{D} \times \mathbf{C}^2$, and that it is simply the (locally finite) sum over p of the zero divisor of $\tilde{M}_p(s; z_1, z_2)$. The zero divisor of the local factor

$$\tilde{M}_p(s; z_1, z_2) = F(x_1, x_2; 1;\, t^2) \tag{6.1}$$

($z_\nu = i x_\nu/2,\ t = t_p = p^{-s}$) is smooth, because of the Gauss differential equation. Its property has not been analyzed systematically. But the intersection with the hyperplane defined by $x_1 + x_2 = 0$ can be analyzed as follows. For $|t| < 1$, consider the "locally normalized" function

$$f_t(x) = F(x/(2\arcsin(t)), -x/(2\arcsin(t)); 1;\, t^2). \tag{6.2}$$

Then $f_0(x) = J_0(x)$, the Bessel function of order 0. Let $\pm\{\gamma_\nu\}_{\nu=1}^{\infty}$ with $0 < \gamma_1 < \gamma_2 < \cdots$ denote all the zeros of $J_0(x)$, so that $\gamma_\nu \in ((\nu - 1/2)\pi, \nu\pi)$. Then we can prove:

Proposition 1 *There exists* $0 < t_0 < 1$ *such that for* $|t| \leq t_0$, (i) *each* γ_ν *extends uniquely and holomorphically to a zero* $\gamma_\nu(t)$ *of* $f_t(x)$ *satisfying* $\mathrm{Re}(\gamma_\nu(t)) \in ((\nu - 1/2)\pi, \nu\pi)$ *and* $|\mathrm{Im}(\gamma_\nu(t))| < 1$*, and* (ii) *there are no zeros of* $f_t(x)$ *other than* $\pm\{\gamma_\nu(t)\}$.

These lead directly to the Weierstrass decomposition

$$f_t(x) = \prod_{\nu=1}^{\infty} \left(1 - \frac{x^2}{\gamma_\nu(t)^2}\right) \tag{6.3}$$

of $f_t(x)$, from which follows the second infinite product decomposition of $\tilde{M}(s; z_1, z_2)$ on $z_1 + z_2 = 0$:

Theorem 5 *We have*

$$\tilde{M}(s; z, -z) = \prod_p \prod_{\nu=1}^{\infty} \left(1 + \left(\frac{\arcsin(p^{-s})}{\gamma_\nu(p^{-s})}\right)^2 z^2\right) = \prod_{\mu=1}^{\infty} (1 + \theta_\mu(s)^2 z^2), \tag{6.4}$$

$\{\theta_\mu(s)\}_\mu$ *being a reordering of* $\{\arcsin(p^{-s})/\gamma_\nu(p^{-s})\}_{p,\nu}$ *according to the absolute values.*

Remark Here, in order to assure that each $\gamma_\nu(p^{-s})$ makes clear sense, we need to assume that $\mathrm{Re}(s)$ is sufficiently large. On the other hand, (6.3) itself holds for each fixed t if we simply let $\pm\gamma_\nu(t)$ denote all the zeros of $f_t(x)$. So, (6.4) remains valid for each fixed s with $\mathrm{Re}(s) > 1/2$ after suitable modifications of local factors for small p's. We might add here that $\lim_{t\to 1} f_t(x) = \sin x/x$.

We shall indicate here the main ingredients for the proofs of the above statements on the zeros of $f_t(x)$, in order to supplement [7]§0.4 and explain why $\arcsin(t)$ should appear. First we need:

Key lemma A *The function $f_t(x)$ admits a Neumann series expansion*

$$\sum_{n=0}^{\infty} a_{2n}(t) J_{2n}(x), \tag{6.5}$$

where $J_{2n}(x)$ is the Bessel function of order $2n$, and $a_{2n}(t)$ is a holomorphic function of t^2 on $|t| < 1$ divisible by t^{2n}, with $a_0(t) = 1$ and $a_{2n}(t) \ll |t|^{2n}$, with $\ll$ independent of n (depending only on the compact subdomain of $|t| < 1$ considered).

To prove this lemma, we may assume that t is positive real. Then the key parameter $\arcsin(t)$ appears as the maximal value of $|\mathrm{Arg}(1 - te^{-i\theta})|$ for $\theta \in \mathbf{R}/2\pi$. By using the new argument θ' defined via

$$\mathrm{Arg}(1 - te^{-i\theta})/\arcsin(t) = \sin\theta', \tag{6.6}$$

we may express $f_t(x)$ as

$$\begin{aligned} f_t(x) &= \frac{1}{2\pi} \int_0^{2\pi} e^{ix\sin\theta'} (d\theta/d\theta') d\theta' \\ &= \frac{2}{\pi} \int_0^{\pi/2} K_\tau(\theta') \cos(x \sin\theta') \cos(\tau \sin\theta') d\theta', \end{aligned} \tag{6.7}$$

where $\tau = \arcsin(t)$ and

$$K_\tau(\theta') = \frac{\tau \cos\theta'}{\sqrt{\sin^2\tau - \sin^2(\tau\sin\theta')}} \tag{6.8}$$
$$= \sum_{\mu=0}^{\infty} \alpha_{2\mu}(\tau)\cos(2\mu\theta'),$$

with $\alpha_{2\mu}(\tau)$ given explicitly and divisible by $\tau^{2\mu}$. We thus obtain

$$f_t(x) = \frac{1}{2}\sum_{\mu=0}^{\infty} \alpha_{2\mu}(\tau)(J_{2\mu}(x+\tau) + J_{2\mu}(x-\tau)), \tag{6.9}$$

from which follows the lemma by the addition formula for Bessel functions.

By this lemma, $f_t(x)$ and $df_t(x)/dx$ are "close to" $J_0(x)$ and $-J_1(x)$ (respectively) of which the asymptotic behaviors away from zeros are well-understood ([21]§7.21). A quantitative closeness is guaranteed by:

Key lemma B

$$|J_n(x)| \ll_{abs.} (n+1)^{1/2}|x|^{-1/2}e^{|\mathrm{Im}(x)|} \qquad (n = 0, 1, 2, \cdots; x \in \mathbf{C}). \tag{6.10}$$

This proof is parallel to that of Lemma 3.3.4 of [5] which was for $x \in \mathbf{R}$; just replace $J_n(x)$ there by $e^{-|\mathrm{Im}(x)|}J_n(x)$.

§ 7. Comparisons

We thus have two decompositions related to $\tilde{M}(s; z_1, z_2)$: The first one

$$\tilde{M}(s; z_1, z_2) = \prod_{n=1}^{\infty} \zeta(2ns)^{P_n(x_1,x_2)} \tag{7.1}$$

is similar to the Riemann-Hadamard decomposition (2.3) of $\zeta(s)$ in the sense that it is related to analytic continuation with respect to s, but is similar to the Euler product decomposition (2.1) of $\zeta(s)$ in the sense that it tells us nothing about the zeros. The second,

$$\tilde{M}(s; z, -z) = \prod_p \prod_{\nu=1}^{\infty} \left(1 + \left(\frac{\arcsin(p^{-s})}{\gamma_\nu(p^{-s})}\right)^2 z^2\right) = \prod_{\mu=1}^{\infty}(1 + \theta_\mu(s)^2 z^2), \tag{7.2}$$

is similar to (2.1) in the sense that it is firstly the product over p, while in the sense that it is the Weierstrass decomposition according to zeros, it is similar to (2.3). It is still mysterious, but we hope that the comparison of these two decompositions will bring us some new insight.

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