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Multiplicative functions on $\mathbb{Z}_+^n$ and the Ewens Sampling Formula

By

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Abstract

We deal with multiplicative functions defined on the additive semigroup $\mathbb{Z}_+^n$. The purpose is to obtain lower bounds for their mean values with respect to the Ewens Probability Measure. They imply useful estimates of probabilities of random permutations missing some cycles. The results are analogues to that obtained by P. Erdős, I.Z. Ruzsa, and K. Alladi for the number theoretical functions.

§ 1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of natural, nonnegative integer, real and complex numbers, $n \in \mathbb{N}$, and let $\mathbb{Z}_+^n$ be the set of vectors $\bar{s} := (s_1, \dots, s_n)$, where $s_j \in \mathbb{Z}_+$ and $1 \le j \le n$. Define the mapping $\ell : \mathbb{Z}_+^n \to \mathbb{Z}_+$ by $\ell(\bar{s}) = 1s_1 + \cdots + ns_n$ and set $\Omega(n) = \ell^{-1}(n)$. The *Ewens Sampling Formula* was introduced in [6] as the probability measure on the subsets of $\Omega(n)$ so that

$$P_{n,\theta}(\bar{s}) := P_{n,\theta}\big(\{\bar{s}\}\big) = \frac{n!}{\theta^{(n)}} \prod_{j \le n} \left(\frac{\theta}{j}\right)^{s_j} \frac{1}{s_j!}, \quad \bar{s} \in \Omega(n), \tag{1.1}$$

where $\theta > 0$ is a parameter and $\theta^{(n)} := \theta(\theta+1)\cdots(\theta+n-1)$. Since its introduction in the field of mathematical genetics $P_{n,\theta}$ is serving in various statistical models and probabilistic combinatorics (see, for instance [8] and [2]).

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Let us recall another expression of (1.1). If ξ_j, $1 \le j \le n$, are mutually independent Poisson random variables (r. vs) with $\mathbf{E}\xi_j = \theta/j$ given on some probability space $(\Omega, \mathcal{F}, P)$ and $\bar{\xi} := (\xi_1, \ldots, \xi_n)$, then

$$P_{n,\theta}(\bar{s}) = P\big(\bar{\xi} = \bar{s} \mid \ell(\bar{\xi}) = n\big), \quad \bar{s} \in \Omega(n).$$

This clearly shows the dependence of coordinates s_j, $1 \le j \le n$, under the probability measure $P_{n,\theta}$. Despite to it, some recent results on the asymptotic behavior as $n \to \infty$ of distributions of the linear statistics $a_{n1}s_1 + \cdots + a_{nn}s_n$, where $a_{nj} \in \mathbb{R}$ and $1 \le j \le n$, give general conditions for weak convergence or sharp estimates of the convergence rates. They are mainly formulated in the terminology of the theory of random permutations; therefore, we now present the connections to the latter.

Let $\mathbb{S}_n$ denote the symmetric group of permutations σ acting on $n \ge 1$ letters. Each $\sigma \in \mathbb{S}_n$ has a unique representation (up to the order) by the product of independent cycles $\varkappa_i$:

$$\sigma = \varkappa_1 \cdots \varkappa_w, \tag{1.2}$$

where $w = w(\sigma)$ denotes the number of cycles. Denote by $k_j(\sigma) \ge 0$ the number of cycles in (1.2) of length j for $1 \le j \le n$ and $\bar{k}(\sigma) := (k_1(\sigma), \ldots, k_n(\sigma))$. The latter is called a *cycle vector* of the permutation σ. The *Ewens Probability Measure* on $\mathbb{S}_n$ is defined by

$$\nu_{n,\theta}\big(\{\sigma\}\big) = \theta^{w(\sigma)}/\theta^{(n)}, \quad \sigma \in \mathbb{S}_n,$$

where $\theta > 0$ is a parameter. An easy combinatorial argument (see [2]) gives the distribution of the cycle vector and the coincidence:

$$\nu_{n,\theta}\big(\bar{k}(\sigma) = \bar{s}\big) = P_{n,\theta}(\bar{s})$$

if $\bar{s} \in \Omega(n)$. Thus, dealing with statistics of random permutations expressed via $\bar{k}(\sigma)$, we may examine corresponding statistics of random vectors $\bar{s} \in \Omega(n)$ taken with probabilities (1.1).

The main advantage of such imbedding is the fact that $\mathbb{Z}_+^n$ has an additive semigroup structure as well as the partial order defined by $\bar{s} = (s_1, \ldots, s_n) \le \bar{t} = (t_1, \ldots, t_n)$ meaning that $s_j \le t_j$ for each $1 \le j \le n$. Moreover, we may introduce the orthogonality of $\bar{s}, \bar{t} \in \mathbb{Z}_+^n$, denoted by $\bar{s} \perp \bar{t}$, meaning that $s_1t_1 + \cdots + s_nt_n = 0$. In this way, we come closer to probabilistic number theory dealing with random numbers taken from the multiplicative semigroup $\mathbb{N}$ (see [9] and [4]) having the partial order defined by division. The semigroup structures and the partial orders in $\mathbb{Z}_+^n$ and $\mathbb{N}$ could play the crucial role in developing parallel theories. Nevertheless, the advance in probabilistic number theory has not been adequately followed by the corresponding results in probabilistic combinatorics. For instance, the results exposed in the recent book [2] concerning the value distribution of additive functions defined on the decomposable structures do not reach the level of their analogs in $\mathbb{N}$ (compare with [4]). In the recent papers [14], [15], and [16] (see also the references therein), the second author did some attempt to fill up this gap.

The same could be said about the development of a theory of multiplicative functions in combinatorics. Having this in mind, we now raise reader's attention to the lower estimates of the mean values of multiplicative functions related to the so-called small sieve problem. The results established in the present paper are analogous to that achieved by P. Erdős and I.Z. Ruzsa [5] and K. Alladi [1] in number theory.

Let us recall necessary definitions. A mapping $G : \mathbb{Z}_+^n \to \mathbb{C}$, $G(\bar{0}) = 1$, is called a *multiplicative function* if $G(\bar{s}+\bar{t}) = G(\bar{s})G(\bar{t})$ for every pair $\bar{s}, \bar{t} \in \mathbb{Z}_+^n$ such that $\bar{s} \perp \bar{t}$. If $\bar{e}_j := (0, \dots, 1, \dots, 0)$, where the only 1 stands at the jth place, then the multiplicative function G has the decomposition

$$G(\bar{k}) = \prod_{j \le n} G(k_j \bar{e}_j) =: \prod_{j \le n} g_j(k_j).$$

Conversely, given a complex two-dimensional array $\{g_j(k)\}$, $1 \le j \le n, k \ge 0$, satisfying the condition $g_j(0) \equiv 1$, by the last equality, we can define a multiplicative function. If $g_j(k) = g_j(1) =: g_j$ for all $k \ge 1$ and $j \le n$, the function G is called *strongly* multiplicative and, similarly, if $g_j(k) = g_j^k$ and $0^0 := 1$, then G is called *completely* multiplicative. Denote, respectively, by $\mathfrak{M}$, $\mathfrak{M}_s$, and $\mathfrak{M}_c$ the sets of just introduced multiplicative functions. Observe that if $G \in \mathfrak{M}_c$ and $g_j \in \{0, 1\}$, then $G \in \mathfrak{M}_s$ and, conversely, the latter together with $g_j \in \{0, 1\}$ implies $G \in \mathfrak{M}_c$. The multiplicative function

$$\Pi(\bar{k}) := \prod_{j \le n} \left(\frac{\theta}{j}\right)^{k_j} \frac{1}{k_j!},$$

depending on θ, plays a special role in the sequel.

If $G \in \mathfrak{M}$, then its mean value with respect to $P_{n,\theta}$ is

$$\begin{aligned} M_{n,\theta}(G) &:= \sum_{\bar{k} \in \Omega(n)} G(\bar{k}) P_{n,\theta}(\bar{k}) = \frac{n!}{\theta^{(n)}} \sum_{\bar{k} \in \Omega(n)} G(\bar{k}) \Pi(\bar{k}) \\ &= \frac{n!}{\theta^{(n)}} \sum_{\bar{k} \in \Omega(n)} \prod_{j \le n} \left(\frac{\theta}{j}\right)^{k_j} \frac{g_j(k_j)}{k_j!} = \frac{n!}{\theta^{(n)}} [x^n] Z(x; G), \end{aligned} \tag{1.3}$$

where

$$Z(x; G) = \prod_{j \ge 1} \left(1 + \sum_{r \ge 1} \left(\frac{\theta}{j}\right)^r \frac{g_j(r)}{r!} x^{jr}\right)$$

and $[x^n]Z(x)$ denotes the nth coefficient of the formal power series $Z(x)$. We also assume that $M_{0,\theta}(G) \equiv 1$ for every $G \in \mathfrak{M}$.

We are interested in estimates of $M_{n,\theta}(G)$ holding uniformly in G belonging to some subclass of $G \in \mathfrak{M}$. If $G \in \mathfrak{M}_c$ and $0 < \theta^- \le g_j \le \theta^+ < \infty$ for all $j \le n$, then we have [12]

$$M_{n,1}(G) \asymp \exp\Big\{ \sum_{j \le n} \frac{g_j - 1}{j} \Big\}, \quad n \ge 1. \tag{1.4}$$

Here and afterwards $a \asymp b$ means that $a \ll b$ and $b \ll a$ while $a \ll b$ or $b \gg a$ are the analogs of $a = \mathrm{O}(b)$. In (1.4), the involved constants depend on θ^- and θ^+. Afterwards, the constants in these symbols as well as positive constants c and c_i, $i \ge 0$, will be dependent at most on θ.

If $G(\bar{k})$ takes the zero value rather often, the lower estimation of $M_{n,1}(G)$ becomes rather involved and, in general, the lower bound as it is stated in (1.4) is false. The second author has achieved a satisfactory result only for $G(\bar{k}) \in \{0, 1\}$ (see [10] and [11]). We now extend these results. For simplicity, we assume that $G \in \mathfrak{M}_s$. As it is demonstrated in Corollaries, an extension to general multiplicative functions can be achieved by some convolution argument.

§ 1.1. Results

Let us start from an easier problem to estimate the averaged mean values

$$\widetilde{M}_{n,\theta}(G) := \frac{1}{\Gamma_{n,\theta}} \sum_{0 \le m \le n} \frac{\theta^{(m)}}{m!} M_{m,\theta}(G),$$

where

$$\Gamma_{n,\theta} := \sum_{0 \le m \le n} \frac{\theta^{(m)}}{m!} = \frac{n^\theta}{\Gamma(\theta+1)} \Big(1 + \mathrm{O}\Big(\frac{1}{n}\Big)\Big)$$

for $n \ge 1$ and $\theta^{(0)} := 1$. The quantity $\widetilde{M}_{n,\theta}(G)$ is just the mean value of G with respect to the measure defined via $\widetilde{P}_{n,\theta}(\bar{s}) = \Pi(\bar{s})/\Gamma_{n,\theta}$ and supported by the set $\{\bar{s} \in \mathbb{Z}_+^n : 0 \le \ell(\bar{s}) \le n\}$. To check this, it suffices to observe that $s_j = 0$ if $\ell(\bar{s}) < j \le n$ and apply an appropriate combinatorial identity.

Theorem 1.1. *Let $\theta > 0$ and $G \in \mathfrak{M}_s$ be defined via sequence $0 \le g_j \le 1$ where $1 \le j \le n$. Then*

$$\widetilde{M}_{n,\theta}(G) \asymp \exp\Big\{ \theta \sum_{j \le n} \frac{g_j - 1}{j} \Big\}.$$

The main result of the paper is the next theorem.

Theorem 1.2. *Let $\theta \ge 1$ and $G \in \mathfrak{M}_s$ be defined via sequence $0 \le g_j \le 1$ where $1 \le j \le n$. If*

$$\sum_{j \le n} \frac{1 - g_j}{j} \le K \tag{1.5}$$

for some $K > 0$*, then there exist positive constants* c_0 *and* c *together with a function* $\mathcal{N} : \mathbb{R}_+ \to \mathbb{N}$ *such that*

$$\gamma(K) := \inf\{M_{n,\theta}(G) : n \geq \mathcal{N}(K)\} \geq c_0 \exp\{-\mathrm{e}^{cK}\}. \tag{1.6}$$

Remark. An instance given in [10] shows that apart of the constants the estimate (1.6) is sharp if $\theta = 1$. Also, the lower bound for n, that is, the use of $n \geq \mathcal{N}(K)$ in (1.6) is unavoidable. Without such a bound, given $K \geq 1$, one can assure condition (1.5) for some function $G \in \mathfrak{M}_s$ such that $g_j = 0$ for each $1 \leq j \leq \mathrm{e}^{K-1}$. Then $M_{n,\theta}(G) = 0$ for each $1 \leq n \leq \mathrm{e}^{K-1}$.

One can now derive lower bounds of probabilities of the vectors in $\Omega(n)$ with some zero coordinates. Speaking in the other terminology (see the recent book by A.L. Yakimyv [17]), they concern the probabilities of A-permutations. When $\theta = 1$, the Corollaries presented below have proved to be very useful in [13] and [14].

Let $J \subset \{1, \ldots, n\}$ and $\Omega(n; J) = \{\bar{k} \in \Omega(n) : \ k_j = 0 \ \forall\, j \in J\}$.

Corollary 1.3. *Let* $\theta \geq 1$*,* $K > 0$*, and* J *be such that*

$$\sum_{j \in J} \frac{1}{j} \leq K < \infty. \tag{1.7}$$

Then

$$P_{n,\theta}\big(\Omega(n; J)\big) \geq c_0 \exp\big\{ - \mathrm{e}^{cK}\big\}$$

for $n \geq \mathcal{N}(K)$*. Here* c*,* c_0*, and* $\mathcal{N}(K)$ *are the same as in Theorem* 1.2.

Proof. Apply Theorem 1.2 for the strongly multiplicative indicator function $G(\bar{k})$ defined via $g_j = 0$ if $j \in J$ and $g_j = 1$ otherwise. □

The next corollary involves two types of sifting (one with respect to the indexes and another with respect to the value of coordinates) of the vectors from $\Omega(n)$. The following result for $\theta = 1$ has been stated without a proof as Lemma 5 in [14].

Corollary 1.4. *Let* $\theta \geq 1$*,* $K > 0$*, and* J *be as in Corollary* 1.3*. Denote* $I = \{1, \ldots, n\} \setminus J$*. Then there exists a positive constant* $R(K)$ *such that*

$$P_{n,\theta}\big(\bar{k} \in \Omega(n; J) : \ k_i \leq 1 \ \forall\, i \in I\big) \geq R(K), \tag{1.8}$$

provided that $n \geq \mathcal{N}_1(K)$ *is sufficiently large.*

Now, the indicator function of the examined event is not strongly multiplicative; therefore, we leave the proof of this corollary to the end of the paper.

§ 2. Proof of Theorem 1.1

Proof. Estimating from above, we examine an arbitrary $G \in \mathfrak{M}$ such that $0 \le g_j(k) \le 1$ for $j, k \ge 1$ with $g_j := g_j(1)$. Applying (1.3), we obtain

$$\widetilde{M}_{n,\theta}(G) = \frac{1}{\Gamma_{n,\theta}} \sum_{0 \le m \le n} [x^m] Z(x; G) \le \frac{1}{\Gamma_{n,\theta}} \prod_{j \le n} \left(1 + \sum_{r \ge 1} \left(\frac{\theta}{j}\right)^r \frac{g_j(r)}{r!}\right).$$

Since $0 \le G(\bar{s}) \le 1$, the infinite product

$$\prod_{j \ge 1} \left(1 + \sum_{r \ge 1} \left(\frac{\theta}{j}\right)^r \frac{g_j(r)}{r!}\right) \mathrm{e}^{-\theta g_j / j}$$

converges uniformly in G. Thus the previous estimate implies

$$\widetilde{M}_{n,\theta}(G) \ll \frac{1}{\Gamma_{n,\theta}} \exp\left\{\theta \sum_{j \le n} \frac{g_j}{j}\right\} \ll \exp\left\{\theta \sum_{j \le n} \frac{g_j - 1}{j}\right\}$$

as claimed.

To obtain the desired lower estimate, we define the Möbius function $\mu(\bar{k})$ on $\mathbb{Z}_+^n$ related to the mentioned partial order. If $\bar{k} = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$, then we set

$$\mu(\bar{k}) = \prod_{j \le n} \mu_j(k_j), \qquad \text{where} \quad \mu_j(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \ge 2, \\ -1 & \text{if } k = 1. \end{cases}$$

Given $G \in \mathfrak{M}_s$, we introduce its dual function $G^* \in \mathfrak{M}_s$ defined by $g_j^* = 1 - g_j$ for each $1 \le j \le n$. Then

$$G^*(\bar{k}) = \sum_{\bar{t} \le \bar{k}} \mu(\bar{t}) G(\bar{t}) = \prod_{\substack{j \le n \\ k_j \ge 1}} (1 - g_j), \qquad G(\bar{k}) = \sum_{\bar{t} \le \bar{k}} \mu(\bar{t}) G^*(\bar{t}).$$

By (1.3),

$$M_{m,\theta}(\mu^2) = \frac{m!}{\theta^{(m)}} [x^m] \prod_{j \ge 1} \left(1 + \frac{\theta x^j}{j}\right).$$

Hence, as it has been shown in [12], $M_{m,\theta}(\mu^2) \asymp 1$ for $m \ge 0$. This implies $\widetilde{M}_{n,\theta}(\mu^2) \asymp 1$ for $n \ge 1$.

If $\ell(\bar{k}) = m \le n$ and $\mu^2(\bar{k}) = 1$, then $\bar{t} \le \bar{k}$ implies $\bar{t} \perp \bar{k} - \bar{t} =: \bar{s}$. Hence

$$\sum_{\bar{t} \le \bar{k}} G(\bar{t}) G^*(\bar{k} - \bar{t}) = \prod_{\substack{j \le n \\ k_j = 1}} (g_j + g_j^*) = 1$$

and $\Pi(\bar{k}) = \Pi(\bar{t}+\bar{s}) = \Pi(\bar{t})\Pi(\bar{s})$. Consequently,

$$\begin{aligned}
1 \ll \widetilde{M}_{n,\theta}(\mu^2) &= \frac{1}{\Gamma_{n,\theta}} \sum_{\ell(\bar{k})\le n} \mu^2(\bar{k})\Pi(\bar{k}) \sum_{\bar{t}\le\bar{k}} G(\bar{t})G^*(\bar{k}-\bar{t}) \\
&\le \frac{1}{\Gamma_{n,\theta}} \sum_{\ell(\bar{t})\le n} G(\bar{t})\Pi(\bar{t}) \times \sum_{\ell(\bar{s})\le n} \mu^2(\bar{s})G^*(\bar{s})\Pi(\bar{s}) \\
&= \widetilde{M}_{n,\theta}(G)\Gamma_{n,\theta}\widetilde{M}_{n,\theta}(G^*\mu^2) \ll \widetilde{M}_{n,\theta}(G) \exp\Big\{\theta \sum_{j\le n} \frac{1-g_j}{j}\Big\}.
\end{aligned}$$

In the last step we applied already proved upper estimate.

The theorem is proved. □

§ 3. Proof of Theorem 1.2

Proof. Let the truncated strongly multiplicative function G_r be defined from $G \in \mathfrak{M}_s$ by setting $g_j = 1$ for each $r \le j \le n$, where $1 \le r \le n+1$. Then $G_{n+1} = G$, and $G_1(\bar{k}) \equiv 1$. Apart from the vectors $\bar{e}_j$, we introduce $\bar{e}^r = (1,\dots,1,0,\dots 0) \in \mathbb{Z}_+^n$, where the zeroes start at the r-th, $r \ge 1$, place. By $\bar{k}\wedge\bar{t}$ we denote the vector with the coordinates $\min\{k_j, t_j\}$ for all $1 \le j \le n$. Observe that

$$G_r(\bar{k}) = \sum_{\bar{t}\le\bar{k}\wedge\bar{e}^r} \mu(\bar{t})G^*(\bar{t}).$$

If $n/2 < j \le n$ and $\bar{k} = \bar{e}_j + \bar{t} \in \Omega(n)$, then $\bar{t} \in \Omega(n-j)$, $\bar{e}_j \perp \bar{t}$, and

$$P_{n,\theta}(\bar{k}) = \frac{n!}{\theta^{(n)}}\Pi(\bar{e}_j)\Pi(\bar{t}) = \frac{n!}{\theta^{(n)}}\frac{\theta}{j}\Pi(\bar{t}). \tag{3.1}$$

Now, if $1/n \le \delta < 1/2$ is arbitrary and $r = m := [(1-\delta)n]$, then, summing over a part of vectors, we obtain

$$\begin{aligned}
M_{n,\theta}(G_m) &\ge \sum_{\substack{\ell(\bar{e}_j+\bar{t})=n \\ m\le j\le n}} G_m(\bar{e}_j+\bar{t})P_{n,\theta}(\bar{e}_j+\bar{t}) \\
&= \frac{n!}{\theta^{(n)}} \sum_{m\le j\le n} \frac{\theta}{j} \sum_{\ell(\bar{t})=n-j} G(\bar{t})\Pi(\bar{t}) \\
&\ge \frac{\theta}{n}\frac{n!}{\theta^{(n)}}\Gamma_{[\delta n],\theta}\,\widetilde{M}_{[\delta n],\theta}(G) \\
&\ge c_1\delta^\theta \exp\{-\theta K\}, \qquad (3.2)
\end{aligned}$$

where $K > 0$ is as in (1.5). In the last step we used $\theta^{(n)}/n! \ll n^{\theta-1}$ if $n \ge 1$ and Theorem 1.1. Recalling our agreement that $M_{0,\theta}(G_m) = 1$, we observe that (3.2) also holds for $0 \le \delta < 1/n$.

The next identity is crucial in the forthcoming induction argument. We have

$$M_{n,\theta}(G_r) = M_{n,\theta}(G) + \frac{\theta n!}{\theta^{(n)}} \sum_{r\le j\le n} \frac{g_j^*}{j} \frac{\theta^{(n-j)}}{(n-j)!} M_{n-j,\theta}(G_j) \tag{3.3}$$

for an arbitrary $G \in \mathfrak{M}_s$ and $n/2 < r \le n$. Indeed, the formal identity

$$1 - \prod_{j=1}^{s}(1-\alpha_j) = \sum_{j=1}^{s} \alpha_j \prod_{i=1}^{j-1}(1-\alpha_i), \quad s \ge 1,$$

implies

$$\begin{aligned} G_r(\bar{k}) - G(\bar{k}) &= G_r(\bar{k})\Big(1 - \prod_{\substack{r\le j\le n\\ k_j=1}} g_j\Big) = G_r(\bar{k})\Big(1 - \prod_{\substack{r\le j\le n\\ k_j=1}} (1-g_j^*)\Big) \\ &= G_r(\bar{k}) \sum_{\substack{r\le j\le n\\ k_j=1}} g_j^* \prod_{\substack{r\le i\le j-1\\ k_i=1}} (1-g_i^*) = \sum_{\substack{r\le j\le n\\ k_j=1}} g_j^* G_j(\bar{k}) \end{aligned}$$

for each $\bar{k} \in \Omega(n)$. By virtue of (3.1), we obtain the mean value of the last sum:

$$\begin{aligned} \sum_{r\le j\le n} g_j^* \sum_{\bar{k}\in\Omega(n)} P_{n,\theta}(\bar{k}) \mathbf{1}\{k_j=1\} G_j(\bar{k}) &= \frac{\theta n!}{\theta^{(n)}} \sum_{r\le j\le n} \frac{g_j^*}{j} \sum_{\bar{t}\in\Omega(n-j)} G_j(\bar{t})\Pi(\bar{t}) \\ &= \frac{\theta n!}{\theta^{(n)}} \sum_{r\le j\le n} \frac{g_j^*}{j} \frac{\theta^{(n-j)}}{(n-j)!} M_{n-j,\theta}(G_j). \end{aligned} \tag{3.4}$$

Combining this with the equality above, we complete the proof of (3.3).

Up to the end of the proof of Theorem 1.2, we fix the notation $m = [(1-\delta)n]$, where $\delta = \mathrm{e}^{-K-C}$ and $C \ge 1$ is a constant to be chosen later. For $\theta \ge 1$, we have

$$\big(n!/\theta^{(n)}\big) \cdot \big(\theta^{(n-j)}/(n-j)!\big) \le 1. \tag{3.5}$$

Hence and from (1.5) and (3.2) we obtain

$$\begin{aligned} M_{n,\theta}(G) &\ge M_{n,\theta}(G_m) - \theta \sum_{m\le j\le n} \frac{g_j^*}{j} M_{n-j,\theta}(G_j) \\ &\ge c_1 \mathrm{e}^{-\theta C} \mathrm{e}^{-2\theta K} - \theta \sum_{m\le j\le n} \frac{g_j^*}{j} \ge \alpha(C)\mathrm{e}^{-2\theta K}, \end{aligned} \tag{3.6}$$

where $\alpha(C) := (c_1/2)\mathrm{e}^{-\theta C}$, provided that

$$\lambda := \sum_{m\le j\le n} \frac{g_j^*}{j} \le (\alpha(C)/\theta)\mathrm{e}^{-2\theta K}.$$

If $c \geq 2\theta$, the bound (3.6) for all $K > 0$ is better than that given in Theorem 1.2 with $\mathcal{N}(K) \equiv 2$ and $c_0 \leq \alpha(C)/\theta$.

In what follows, we assume that $\lambda \geq (\alpha(C)/\theta)\mathrm{e}^{-2\theta K}$. We will bound $\gamma(K)$ from below applying the real type induction on K. To verify the initial step, we argue as in obtaining (3.3). We firstly notice that

$$\sum_{\bar{k}\in\Omega(n)} P_{n,\theta}(\bar{k})\mathbf{1}\{k_j \geq 1\} \leq \sum_{\bar{k}\in\Omega(n)} k_j P_{n,\theta}(\bar{k}) = \frac{n!}{\theta^{(n)}} \cdot \frac{\theta^{(n-j)}}{(n-j)!} \cdot \frac{\theta}{j} \leq \frac{\theta}{j}\,,$$

where the first moment formula found in [2] (p. 96, (5.6)) and inequality (3.5) are used. Using this, we obtain

$$\begin{aligned} M_{n,\theta}(G) &= \sum_{\bar{k}\in\Omega(n)} P_{n,\theta}(\bar{k}) \prod_{\substack{j\leq n\\ k_j\geq 1}} \big(1-(1-g_j)\big) \\ &= \sum_{\bar{k}\in\Omega(n)} P_{n,\theta}(\bar{k})\Bigg(1-\sum_{\substack{j\leq n\\ k_j\geq 1}} g_j^* \prod_{\substack{i\leq j-1\\ k_i\geq 1}} g_i\Bigg) \\ &\geq 1-\sum_{j\leq n} g_j^* \sum_{\bar{k}\in\Omega(n)} P_{n,\theta}(\bar{k})\mathbf{1}\{k_j\geq 1\} \\ &\geq 1-\theta\sum_{j\leq n}\frac{g_j^*}{j} \geq 1-\theta K \end{aligned}$$

for $n \geq 1$. If $\theta K \leq 1/2$, this is better than the desired estimate (1.6) with any $c > 0$ and $c_0 \leq 1$.

Let $\theta K > 1/2$ and $n \geq 1/\delta$. We further examine the set Ω' of vectors $\bar{k} \in \Omega(n)$ having a coordinate $k_j \geq 1$ for some $\delta n \leq j \leq n/2$. The indicator function of this set is

$$\mathbf{1}\{\bar{k}\in\Omega'\} = \max\big\{\mathbf{1}\{\bar{k}:\ k_j\geq 1\}:\ \delta n\leq j\leq n/2\big\}.$$

By virtue of $\ell(\bar{k}) = n$, the equality $\mathbf{1}\{\bar{k}:\ k_j \geq 1\} = 1$ holds for at most $1/\delta$ of $j \in [\delta n, n/2]$. Hence

$$\mathbf{1}\{\bar{k}\in\Omega'\} \geq \delta \sum_{\delta n\leq j\leq n/2} \mathbf{1}\{\bar{k}:\ k_j\geq 1\}.$$

If $\bar{k} \in \Omega'$, then $\bar{k} = \bar{e}_j + \bar{t}$, where $\bar{t} \in \Omega(n-j)$. Moreover,

$$G(\bar{k}) = g_j \prod_{\substack{i\leq n-j\\ i\neq j}} g_i^{\mathbf{1}\{t_i\geq 1\}} \geq g_j G(\bar{t}).$$

Similarly, due to $n \geq j(t_j+1) \geq \delta n(t_j+1)$, we have $t_j + 1 \leq 1/\delta$ and

$$P_{n,\theta}(\bar{e}_j+\bar{t}) = \frac{\theta}{j(t_j+1)}\frac{n!}{\theta^{(n)}}\frac{\theta^{(n-j)}}{(n-j)!}P_{n-j,\theta}(\bar{t}) \geq \frac{c_2\delta}{j}P_{n-j,\theta}(\bar{t})$$

for $\delta n \le j \le n/2$. Here we have applied the estimate $\theta^{(r)}/r! \asymp r^{\theta-1}$ if $r \in \mathbb{N}$. Hence

$$\begin{aligned} M_{n,\theta}(G) &\ge \sum_{\bar{k}\in\Omega(n)} G(\bar{k})P_{n,\theta}(\bar{k})\mathbf{1}\{\bar{k}\in\Omega'\} \\ &\ge \delta \sum_{\delta n\le j\le n/2} g_j \sum_{\bar{t}\in\Omega(n-j)} G(\bar{t})P_{n,\theta}(\bar{e}_j+\bar{t}) \\ &\ge c_2\delta^2 \sum_{\delta n\le j\le n/2} \frac{g_j}{j} \sum_{\bar{t}\in\Omega(n-j)} G(\bar{t})P_{n-j,\theta}(\bar{t}) \\ &= c_2\delta^2 \sum_{\delta n\le j\le n/2} \frac{g_j}{j} M_{n-j,\theta}(G). \end{aligned} \tag{3.7}$$

We now assume that the claim of Theorem 1.2 is proved for $K-\Delta =: K-(\alpha(C)/\theta)\mathrm{e}^{-2\theta K}$, that is,

$$\gamma(K-\Delta) \ge c_0 \exp\{-\mathrm{e}^{c(K-\Delta)}\} \tag{3.8}$$

and $\mathcal{N}(K-\Delta)$ is found in the latter. Here $c \ge 2\theta$ and $0 < c_0 \le \min\{1, \alpha(C)/\theta\}$ are constants. The task now is to extend this lower estimate for K and define $\mathcal{N}(K)$. We apply (3.8) for the mean values on the right-hand side of (3.7).

If $\delta n \le j \le n/2$, then

$$\sum_{i\le n-j} \frac{g_i^*}{i} \le K - \sum_{m<i\le n} \frac{g_i^*}{i} = K-\lambda \le K-\Delta$$

by our earlier agreement on λ and the definition of m. Set

$$\mathcal{N}(x) = \max\{\mathrm{e}^{K+C}, 2\mathcal{N}(K-\Delta)\}$$

for $K-\Delta < x \le K$. If $n \ge \mathcal{N}(K)$ and $\delta n \le j \le n/2$, then $n-j \ge \mathcal{N}(K-\Delta)$. Hence, by (3.8)

$$M_{n-j,\theta}(G) \ge \gamma(K-\Delta) \ge c_0 \exp\{-\mathrm{e}^{c(K-\Delta)}\}.$$

Consequently, (3.7) implies

$$\begin{aligned} M_{n,\theta}(G) &\ge c_0c_2\delta^2 \exp\{-\mathrm{e}^{c(K-\Delta)}\} \cdot \sum_{\delta n\le j\le n/2} \frac{1-g_j^*}{j} \\ &\ge c_0c_2\delta^2 \exp\{-\mathrm{e}^{c(K-\Delta)}\}\Big(-\log(2\delta) - \frac{C_1}{\delta n} - K\Big) \\ &\ge c_0c_2\delta^2 \exp\{-\mathrm{e}^{c(K-\Delta)}\}\big(C-\log 2 - C_1\big) \end{aligned}$$

where $C_1 > 0$ is an absolute constant. The choice of $C = (\log 2 + C_1 + 1)/c_2$ is at our disposal. It gives

$$M_{n,\theta}(G) \ge c_0\delta^2 \exp\{-\mathrm{e}^{c(K-\Delta)}\}.$$

Now, if

$$\mathrm{e}^{-2K-2C}\exp\{-\mathrm{e}^{c(K-\Delta)}\} \geq \exp\{-\mathrm{e}^{cK}\} \tag{3.9}$$

for all $K \geq 1/(2\theta)$ and for some sufficiently large $c \geq 2\theta$, from the last inequality, we obtain the desired estimate (1.6) with this very c.

Inequality (3.9) is equivalent to

$$\mathrm{e}^{cK}\big(1-\mathrm{e}^{-c\Delta}\big) \geq 2K+2C.$$

Assuming that $c \geq \theta\alpha(C)^{-1}$, we see that the last inequality follows from

$$\mathrm{e}^{cK}\Big(1-\mathrm{e}^{-\mathrm{e}^{-2\theta K}}\Big) \geq 2K+2C.$$

Furthermore, due to $x\mathrm{e}^{-x} \leq 1-\mathrm{e}^{-x}$ for $x \geq 0$, this is implied by

$$\mathrm{e}^{(c-2\theta)K}\mathrm{e}^{-\mathrm{e}^{-2\theta K}} \geq 2K+2C$$

and, further, by

$$\mathrm{e}^{(c-2\theta)K}\mathrm{e}^{-1} = \mathrm{e}^{2K+2C}\exp\{(c-2\theta-2)K-2C-1\} \geq 2K+2C.$$

It is evident that the last inequality holds for all $K \geq 1/(2\theta)$ if $(c-2\theta-2)/(2\theta) \geq 2C+1$. Therefore, to assure this and validity of the previous cases, it suffices to chose

$$c = \max\{\theta\alpha(C)^{-1}, 2+4\theta(C+1)\}.$$

The theorem is proved. □

§ 4. Proof of Corollary 1.4

As we have mentioned, the proof of Corollary 1.4 is based on the convolution argument combined with a few simple lemmas.

Lemma 4.1. *Assume that*

$$\chi_j(z) = \sum_{n\geq 2} a_{jn}z^n, \quad j \geq 1,$$

are entire functions satisfying $|a_{jn}| \leq C_2^n/n!$ for all $j \geq 1$ and $n \geq 0$, where $C_2 > 0$ is a constant. Then

$$[z^k]\prod_{j\geq 1}\big(1+\chi_j(z^j/j)\big) \leq \frac{C_3}{k^2}, \quad k \geq 1,$$

where C_3 is a positive constant depending on C_2 only.

Proof. This is essentially Lemma 6 from [7], where the case of $\chi_j(z)$ not depending on j has been examined. The proof in more general case goes by the repetition of the same argument. □

Lemma 4.2. *Let $F(\bar{k})$ be a complex valued multiplicative function defined via $f_j(s)$ such that $|f_j(s)| \le 1$ for all $j \ge 1$ and $s \ge 1$. Define the completely multiplicative function $G(\bar{k})$ by setting $g_j = f_j(1)$, $j \ge 1$. If $Z(z;F)$ and $Z(z;G)$ are the corresponding generating functions, then*

$$[z^k]H(z) = [z^k]\big(Z(z;F)/Z(z;G)\big) \ll k^{-2}, \quad k \ge 1. \tag{4.1}$$

Proof. We write

$$H(z) = \prod_{j\ge1} \mathrm{e}^{-\theta g_j z^j/j}\left(1 + \sum_{s\ge1} \frac{\theta^s f_j(s)}{j^s s!} z^{sj}\right) =: \prod_{j\ge1}\big(1 + \chi_j(z^j/j)\big).$$

Here

$$\chi_j(z) = \sum_{n\ge2} \frac{\theta^n z^n}{n!} \sum_{\substack{r+s=n\\ r,s\ge0}} \binom{n}{r} (-g_j)^r f_j(s) =: \sum_{n\ge2} a_{jn} z^n$$

are entire functions. Moreover, $|a_{jn}| \le (2\theta)^n/n!$. By Lemma 4.1, this implies (4.1).

The lemma is proved. □

Lemma 4.3. *Let $G \in \mathfrak{M}_c$ be as in Theorem 1.2 and $2 \le T \le \sqrt{n}$ be arbitrary. Then there exist positive constants c_3 and $R_1(K)$ such that*

$$M_{n,\theta}(G) = \frac{\Gamma(\theta)}{2\pi i}\int_{1-iT}^{1+iT} \frac{\mathrm{e}^z}{z^\theta} \exp\left\{\theta \sum_{j\le n} \frac{g_j - 1}{j}\,\mathrm{e}^{-zj/n}\right\}\mathrm{d}z + \mathrm{O}\Big(R_1(K)T^{-c_3}\Big).$$

Proof. This is a corollary of Proposition in [3]. Checking its proof, one could find an expression of $R_1(K)$.

Lemma 4.4. *Suppose $G \in \mathfrak{M}_c$ be as in Theorem 1.2. Then*

$$M_{m,\theta}(G) - M_{n,\theta}(G) \ll n^{-c_4} R_2(K) \tag{4.2}$$

uniformly in $n - \sqrt{n} \le m \le n$. Here $R_2(K) = \max\{R_1(K), \mathrm{e}^{\theta K}\}$.

Proof. We apply twice the integral representation given in the last lemma and compare the integrands. Let $z = 1 + it$, $t \in \mathbb{R}$, $|t| \le T$, and $2 \le T \le \sqrt{n}$, then

$$\begin{aligned}\sum_{j\le n} \frac{g_j - 1}{j}\,\mathrm{e}^{-zj/n} - \sum_{j\le m} \frac{g_j - 1}{j}\,\mathrm{e}^{-zj/m} &\ll \frac{1}{\sqrt{n}} + \sum_{j\le m} \frac{1}{j}\Big|1 - \mathrm{e}^{-zj(n-m)/mn}\Big| \\ &\ll \frac{T\log n}{\sqrt{n}}\end{aligned}$$

for $n-\sqrt{n}\le m\le n$. If $T\le n^{1/3}$, this and Lemma 4.3 imply

$$M_{m,\theta}(G)-M_{n,\theta}(G)\ll \frac{T(\log T)\log n}{\sqrt{n}}\exp\left\{\theta\sum_{j\le n}\frac{1-g_j}{j}\right\}+R_1(K)T^{-c_3}.$$

Now we chose $T=n^{1/4}$ to complete the proof of (4.2).

The lemma is proved. □

We now prove Corollary 1.4.

Proof. The indicator of the event in (1.8) is the multiplicative function $F(\bar{k})$ defined by

$$f_j(k)=\begin{cases}0 & \text{if } j\in J,\\ 0 & \text{if } j\in I \text{ and } k\ge 2,\\ 1 & \text{otherwise.}\end{cases}$$

Introduce also the multiplicative indicator function $G\in\mathfrak{M}_c\cap\mathfrak{M}_s$ so that $g_j=f_j(1)$ where $j\le n$. The corresponding generating functions satisfy the following relation

$$Z(z;F)=Z(z;G)H(z),$$

where, by Lemma 4.2, $h_k:=[z^k]H(z)\ll k^{-2}$ for $k\ge 1$.

Applying Lemma 4.4, we obtain

$$\begin{aligned}M_{n,\theta}(F)&=\Bigg(\sum_{k\le\sqrt{n}}+\sum_{\sqrt{n}<k\le n}\Bigg)h_kM_{n-k,\theta}(G)\\ &=\big(M_{n,\theta}(G)+\mathrm{O}\big(n^{-c_4}R_2(K)\big)\big)\sum_{k\le\sqrt{n}}h_k+\mathrm{O}\Bigg(\sum_{\sqrt{n}<k\le n}\frac{1}{k^2}\Bigg)\\ &=H(1)M_{n,\theta}(G)+\mathrm{O}\big(n^{-c_4}R_2(K)\big)+\mathrm{O}(n^{-1/2}).\end{aligned}$$

By definition,

$$H(1)=\prod_{j\in I}\mathrm{e}^{-1/j}\Big(1+\frac{1}{j}\Big)\ge\frac{2}{\mathrm{e}}\prod_{j\ge 2}\Big(1-\frac{1}{j^2}\Big)=\frac{1}{\mathrm{e}}.$$

Inserting this and the estimate obtained in Corollary 1.3 into the previous inequality, we complete the proof.

Corollary 1.4 is proved. □

Remark. It would be interesting to extend the claim of Theorem 1.2 for $\theta<1$. Unfortunately, so far, we could prove it if the stronger condition

$$\sum_{j\le n}\frac{1-g_j}{j^\theta}\le K$$

is satisfied.

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