RIMS Kôkyûroku Bessatsu
B34 (2012), 153–189

The Takagi function and its properties

By

Jeffrey C. Lagarias*

Abstract

The Takagi function $\tau(x)$ is a continuous non-differentiable function introduced by Teiji Takagi in 1903. It has appeared in a surprising number of different mathematical contexts, including mathematical analysis, probability theory and number theory. This paper surveys properties of this function.

§ 1. Introduction

The Takagi function $\tau(x)$ was introduced by T.Takagi [81] in 1903 as an example of an everywhere non-differentiable function on $[0,1]$. It can be defined on the unit interval $x \in [0,1]$ by

$$\tau(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} \langle\langle 2^n x \rangle\rangle \tag{1.1}$$

where $\langle\langle x \rangle\rangle$ is the distance from x to the nearest integer. Takagi defined it using binary expansions, and showed that his definition was consistent for numbers having two binary expansions (dyadic rationals). The function is pictured in Figure 1.

An immediate generalization of the Takagi function is to set, for integer $r \geq 2$,

$$\tau_r(x) := \sum_{n=0}^{\infty} \frac{1}{r^n} \langle\langle r^n x \rangle\rangle. \tag{1.2}$$

In 1930 van der Waerden [85] studied the function τ_{10} and proved its non-differentiability. In 1933 Hildebrandt [40] simplified his construction to rediscover the Takagi function. Another rediscovery of the Takagi function was made by of de Rham [73] in 1957.

Received August 31, 2011. Revised September 13, 2011. Revised October 24, 2011. Accepted November 18, 2011.
2000 Mathematics Subject Classification(s):
The work of the author was partially supported by NSF Grants DMS-0801029 and DMS-1101373.
*Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043, USA.
e-mail: `lagarias@umich.edu`

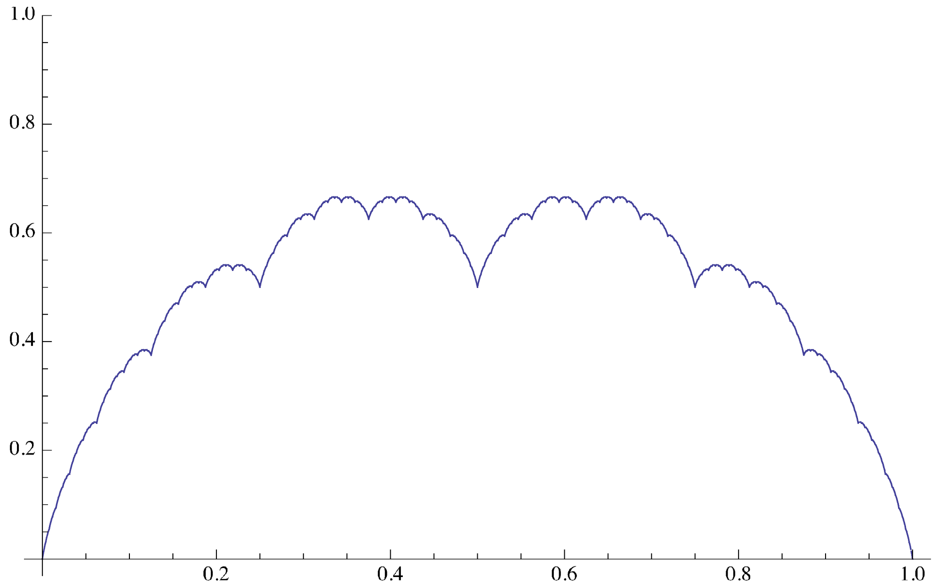

Figure 1. Graph of the Takagi function $\tau(x)$.

The Takagi function has appeared in a number of different mathematical contexts, including analysis, probability theory and number theory. It is a prescient example of a self-similar construction. An important feature of this function is that it satisfies many self-similar functional equations, some of which are related to the dilation equations that appear in wavelet theory. It serves as a kind of exactly solvable "toy model" representing a solution to a discrete version of the Laplacian operator. It serves as an interesting test case for determining various measures of irregularity of behavior of a function. It also turns out to be related to a number of interesting singular functions.

This object of this paper is to survey properties of Takagi function across all these fields. In Section 2 we present some history of work on the Takagi function. In Sections 3 and 4 we review basic analytic properties of the Takagi function. In Section 5 we discuss its connection to dynamical systems. In Section 6 we treat its Fourier transform, and in Section 7 we describe its relation with Bernoulli convolutions in probability theory. Section 8 summarizes analytic results quantifying the local oscillatory behavior of the Takagi function at different scales, which imply its non-differentiabilty. Sections 9 and 10 present results related to number theory, connecting it with binary digit sums and with the Riemann hypothesis, respectively. In Section 11 we describe properties of the graph of the Takagi function. In Sections 12 - 14 we present results concerning the level sets of the Takagi function. These include results recently obtained jointly with Z. Maddock ([56], [57]). Many of the results in these sections are based on properties specific to the Takagi function. They exhibit its value as a "toy model", where exact calculations are possible.

In this paper most results are given without proof. However we have included proofs of several results not conveniently available, and some of these may not have been noted before, e.g. Theorem 6.2 and Theorem 11.4. We also draw the reader's attention to another recent survey of results on the Takagi function given by Allaart and Kawamura [6], which is somewhat complementary to this one.

§ 2. History

The Takagi function was introduced during a period when the general structure of non-differentiable functions was being actively explored. This started from Weierstrass's discovery of an everywhere non-differentiable function, which he lectured on as early as 1861, but which was first published (and attributed to Weierstrass) by Du Bois-Reymond [15] in 1875. Weierstrass's example has no finite or infinite derivative at any point. The example of Takagi is much simpler, and has no point with a finite derivative, but it does have some points with a well-defined infinite derivative, see Theorem 8.6 below. Pinkus [70] gives additional history on this problem.

In 1918 Knopp [50] studied many variants of non-differentiable functions and reviewed earlier work, including that of Faber [28], [29] and Hardy [36], among others. He considered functions of the general form

$$F(x) := \sum_{n=0}^{\infty} a^n \, \phi(b^n x), \tag{2.1}$$

where $\phi(x)$ is a given periodic continuous function of period one, for real numbers $0 < a < 1$ and $b > 1$. This general form includes the Takagi function as well as functions in the Weierstrass non-differentiable function family

$$W_{a,b}(x) := \sum_{n=0}^{\infty} a^n \cos(\pi b^n x).$$

Weierstrass showed this function has no finite or infinite derivative when $0 < a < 1$ and b is an odd integer with $ab > 1 + \frac{3\pi}{2}$. In 1916 Hardy [36, Theorem 1.31] proved that the Weierstrass function has no finite derivative for real a, b satisfying $0 < a < 1$ and $ab \geq 1$. For the special case where $\phi(x) = \langle\langle x \rangle\rangle$, relevant to the Takagi function, Knopp proved ([50, p. 18]) the non-differentiability for real $0 < a < 1$ and b being an even integer with $ab > 4$. The Takagi function has $ab = 1$, so is not covered by Knopp's result; however a later result of Behrend [13, Theorem III] in 1948, applies to $\phi(x) = \langle\langle x \rangle\rangle$ and establishes no finite derivative for integer b and $0 < a < 1$, having $ab \geq 1$, with some specific exceptions.

In the 1930's there were significant developments in probability theory, including its formalization in terms of measure theory. The expression of Lebesgue measure $[0, 1]$ in terms of the induced measure on the binary coefficients, reveals that these measures are given by independent coin flips (Bernoulli trials). This measure density can be expressed as an infinite product (Bernoulli convolution), see Kac [45] for a nice treatment. In 1934, Lominicki and Ulam [59] studied related measures where biased coin flips are allowed. These measures are generally singular with respect to Lebesgue measure. In 1984 Hata and Yamaguti [38] noted a relation of the Takagi function to this family of measures, stated below in Theorem 7.2.

There has been much further work studying properties of Bernoulli convolutions, as well as more general infinite convolutions, and their associated measures. Additional motivation comes from work of Jessen and Wintner [44] concerning the Riemann zeta function, which is described at length in 1938 lecture notes of Wintner [86]. Peres, Schlag and Solomyak [69] give a recent progress report on Bernoulli convolutions, and Hilberdink [39] surveys connections of Bernoulli convolutions with analytic number theory.

In the 1950's Georges de Rham ([71], [72], [73]) considered self-similar constructions of geometric objects, again constructing a function equivalent to the Takagi function. In this context Kahane [46] noted an important property of the Takagi function. Similar constructions appear in the theory of splines, of functions iteratively constructed using control points.

The Takagi function appeared in number theory in connection with the summatory functions of various arithmetic functions associated to binary digits. The analysis of such sums began with Mirsky [66] in 1949, but the connection with the Takagi function was first observed by Trollope [83] in 1968. It was further explained in a very influential paper of Delange [26] in 1975. A reformulation of the theory in terms of Mellin transforms was given by Flajolet et al [31]. These results are discussed in Section 8.

The Takagi function can also be viewed in terms of a dynamical system, involving iterations of the tent map. This viewpoint was taken in 1984 by Hata and Yamaguti [38]. Here the Takagi function can be seen as a kind of fractal. For further information on the fractal interpretation see Yamaguti, Hata and Kigami [90].

In the 1990's the construction of compactly-supported wavelets led to the study of *dilation equations*, which are functional equations which linearly relate functions at two (or more) different scales, see Daubechies [22]. Basic results on the solution of such equations appear in Daubechies and the author [23], [24]. These functions can be described in terms of infinite products, which are generalizations of Bernoulli convolutions (see [25]). The Takagi function appears in this general context because it satisfies a non-homogenous dilation equation, driven by an auxiliary function, which is stated in Theorem 4.1. Related connections with de Rham's functions were observed by Berg and Krüppel [12] in 2000 and by Krüppel [55] in 2009.

The Takagi function has appeared in additional contexts. In 1995 Frankl, Matsumoto, Rusza and Tokushige [32] gave a combinatorial application. For a family $\mathcal{F}$ of k-element sets of the N-element set $[N] := \{1, 2, ..., N\}$ and for any $\ell < k$ the *shadow* $\Delta_\ell(\mathcal{F})$ of $\mathcal{F}$ on ℓ-element sets is the set of all ℓ-element sets that are contained in some set in $\mathcal{F}$. The Kruskal-Katona theorem asserts that the minimal size of ℓ-shadows of size m families $\mathcal{F}$ of k-element sets is attained by choosing the sets of $\mathcal{F}$ as the first m elements in the k-element subsets of $[N]$ ordered in the co-lexicographic order. The *Kruskal-Katona function* gives this number, namely

$$K_l^k(m) := \min\{\#\Delta_\ell(\mathcal{F}) : \mathcal{F} \text{ consists of k-element sets}, \quad |\mathcal{F}| = m\}.$$

This number is independent of the value of N, requiring only that N be sufficiently large that such families exist, i.e. that $\binom{N}{k} \geq m$. The *shadow function* $S_k(x)$ is a normalized version of the Kruskal-Katona function, taking $\ell = k - 1$, which is given by

$$S_k(x) := \frac{k}{\binom{2k-1}{k}} K_{k-1}^k(\lfloor \binom{2k-1}{k} x \rfloor), \quad 0 \leq x \leq 1.$$

Theorem 4 of [32] states that as $k \to \infty$ the shadow functions $S_k(x)$ uniformly converge to the Takagi function $\tau(x)$. The Takagi function also appears in models of diffusion-reaction processes ([35]) and in basins of attraction of dynamical systems ([87]).

§ 3. Basic Properties: Binary Expansions

Takagi's definition of his function was in terms of binary expansions, which we write

$$x = \sum_{j=1}^{\infty} \frac{b_j}{2^j} = 0.b_1 b_2 b_3 \cdots, \qquad \text{each } b_j \in \{0, 1\}. \tag{3.1}$$

The binary expansion of x is unique except for dyadic rationals $x = \frac{k}{2^n}$, which have two possible expansions. For $0 \leq x \leq 1$ the distance to the nearest integer function $\langle\langle x \rangle\rangle$ is

$$\langle\langle x \rangle\rangle := \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2}, \text{ i.e. } b_1 = 0 \\ 1 - x & \text{if } \frac{1}{2} \leq x \leq 1, \text{ i.e. } b_1 = 1. \end{cases} \tag{3.2}$$

For $n \geq 0$, we have

$$\langle\langle 2^n x \rangle\rangle = \begin{cases} 0.b_{n+1} b_{n+2} b_{n+3} ... & \text{if } b_{n+1} = 0 \\ \\ 0.\bar{b}_{n+1} \bar{b}_{n+2} \bar{b}_{n+3} ... & \text{if } b_{n+1} = 1, \end{cases} \tag{3.3}$$

where we use the bar-notation

$$\bar{b} = 1 - b, \quad \text{for} \quad b = 0 \text{ or } 1, \tag{3.4}$$

to mean complementing a bit.

Definition 3.1. *Let $x \in [0,1]$ have binary expansion $x = \sum_{j=1}^{\infty} \frac{b_j}{2^j} = 0.b_1b_2b_3...$, with each $b_j \in \{0,1\}$. For each $j \geq 1$ we define the following integer-valued functions.*

(1) The digit sum function *$N_j^1(x)$ is*

$$N_j^1(x) := b_1 + b_2 + \cdots + b_j. \tag{3.5}$$

We also let $N_j^0(x) = j - N_j^1(x)$ count the number of 0's in the first j binary digits of x.

(2) The deficient digit function *$D_j(x)$ is given by*

$$D_j(x) := N_j^0(x) - N_j^1(x) = j - 2N_j^1(x) = j - 2(b_1 + b_2 + \cdots + b_j). \tag{3.6}$$

Here we use the convention that x denotes a binary expansion; dyadic rationals have two different binary expansions, and all functions $N_j^0(x)$, $N_j^1(x)$, $D_j(x)$ depend on which binary expansion is used. (The name "deficient digit function" reflects the fact that $D_j(x)$ counts the excess of binary digits $b_k = 0$ over those with $b_k = 1$ in the first j digits, i.e. it is positive if there are more 0's than 1's.)

Takagi's original characterization of his function, which he used to establish non-differentiability, is as follows.

Theorem 3.2. (Takagi 1903) *For $x = 0.b_1b_2b_3...$ the Takagi function is given by*

$$\tau(x) = \sum_{m=1}^{\infty} \frac{\ell_m}{2^m}, \tag{3.7}$$

in which $0 \leq \ell_m = \ell_m(x) \leq m-1$ is the integer

$$\ell_m(x) = \#\{i : \ 1 \leq i < m, \quad b_i \neq b_m\}. \tag{3.8}$$

In terms of the digit sum function $N_m^1(x) = b_1 + b_2 + ... + b_m$,

$$\ell_{m+1}(x) = \begin{cases} N_m^1(x) & \text{if} \, b_{m+1} = 0, \\ \\ m - N_m^1(x) & \text{if} \, b_{m+1} = 1. \end{cases} \tag{3.9}$$

Dyadic rationals $x = \frac{k}{2^m}$ have two binary expansions, and in consequence the formulas above give two expansions for $\tau(x)$. Theorem 3.2 asserts that these expansions give the same value; one may verify that $\tau(x)$ itself will then be another dyadic rational, with the same or smaller denominator. See [56, Lemma 2.1] for a proof of this result.

We deduce some basic properties of the Takagi function from its definition.

Theorem 3.3. *(1) The Takagi function $\tau(x)$ maps rational numbers x to rational numbers $\tau(x)$.*

(2) The values of the Takagi function satisfy $0 \le \tau(x) \le \frac{2}{3}$. The minimal value $y = 0$ is attained only at $x = 0, 1$. The maximal value $y = \frac{2}{3}$ is also attained at some rational x, in particular $\tau(\frac{1}{3}) = \frac{2}{3}$.

Proof. (1) This follows from (1.1) because for rational x the sequence $\langle\langle 2^n x \rangle\rangle$ takes rational values with bounded denominators, and becomes eventually periodic. Summing geometric series then gives the rationality.

(2) The lower bound case is clear, and by inspection is attained for $x = 0, 1$ only. The upper bound is proved by checking that $\langle\langle x \rangle\rangle + \frac{1}{2}\langle\langle 2x \rangle\rangle \le \frac{1}{2}$ holds for all $x \in [0, 1]$, and using this on successive pairs of terms in (1.1) to get $\tau(x) \le \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = \frac{2}{3}$. One checks from (1.1) that $\tau(\frac{1}{3}) = \frac{2}{3}$. $\square$

Remark. The set of values x having $\tau(x) = \frac{2}{3}$ is quite large; see Theorem 12.2. Concerning the converse direction to Theorem 3.3 (1): It is not known which rationals y with $0 \le y \le \frac{2}{3}$ have the property that there is some rational x such that $\tau(x) = y$.

The Takagi function $\tau(x)$ can be constructed as a limit of piecewise linear approximations. The *partial Takagi function* of level n is given by:

$$\tau_n(x) := \sum_{j=0}^{n-1} \frac{\langle\langle 2^j x \rangle\rangle}{2^j}, \tag{3.10}$$

See Figure 2 for $\tau_2(x), \tau_3(x)$ and $\tau_4(x)$.

Theorem 3.4. *The piecewise linear function $\tau_n(x) = \sum_{j=0}^{n-1} \frac{\langle\langle 2^j x \rangle\rangle}{2^j}$ is linear on each dyadic interval $[\frac{k}{2^n}, \frac{k+1}{2^n}]$.*

(1) On each such interval $\tau_n(x)$ has integer slope between $-n$ and n given by the deficient digit function

$$D_n(x) = N_n^0(x) - N_n^1(x) = n - 2(b_1 + b_2 + \cdots + b_n),$$

Here $x = 0.b_1 b_2 b_3 ...$ may be any interior point on the dyadic interval, and can also be an endpoint provided the dyadic expansion ending in 0's is taken at the left endpoint $\frac{k}{2^n}$ and that ending in $1'$s is taken at the right endpoint $\frac{k+1}{2^n}$.

(2) The functions $\tau_n(x)$ approximate the Takagi function monotonically from below

$$\tau_1(x) \le \tau_2(x) \le \tau_3(x) \le ... \tag{3.11}$$

The values $\{\tau_n(x) : n \ge 1\}$ converge uniformly to $\tau(x)$, with

$$|\tau_n(x) - \tau(x)| \le \frac{2}{3} \cdot \frac{1}{2^n}. \tag{3.12}$$

(3) For a dyadic rational $x = \frac{k}{2^n}$, perfect approximation occurs at the n-th step, and

$$\tau(x) = \tau_m(x), \quad \text{for all } \; m \geq n. \tag{3.13}$$

Proof. All statements follow easily from the observation that each function $f_n(x) := \frac{\langle\langle 2^n x \rangle\rangle}{2^n}$ is a piecewise linear sawtooth function, linear on dyadic intervals $[\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}]$, with slope having value $+1$ if the binary expansion of x has $b_{n+1} = 0$ and slope having value -1 if $b_{n+1} = 1$. The inequality in (3.12) also uses the fact that $\max_{x \in [0,1]} \tau(x) = \frac{2}{3}$. $\square$

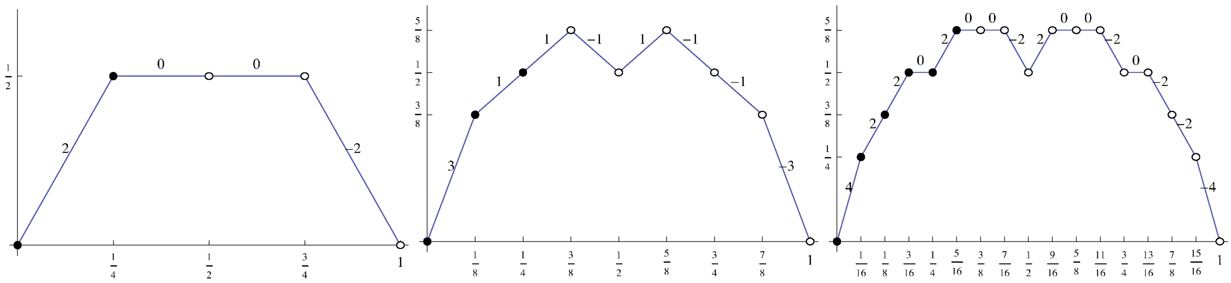

Figure 2. Approximants to Takagi function: (left to right) $\tau_2(x), \tau_3(x), \tau_4(x)$. Slopes of linear segments are labelled on graphs. A vertex (x, y) is marked by a solid point if and only if $x \in \Omega^L$, as defined in (13.1).

The Takagi function itself can be directly expressed in terms of the deficient digit function. The relation (3.6) compared with the definition (3.9) of $\ell_m(x)$ yields

$$\ell_{m+1}(x) = \frac{m}{2} - \frac{1}{2}(-1)^{b_{m+1}} D_m(x).$$

Substituting this in Takagi's formula (3.7) and simplifying (noting that $\ell_1(x) = D_0(x) = 0$) yields the formula

$$\tau(x) = \frac{1}{2} - \frac{1}{4}\left(\sum_{m=1}^{\infty} (-1)^{b_{m+1}} \frac{D_m(x)}{2^m} \right). \tag{3.14}$$

§ 4. Basic Properties: Functional Equations and Self-Affine Rescalings

We next recall two basic functional equations that the Takagi function satisfies. These have been repeatedly found, see Kairies, Darslow and Frank [48] and Kairies [47].

Theorem 4.1. *(1) The Takagi function satisfies two functional equations, each valid for*
$0 \le x \le 1$, *the reflection equation*

$$\tau(x) = \tau(1-x), \tag{4.1}$$

and the dyadic self-similarity equation

$$\tau(\frac{x}{2}) = \frac{1}{2}x + \frac{1}{2}\tau(x). \tag{4.2}$$

(2) The Takagi function on $[0,1]$ *is the unique continuous function on* $[0,1]$ *that satisfies both these functional equations.*

Proof. (1) Here (4.1) follows directly from (1.1), since $\langle\langle kx \rangle\rangle = \langle\langle k(1-x) \rangle\rangle$ for $k \in \mathbb{Z}$. To obtain (4.2), let $x = 0.b_1b_2b_3...$ and set $y := \frac{x}{2} = 0.0b_1b_2b_3....$ Then $\langle\langle y \rangle\rangle = y$, whence (1.1) gives

$$2\tau(y) = 2\langle\langle y \rangle\rangle + 2\left(\sum_{n=1}^{\infty} \frac{\langle\langle 2^n y \rangle\rangle}{2^n}\right) = x + \sum_{m=0}^{\infty} \frac{\langle\langle 2^m x \rangle\rangle}{2^m} = x + \tau(x).$$

(2) The uniqueness result follows by showing that the functional equations determine the value at all dyadic rationals. Indeed, the dyadic self-similarity equation first gives $\tau(0) = 0$, whence $\tau(1) = 0$ by the reflection equation. Then $\tau(\frac{1}{2}) = \frac{1}{2}$ by the self-similarity equation, and now we iterate to get all other dyadic rationals. Since the dyadic rationals are dense, there is at most one continuous interpolation of the function. The fact that a continuous interpolation exists follows from Theorem 3.4. (This result was noted by Knuth [51, Exercise 82, solution p. 103]. $\square$

The functional equations yield a self-affine property of the Takagi function associated to shifts by dyadic rationals $x = \frac{k}{2^n}$.

Theorem 4.2. *For an arbitrary dyadic rational* $x_0 = \frac{k}{2^n}$ *then for* $x \in [\frac{k}{2^n}, \frac{k+1}{2^n}]$ *given by* $x = x_0 + \frac{y}{2^n}$, *there holds*

$$\tau(x_0 + \frac{y}{2^n}) = \tau(x_0) + \frac{1}{2^n}(\tau(y) + D_n(x_0)\, y), \quad 0 \le y \le 1, \tag{4.3}$$

That is, the graph of $\tau(x)$ *on* $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ *is a miniature version of the tilted Takagi function* $\tau(x) + D_n(x_0)x$, *shrunk by a factor* $\frac{1}{2^n}$ *and vertically shifted by* $\tau(x_0)$.

Proof. By Theorem 3.4(1), we have $\tau_n(x_0 + \frac{y}{2^n}) = \tau_n(x_0) + D_n(x_0) \cdot \frac{y}{2^n}$. Therefore, by (1.1) it follows that

$$\begin{aligned}\tau(x) &= \tau_n(x) + \sum_{j=n}^{\infty} \frac{\langle\langle 2^j x\rangle\rangle}{2^k} \\ &= \tau_n(x_0) + D_n(x_0) \cdot \frac{y}{2^n} + \sum_{j=n}^{\infty} \frac{\langle\langle 2^j(\frac{y}{2^n})\rangle\rangle}{2^j} \\ &= \tau(x_0) + \frac{1}{2^n}(\tau(y) + D_n(x_0)y). \quad \square\end{aligned}$$

Theorem 4.2 simplifies in the special case of $x_0 = \frac{k}{2^n}$ having $D_n(x_0) = 0$, which we call a *balanced dyadic rational*; such dyadic rationals can only occur when $n = 2m$ is even. The formula (4.3) then becomes

$$\tau(x_0 + \frac{y}{2^n}) = \tau(x_0) + \frac{\tau(y)}{2^n}, \quad 0 \le y \le 1, \tag{4.4}$$

which a shrinking of the Takagi function together with a vertical shift. Balanced dyadic rationals play a special role in the analysis of the Takagi function.

As a final topic in this section, a convex function is characterized as one that satisfies the condition

$$f(\frac{x+y}{2}) \le \frac{f(x) + f(y)}{2}.$$

The Takagi function is certainly very far from being a convex function. However one can establish the following approximate mid-convexity property of the Takagi function, due to Boros [16].

Theorem 4.3. (Boros 2008) *The Takagi function is* $(0, \frac{1}{2})$*-midconvex. That is, it satisfies the bound*

$$\tau(\frac{x+y}{2}) \le \frac{\tau(x) + \tau(y)}{2} + \frac{1}{2}|x - y|.$$

This result establishes the sharpness of general bounds for approximately midconvex functions established by Házy and Páles [37].

§ 5. The Takagi function and Iteration of the Tent Map

An alternate interpretation of the Takagi function involves iterations of the *symmetric tent map* $T : [0, 1] \to [0, 1]$, given by

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1. \end{cases} \tag{5.1}$$

This function extends to the whole real line by making it periodic with period 1, and is then given by $T(x) = 2\langle\langle x \rangle\rangle$, and it satisfies the functional equation $T(x) = T(1-x)$. A key property concerns its behavior under iteration $T^{\circ n}(x) := T(T^{\circ(n-1)}(x))$ (n-fold composition of functions is here denoted $\circ\, n$).

it satisfies the functional equation under composition

$$T^{\circ n}(x) = T(2^n x). \tag{5.2}$$

The tent map $T(x)$ under iteration is an extremely special map. It defines under iteration a completely chaotic dynamical system, defined on the unit interval, whose symbolic dynamics is the full shift on two letters. This map has Lebesgue measure on $[0,1]$ as an invariant measure, and this measure is the maximal entropy measure among all invariant Borel measures for $T(x)$.

As an immediate consequence of (5.2) we have the following formula for $\tau(x)$, noted by Hata and Yamaguti [38] in 1984.

Theorem 5.1. (Hata and Yamaguti 1984) *The Takagi function is given by*

$$\tau(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} T^{\circ n}(x), \tag{5.3}$$

where $T^{\circ n}(x)$ denotes the n-th iterate of the tent map $T(x)$.

This result differs conceptually from the original definition (1.1) which produces powers of 2 by a rescaling the variable in a fixed map, in that in (5.3) the powers of 2 are produced by iteration of a map.

Hata and Yamaguti [38] defined generalizations of the Takagi function based on iteration of maps. They defined the *Takagi class E_T* to consist of those functions given by

$$f(x) = \sum_{n=1}^{\infty} a_n T^{\circ n}(x).$$

where $\sum_{n=1}^{\infty} |a_n| < \infty$ viewed as members of the Banach space of continuous functions $C^0([0,1])$, under the sup norm. This class contains some piecewise smooth functions, for example

$$\sum_{n=1}^{\infty} \frac{1}{4^n} T^{\circ n}(x) = x(1-x).$$

Hata and Yamaguti showed that continuous functions in E_T can be characterized in terms of their Faber-Schauder expansions in $C^0([0,1])$, and that the Takagi class E_T is a closed subspace of $C^0([0,1])$. The standard Faber-Schauder functions, defined by

Faber [30] in 1910 and generalized by Schauder [75, pp. 48-49] in 1927, are $1, x$ and $\{S_{i,2^j}(x) : j \ge 0, 0 \le i \le 2^j - 1\}$ given by dyadically shrunken and shifted tent maps

$$S_{i,2^j}(x) := 2^j\{|x - \frac{i}{2^j}| + |x - \frac{i+1}{2^j}| - |2x - \frac{2i+1}{2^{j+1}}|\},$$

These functions form a Schauder basis of the Banach space $C^0([0,1])$, in the sup norm, taking them in the order $1, x$, followed by the other functions in the order: $S_{i,2^j}$ precedes $S_{i',2^{j'}}$ if $j < j'$ or $j = j'$ and $i < i'$. (For a discussion of Schauder bases see Megginson [62, Sec. 4.1].) A function $f(x)$ in $C^0([0,1])$ has Faber-Schauder expansion

$$f(x) = a_0 + a_1 x + \sum_{j=0}^{\infty} \sum_{i=0}^{2^j-1} a_{i,j} S_{i,2^j}(x), \tag{5.4}$$

with coefficients $a_0 = f(0),\ a(1) = f(1) - f(0)$, and

$$a_{i,j} := f(\frac{2i+1}{2^{j+1}}) - \frac{1}{2}\Big(f(\frac{i}{2^j}) + f(\frac{i+1}{2^j})\Big).$$

Functions in the Takagi class satisfy the restriction that $a_0 = a_1 = 0$ and their Faber-Schauder coefficients $\{a_{i,j}\}$ depend only on the level j.

Hata and Yamaguti [38, Theorem 3.3] proved the following converse result.

Theorem 5.2. (Hata and Yamaguti 1984) *A function $f(x) \in C^0([0,1])$ belongs to the Takagi class E_T if and only if its Faber-Schauder expansion $f(x) = a_0 + a_1 x + \sum_{j=0}^{\infty} \sum_{i=0}^{2^j-1} a_{i,j} S_{i,2^j}(x)$ satisfies*

1. *The coefficients $a_0 = a_1 = 0$ and*
$$a_{0,j} = a_{i,j} \quad \text{for all} \quad j \ge 0,\ 0 \le i \le 2^{j-1}.$$
2. *If we set $c_j = a_{0,j}$, then*
$$\sum_{j=0}^{\infty} |c_j| < \infty.$$

Note that if the conditions 1, 2 above hold, then $f(x)$ is given by the expansion

$$f(x) = \sum_{j=0}^{\infty} c_j\, T^{\circ(j+1)}(x).$$

Hata and Yamaguti also viewed functions in the Takagi class as satisfying a difference analogue of Laplace's equation, using the scaled central second difference operators

$$\Delta_{i,2^j}(f) := f(\frac{i}{2^j}) + f(\frac{i+1}{2^j}) - 2f(\frac{2i+1}{2^{j+1}}),$$

for $0 \le i \le 2^j - 1$, $j \ge 0$, along with Dirichlet boundary conditions. (The Faber-Schauder coefficients $a_{i,j} = -\frac{1}{2}\Delta_{i,2^j}(f)$.) They obtained the following existence and uniqueness result ([38, Theorem 4.1]).

Theorem 5.3. (Hata and Yamaguti 1984) *Given data* $\{c_j : j \geq 0\}$, *the infinite system of linear equations defined for*

$$\Delta_{i,2^j}(f) = c_j, \qquad j \geq 0,\ 0 \leq i \leq 2^j - 1,$$

has a continuous solution $f(x) \in C^0([0,1])$ *satisfying Dirichlet boundary conditions* $f(0) = f(1) = 0$ *if and only if*

$$\sum_{j=0}^{\infty} |c_j| < \infty.$$

In this case $f(x) \in E_T$, *with*

$$f(x) = -\frac{1}{2}\sum_{j=0}^{\infty} c_j\, T^{\circ(j+1)}(x).$$

There are interesting functions obtainable from the Takagi function by monotone changes of variable. The tent map $T(x)$ is real-analytically conjugate on the interval $[0,1]$ to a particular *logistic map*

$$F(y) := 4y(1-y), \qquad 0 \leq y \leq 1. \tag{5.5}$$

That is, $F(y) = \varphi^{-1} \circ T \circ \varphi(y)$ for a monotone increasing real-analytic function $\varphi(x)$, which is

$$\varphi(y) = \frac{2}{\pi}\arcsin\sqrt{y},$$

with its functional inverse $\varphi^{-1}(x)$ given by $\varphi(x) := \sin^2\left(\frac{\pi x}{2}\right)$. The dynamics under iteration of the logistic map (5.5) has been much studied. It is a post-critically finite quadratic polynomial, and it is affinely conjugate[1] to the monic centered quadratic map $\tilde{F}(z) = z^2 - 2$, which specifies a boundary point of the Mandelbrot set.

Its n-th iterate $F^{\circ n}(y)$ is a polynomial of degree 2^n, which is conjugate to the Chebyshev polynomial $T_{2^n}(x)$. The analytic conjugacy above gives

$$\varphi \circ F^{\circ n}(y) = T^{\circ n} \circ \varphi(y), \qquad \text{for}\ \ n \geq 1.$$

The change of variable $x = \varphi(y)$ applied to a function $f(x) = \sum_{n=1}^{\infty} c_n T^{\circ n}(x)$ in the Takagi class yields the rescaled function

$$g(y) = f(\varphi(y)) = \sum_{n=1}^{\infty} c_n\, \varphi \circ F^{\circ n}(y), \tag{5.6}$$

motivating the further study of maps of this form.

[1]The conjugacy is $y = \psi(z) = -\frac{1}{4}z + \frac{1}{2}$ with inverse $z = \psi^{-1}(y) = -4y + 2$.

More generally, given a dynamical system obtained by iterating a map $\tilde{T} : [0,1] \to [0,1]$ of the interval, and a rescaling function $\psi : [0,1] \to \mathbb{C}$, Hata and Yamaguti define the *generating function*

$$F(t,x) := \sum_{n=0}^{\infty} t^n \, \psi \circ \tilde{T}^{\circ n}(x).$$

Here these functions, where t may vary, encode various statistical information about the discrete dynamical system $\tilde{T}$. For more information on this viewpoint, see Yamaguti, Hata and Kigami [90, Chap. 3].

§ 6. Fourier Series of the Takagi Function

The Takagi function defined by (1.1) extends to a continuous periodic function on the real line with period 1, which comes with a Fourier series expansion. The functional equation $\tau(x) = \tau(1-x)$ for $0 \le x \le 1$ implies that the extended function also satisfies

$$\tau(x) = \tau(-x), \tag{6.1}$$

so that it is an even function. As a consequence, its Fourier series only involves $\cos(2\pi n x)$ terms. It is easily computable from the known Fourier series of the symmetric tent map, as follows.

Theorem 6.1. *The Fourier series of the Takagi function is given by*

$$\tau(x) := \sum_{n \in \mathbb{Z}} c_n \, e^{2\pi i n x}, \tag{6.2}$$

in which

$$c_0 = \int_0^1 \tau(x) dx = \frac{1}{2} \tag{6.3}$$

and for $n > 0$ there holds

$$c_n = c_{-n} = -\frac{1}{2^m (2k+1)^2 \pi^2}, \quad \textit{where} \quad n = 2^m(2k+1). \tag{6.4}$$

Proof. The even function $\psi(x) = \langle\langle x \rangle\rangle$ has Fourier series with real coefficients $b_n = b_{-n}$, given as

$$\psi(x) = \sum_{n=-\infty}^{\infty} b_n e^{2\pi i n x},$$

with $b_{-n} = \int_{-\frac{1}{2}}^{\frac{1}{2}} |x| e^{2\pi i n x} dx$. Clearly $b_0 = \frac{1}{4}$ and, integrating by parts,

$$\begin{aligned}\int_0^{\frac{1}{2}} x\, e^{2\pi i n x} dx &= [\frac{x}{2\pi i n} e^{2\pi i n x}] \,|_{x=0}^{x=\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{1}{2\pi i n} e^{2\pi i n x} dx \\ &= -\frac{1}{4\pi i n} - [\frac{1}{(2\pi i n)^2} e^{2\pi i n x}] \,|_{x=0}^{x=\frac{1}{2}} \\ &= \begin{cases} -\frac{1}{4\pi i n} - \frac{1}{2\pi^2 n^2}, & n \quad \text{odd}, \\ -\frac{1}{4\pi i n}, & n \ \ \text{even}. \end{cases}\end{aligned}$$

A similar calculation on the interval $[-\frac{1}{2}, 0]$ gives the complex conjugate value, whence

$$b_n = b_{-n} = \begin{cases} -\frac{1}{\pi^2 n^2} & \text{if } n \ \text{ is odd}, \\ \\ 0 & \text{if } n \neq 0 \text{ is even}. \end{cases}$$

Now $\tau(x) = \sum_{m=0}^{\infty} \frac{1}{2^m} \langle\langle 2^m x \rangle\rangle$, and the Fourier coefficients of $\psi(2^m x)$ are $b_n^m := b_{n/2^m}$ if $2^m | n$ and 0 otherwise. By uniform convergence of the sum we obtain $c_0 = \sum_{n=0}^{\infty} \frac{1}{2^n} b_0 = \frac{1}{2}$, and, for $n = \pm 2^m(2k+1)$, we obtain

$$c_n = \sum_{j=0}^{\infty} \frac{1}{2^j} b_n^j = \sum_{j=0}^{m} \frac{1}{2^j} b_{2^{m-j}(2k+1)} = -\frac{1}{2^m (2k+1)^2 \pi^2},$$

which is the result. $\square$

Note that the decay of the Fourier coefficients as $n \to \infty$ has $\limsup_{n\to\infty} n^2 |c_n| > 0$. This fact directly implies that $\tau(x)$ cannot be a C^2-function. However much more about its oscillatory behavior, including its non-differentiability, can be proved by other methods.

As a direct consequence of this result, the Takagi function is obtainable as the real part of the boundary value of a holomorphic function on the unit disk.

Theorem 6.2. *Let $\{c_n : n \in \mathbb{Z}\}$ be the Fourier coefficients of the Takagi function, and define the power series*

(6.5)
$$f(z) = \frac{1}{2} c_0 + \sum_{n=1}^{\infty} c_n z^n.$$

This power series converges absolutely in the closed unit disk $\{z : |z| \leq 1\}$ to define a holomorphic function in its interior, and it has the unit circle as a natural boundary to analytic continuation. It defines a continuous function on the boundary of the unit disk, and its values there satisfy

(6.6)
$$f(e^{2\pi i\theta}) = \frac{1}{2} \left(X(\theta) + iY(\theta) \right)$$

in which $X(\theta) = \tau(\theta)$ is the Takagi function.

Remark. The imaginary part of $f(x)$ now defines a new function $Y(\theta)$ which we call the *conjugate Takagi function.* It is a periodic function with period 1 which is also an odd function, so that $Y(\theta) = -Y(-\theta) = -Y(1-\theta)$. Its graph is pictured in Figure 3. (This figure is courtesy of G. Alkauskas.)

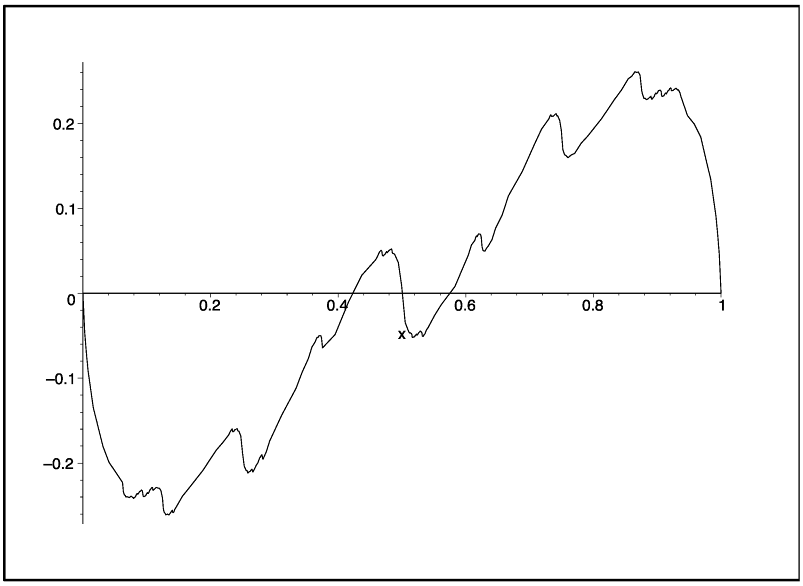

Figure 3. Graph of the conjugate Takagi function $Y(\theta)$, $0 \le \theta \le 1$.

Proof. The Fourier coefficients of the Takagi function satisfy

$$\sum_{n=0}^{\infty} |c_n| = \frac{1}{2} + \sum_{m=0}^{\infty} \frac{1}{2^{m-1}} \left(\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2\pi^2} \right) = \frac{1}{2} + \frac{1}{2} < \infty$$

It follows that the power series for $f(z)$ converges absolutely on the unit circle, so is continuous on the closed unit disk, and holomorphic in its interior. Since the Fourier series for the Takagi function is even, by inspection

$$f(e^{2\pi i\theta}) + f(e^{-2\pi i\theta}) = 2Re(F(e^{2\pi i\theta})) = X(\theta).$$

This justifies (6.6), defining $Y(\theta)$ by

$$f(e^{2\pi i\theta}) - f(e^{-2\pi i\theta}) = 2iIm(F(e^{2\pi i\theta})) = iY(\theta).$$

Since the Takagi function is non-differentiable everywhere, the function $f(z)$ cannot analytically continue across any arc of the unit circle, so that the unit circle is a natural boundary for $f(z)$. $\square$

As mentioned in Section 5, the Takagi function can also be studied using its Faber-Schauder expansion, rather than a Fourier expansion. In 1988 Yamaguti and Kigami [91] defined for $p \geq 1$ the Banach spaces

$$H_p := \overline{\{f(x) = \sum_{i,j} c_{i,j} 2^{-\frac{j}{2}+1} S_{i,2^j}(x)\}},$$

defined using a rescaled Schauder basis and the norm $||f||_p = \left(\sum_{i,j} |c_{i,j}|^p\right)^{1/p}$. They deduced using its Schauder expansion that the Takagi function belongs to the Banach space H_p for all $p > 2$. (It does not belong to H_2, which is a Hilbert space coinciding with a space denoted H^1 in [91].) In 1989 Yamaguti [89] proposed a generalized Schauder basis consisting of polynomials in x, in which to study the Takagi function and other functions.

§ 7. The Takagi Function and Bernoulli Convolutions

The Takagi function also appears in the analysis of non-symmetric Bernoulli convolutions. Let

$$x = \sum_{j=1}^{\infty} \epsilon_j (\frac{1}{2})^j,$$

in which the digits $\epsilon_j \in \{0, 1\}$ are drawn as independent (non-symmetric) Bernoulli random variables taking value 0 with probability α and 1 with probability $1 - \alpha$. Then x is a random real number in $[0, 1]$ with cumulative distribution function

$$L_\alpha(x) = \mu_\alpha([0, x]) = \int_0^x d\mu_\alpha,$$

in which μ_α is a certain Borel measure on $[0, 1]$. These functions were introduced in 1934 by Lomnicki and Ulam [59, pp. 267-269] with this interpretation. The measure μ_α is Lebesgue measure for $\alpha = \frac{1}{2}$, and is a singular measure otherwise.

In 1943 R. Salem [74] gave a geometric construction of monotonic increasing singular functions that includes these functions as a special case.

Theorem 7.1. (Salem 1943) *The function $L_\alpha(x)$ has Fourier series $L_\alpha(x) \sim \sum_{n=1}^{\infty} c_n \cos 2\pi n x$ using the formula (for real t)*

$$\int_0^1 e^{2\pi i t x} dL_\alpha(x) = \prod_{k=1}^{\infty} \left(\alpha + (1-\alpha) e^{\frac{2\pi i t}{2^k}}\right).$$

The Fourier coefficients are given by infinite products

$$c_n = e^{-\pi i n} \prod_{k=1}^{\infty} \left(\cos \frac{\pi n}{2^k} + i(2\alpha - 1) \sin \frac{\pi n}{2^k}\right).$$

An interesting property of the Lomnicki-Ulam function $L_\alpha(x)$ is that it satisfies a *two-scale dilation equation*

$$L_\alpha(x) = \begin{cases} \alpha L_\alpha(2x) & \text{for } 0 \le x \le \frac{1}{2}, \\ (1-\alpha)L_\alpha(2x-1)+\alpha & \text{for } \frac{1}{2} \le x \le 1. \end{cases}$$

In 1956 de Rham ([71] , [72], [73]) studied such functional equations in detail. Dilation equations are relevant to the construction of compactly supported wavelets, and general properties of solutions to such equations were derived in Daubechies and Lagarias [24], [25].

In 1984 Hata and Yamaguti [38, Theorem 4.6] made the following connection of the Lomnicki-Ulam functions $L_\alpha(x)$ to the Takagi function, cf. [38, p. 195].

Theorem 7.2. (Hata and Yamaguti 1984) *For fixed x the function $g_x(\alpha) := L_\alpha(x)$, initially defined for $\alpha \in [0,1]$, extends to an analytic function of α on the lens-shaped region*

$$D = \{\alpha \in \mathbb{C} : \ |\alpha| < 1 \quad and \quad |1-\alpha| < 1\}.$$

The Takagi function appears as the derivative of these functions at the central point $\alpha = \frac{1}{2}$:

$$\frac{d}{d\alpha} L_\alpha(x)|_{\alpha=\frac{1}{2}} = 2\,\tau(x). \tag{7.1}$$

This result permits an interpretation of the Takagi function as a generating function of a chaotic dynamical system (Yamaguti et al. [90, Chapter 3]). This result was generalized further by Sekiguchi and Shiota [76].

§ 8. Oscillatory Behavior of the Takagi Function

The main feature of the Takagi function is that it is non-differentiable everywhere. Takagi [81] showed the function has no two-sided finite derivatives at any point. In 1984 Cater [20] showed that the Takagi function has no one-sided finite derivative at any point. However it does have well-defined (two-sided) improper derivatives equal to $+\infty$ (resp. $-\infty$) at some points, see Theorem 8.6.

The non-differentiability of the Takagi function is bound up with its increasing oscillatory behavior as the scale decreases. Letting $0 < h < 1$ measure a scale size, we have the following elementary estimate bounding the maximal size of oscillations at scale h.

Theorem 8.1. *For $0 \le x \le x+h \le 1$, the Takagi function satisfies*

$$|\tau(x+h) - \tau(x)| \le 2h \log_2 \frac{1}{h}. \tag{8.1}$$

Proof. Suppose $2^{-n} \le h \le 2^{-n+1}$, so that $n \le \log_2 \frac{1}{h}$. Theorem 3.4 gives the estimate that $|\tau(x) - \tau_n(x)| \le \frac{2}{3}\frac{1}{2^n}$, and we know $\tau_n(x)$ has everywhere slope between $-n$ and n. It follows that

$$|\tau(x+h) - \tau(x)| \le \frac{2}{3}\left(\frac{1}{2^n}\right) + nh \le h(n + \frac{2}{3}) \le 2h \log_2 \frac{1}{h}.$$

as required. $\square$.

Theorem 8.1 is sharp to within a multiplicative factor of 2, since for $h = 2^{-n}$,

$$\tau(h) - \tau(0) = \tau(2^{-n}) = \frac{n}{2^n} = h \ \log_2 \frac{1}{h},$$

In fact the multiplicative factor of 2 can be decreased to 1 as the scale decreases, in the following sense (Kôno [52, Theorem 4]).

Theorem 8.2. (Kôno 1987) *Let* $\sigma_u(h) = \log_2 \frac{1}{h}$. *Then there holds*

$$\limsup_{|x-y|\to 0^+} \frac{\tau(x) - \tau(y)}{|x-y|\,\sigma_u(|x-y|)} = 1.$$

and

$$\liminf_{|x-y|\to 0^+} \frac{\tau(x) - \tau(y)}{|x-y|\,\sigma_u(|x-y|)} = -1.$$

The average size of extreme fluctuations for most x is of the smaller order $h\sqrt{\log_2 \frac{1}{h}}\sqrt{2 \log\log\log_2 \frac{1}{h}}$, as given in the following result. (Kôno [52, Theorem 5]).

Theorem 8.3. (Kôno 1987) *Let* $\sigma_l(h) = \sqrt{\log_2 \frac{1}{h}}$. *Then for almost all* $x \in [0,1]$ *there holds*

$$\limsup_{h\to 0^+} \frac{\tau(x+h) - \tau(x)}{h\,\sigma_l(h)\sqrt{2\log\log \sigma_l(h)}} = 1,$$

and

$$\liminf_{h\to 0^+} \frac{\tau(x+h) - \tau(x)}{h\,\sigma_l(h)\sqrt{2\log\log \sigma_l(h)}} = -1.$$

Kôno used expansions of $\tau(x)$ in terms of Rademacher functions to obtain his results.

Finally, if one scales the oscillations by the factor $h\sqrt{\log_2 \frac{1}{h}}$ on the scale size h then one obtains a Gaussian limit distribution of the maximal oscillation sizes at scale h as $h \to 0^+$, in the following sense (Gamkrelidze[34, Theorem 1]).

Theorem 8.4. (Gamkrelidze 1990) *Let* $\sigma_l(h) = \sqrt{\log_2 \frac{1}{h}}$. *Then for each real* y,

$$\lim_{h\to 0^+} Meas\,\{x : \frac{\tau(x+h) - \tau(x)}{h\,\sigma_l(h)} \le y\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}t^2}\,dt. \tag{8.2}$$

Gamkrelidze also observes that Theorem 8.3 can be derived as a consequence of this result. Namely, in this context Theorem 8.3 is interpretable as analogous to the law of the iterated logarithm in probability theory.

The oscillatory behavior of the Takagi function has also been studied in the Hölder sense. We define on $[0,1]$ the class C^0 of continuous functions, the class C^1 of continuously differentiable functions (with one-sided derivatives at the endpoints) and, for $0 < \alpha \leq 1$, the (intermediate) Lipschitz classes

(8.3)

$$Lip^{\alpha} := \{f \in C^0 : \text{ there exists } K > 0 \text{ with } |f(x) - f(y)| < K|x-y|^{\alpha},\ x, y \in [0,1]\}.$$

Theorem 8.5. (de Vito 1985; Brown and Kozlowski 2003) *(1) The Takagi function τ belongs to the function class*

$$\tau \in \bigcap_{0<\alpha<1} Lip^{\alpha}. \tag{8.4}$$

(2) The Takagi function τ does not agree with any function $g \in C^1$ on any set of positive measure. In fact, if M is a subset of $[0,1]$ with positive measure then the set

$$D(\tau, M) := \{\frac{\tau(y) - \tau(x)}{y - x} : \quad x, y \in M \quad \textit{with} \quad x \neq y\} \tag{8.5}$$

is unbounded.

Proof. (1) This was shown by de Vito [84] in 1958. He showed moreover, that for $0 < \alpha < 1$ and $0 \leq x, y \leq 1$, there holds

$$|\tau(x) - \tau(y)| \leq \frac{2^{\alpha-1}}{1 - 2^{\alpha-1}}|x - y|^{\alpha}.$$

Another proof of (8.4) was given by Shidfar and Sabetfakhri [77]. An extension to τ_r for all even $r \geq 2$ in (1.2) follows from results in Shidfar and Sabetfakhri [78].

(2) A proof establishing this and (8.5) was given by Brown and Kozlowski [17] in 2003. □

Concerning the non-differentiability of the Takagi function, Takagi established there is no finite derivative at any point of $[0,1]$. However the Takagi function does have some points where it has a well-defined (two-sided) infinite deriviative. These points were recently classified by Allaart and Kawamura [5], who proved the following result ([5, Corollary 3.9]).

Theorem 8.6. (Allaart and Kawamura 2010) *The set of points where the Takagi function has a well-defined two-sided derivative $\tau'(x) = +\infty$, and the set of points where it has $\tau'(x) = -\infty$ both are dense in $[0,1]$ and have Hausdorff dimension 1.*

There has been much further work studying oscillatory behavior of the Takagi function in various metrics. We mention the work of Buczolich [18] and of Allaart and Kawamura [4], [5]. In addition Allaart [1] has proved analogues of many results above for a wider class of functions in the Takagi class.

§ 9. The Takagi function and Binary Digit Sums

The Takagi function appears in the analysis of binary digit sums. More generally, let $A_r(x)$ denote the sum of the base r digits of all integers below x, so that

$$A_r(x) := \sum_{j=1}^{[x]} S_r(j), \tag{9.1}$$

in which $S_r(j)$ denotes the sum of the digits in the base r expansion of j. In 1949 Mirsky [66] observed that

$$A_r(x) = \frac{1}{2}(r-1)x\log_r x - E_r(x)$$

with remainder term $E_r(x) = O(x)$. In 1968 Trollope [83] observed that, for $r = 2$, this remainder term has an explicit formula given in terms of the Takagi function.

Theorem 9.1. (Trollope 1968) *If the integer n satisfies $2^m \le n < 2^{m+1}$, and if we write $n = 2^m(1+x)$ for a rational number $0 \le x < 1$, with $x = \frac{n}{2^m} - 1$, then*

$$E_2(n) = 2^{m-1}\left((1+x)\log_2(1+x) - 2x + \tau(x)\right).$$

This was generalized in a very influential paper of Delange [26], who observed the following result.

Theorem 9.2. (Delange 1975) *One can write the rescaled error term $E_r(n)$ for base r digit sums at integer values n as*

$$\frac{1}{n}E_r(n) = \frac{1}{2}\log_r n + F_r\left(\log_r n\right),$$

in which $F_r(t)$ is a continuous function which is periodic of period 1. The Fourier series of the function $F_r(t)$ is given by

$$F_r(t) = \sum_{k\in\mathbb{Z}} c_k(r)e^{2\pi ikt},$$

The Fourier coefficients $c_k(r)$ involve the values of the Riemann zeta function $\zeta(s)$ at the points $\frac{2\pi ki}{\log r}$ on the imaginary axis.

In view of Trollope's result, in the case $r = 2$ the Takagi function $\tau(x)$ appears in the function $F_2(t)$, namely this function is

$$F_2(t) = \frac{1}{2}(1 + \lfloor t \rfloor - t) + 2^{1+\lfloor t \rfloor - t}\tau(2^{t-\lfloor t \rfloor -1}). \tag{9.2}$$

Also for $r = 2$, and $k \neq 0$, the Fourier coefficients of $F_2(t)$ are

$$c_k = \frac{\frac{i}{2k\pi}}{1 + \frac{2k\pi i}{\log 2}}\,\zeta(\frac{2k\pi i}{\log 2}),$$

and for $k = 0$,

$$c_0 = \frac{1}{2\log 2}(\log 2\pi - 1) - \frac{3}{4}.$$

Results on k-th powers of binary digit sums were given in 1977 by Stolarsky [80], who also provides a summary of earlier literature. A connection of these sums with a series of more complicated oscillatory functions was obtained in 1986 in Coquet [21]. Further work was done by Okada, Sekiguchi and Shiota [67] in 1995. Recently Krüppel [54] gave another derivation of these results and further extended them.

For a related treatment of similar functions, using Mellin transforms, see Flajolet et al [31]. A general survey of dynamical systems in numeration, including these topics, is given by Barat, Berthé, Liardet and Thuswaldner [11].

§ 10. The Takagi Function and the Riemann Hypothesis

The Riemann hypothesis can be formulated purely in terms of the Takagi function, as was observed by Kanemitsu and Yoshimoto [49, Corollary to Theorem 5]. Their encoding of the Riemann hypothesis concerns the values of the function restricted to the Farey fractions $\mathcal{F}_N$.

The Farey series of level N consists of all reduced rational fractions $0 \le \frac{p}{q} < 1$ having denominator at most N, their number being $|\mathcal{F}_N| = \frac{3}{\pi^2}N^2 + O(N \log N)$. The connection of the approximately uniform spacing of Farey fractions and the Riemann hypothesis starts with Franel's theorem, cf. Franel [33], proved in 1924. (cf. Huxley [41, p. 36]). Since that time many variants of Franel's result have been established. A particular version of Franel's theorem that is relevant here was given by Mikolas [63], [64], [65].

Uniformity of distribution of a set of points (here the Farey fractions) can be measured in terms of the efficiency of numerical integration of functions on the unit interval obtained by sampling their values at these points. This is the framework taken in the result of Kanemitsu and Yoshimoto. The particular interest of their result is that it applies to the numerical integration of continuous functions that are not necessarily differentiable, but which belong to the Lipschitz class $Lip^{1-\epsilon}$ for each $\epsilon > 0$. The role of the Takagi function here is to be a particularly interesting example to which their general theorem applies.

Theorem 10.1. (Kanemitsu and Yoshimoto 2000) *The Riemann hypothesis is equivalent to the statement that for each* $\epsilon > 0$ *there holds, for the Farey sequence* $\mathcal{F}_N$*,*

$$\sum_{\rho \in \mathcal{F}_N} \tau(\rho) - |\mathcal{F}_N| \int_0^1 \tau(x)dx = O\left(N^{\frac{1}{2}+\epsilon}\right) \tag{10.1}$$

as $N \to \infty$.

Proof. This result was announced in Kanemitsu and Yoshimoto [49, Corollary to Theorem 5], and a detailed proof was given in 2006 in Balasubramanian, Kanemitsu and Yoshimoto [10]. According to the Principle stated in [10, p. 4], the exponent $\frac{1}{2} + \epsilon$ in the remainder term in (10.1) depends on the fact that the Takagi function is in the Lipschitz class Lip^{α} for any $\alpha < 1$, see Theorem 12.4 below. (Note that in [10, Sec. 2.2] their Takagi function $T(x)$ is defined by a Fourier series, which when compared with Theorem 6.1 suggests it needs a rescaling plus a constant term added to agree with $\tau(x)$; this does not affect the general argument.) □

An interesting feature of this result is that for each N the left side of (10.1) is a rational number. This follows since the Takagi function takes rational values at rational numbers, and because with our scaling of the Takagi function we have $\int_0^1 \tau(x)dx = \frac{1}{2}$.

§ 11. Graph of the Takagi Function

We next consider properties of the graph of the Takagi function

$$\mathcal{G}(\tau) := \{(x, \tau(x)) : \ 0 \le x \le 1\}.$$

The local extreme points of the graph of the Takagi function were determined by Kahane [46].

Theorem 11.1. (Kahane 1959)

(1) The set of local minima of the Takagi function are exactly the set of all dyadic rational numbers in $[0, 1]$.

(2) The set of local maxima of the Takagi function are exactly those points x *such that the binary expansion of* x *have deficient digit function* $D_{2n}(x) = 0$ *for all sufficiently large* n.

Proof. This is shown in [46, Sec. 1]. Kahane states condition (2) as requiring that the binary expansion of x have $b_1 + b_2 + \cdots + b_{2n} = n$ holding for all sufficiently large n. $\square$

There are uncountably many x that satisfy condition (2) above. It is easy to deduce from this result that the graph of the Takagi function contains a dense set of local minima and local maxima, viewed from either of the abscissa or ordinate directions. Allaaert and Kawamura [4] further study the extreme values of functions related to the Takagi function.

The Hausdorff dimension of the graph of the Takagi function was determined by Mauldin and Williams [61].

Theorem 11.2. (Mauldin and Williams 1986) *The graph $\mathcal{G}(\tau)$ of the Takagi function has Hausdorff dimension* 1 *as a subset of* $\mathbb{R}^2$.

Proof. This result is given as [61, Theorem 7]. $\square$

Mauldin and Williams [61] raised the question of whether the graph of the Takagi function is σ-finite. This was answered in the affirmative by Anderson and Pitt [8].

Theorem 11.3. (Anderson and Pitt 1989) *The graph $\mathcal{G}(\tau)$ of the Takagi function has σ-finite linear measure.*

Proof. This result is given as [8, Thm. 6.4, and Remark p. 588]. $\square$

In the opposite direction, one can show the graph of the Takagi function has infinite length. We have the following result.

Theorem 11.4. *The graph $\mathcal{G}(\tau)$ has infinite length locally. That it, it has infinite length over any nonempty open interval $x_1 < x < x_2$ in* $[0, 1]$.

Proof. This is proved using the piecewise linear approximations $\tau_n(x)$ to the Takagi function. This function has control points at $x = \frac{k}{2^n}$ where it takes values agreeing with $\tau(x)$. It suffices to show that the length of $\tau_n(x)$ becomes unbounded as $n \to \infty$. This shows the whole graph has infinite length. The self-similar functional equation then implies that any little piece of the graph over the interval $x_1 \le x \le x_2$ also has infinite length.

To show the unboundedness of the length of $\tau_n(x)$, as $n \to \infty$, it suffices to show that the average size of the (absolute value) of the slope of $\tau_n(x)$ becomes unbounded. To do this we note that the subdivision from level n to level $n+1$ replaces each slope m with two intervals of slopes $m+1$ and $m-1$. These do not change the average value of the (absolute value) of slope, except when $m = 0$. Such slopes occur only on even levels $2n$ and there are $\binom{2n}{n}$ of them (they are the balanced dyadic rationals at that level). Consequently the average value of the slope at level $2n$ (resp. $2n+1$) obeys the recursion: $a_{2n} = a_{2n-1}$ and $a_{2n+1} = a_{2n} + \frac{1}{2^{2n}}\binom{2n}{n}$. Since

$$\frac{1}{2^{2n}}\binom{2n}{n} \sim \frac{1}{\sqrt{4\pi n}}$$

and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4\pi n}}$ diverges, we have $a_n \to \infty$ as $n \to \infty$, proving the result. $\square$.

In 1997 Tricot [82] defined the notion of *irregularity degree* of a function f to be the Hausdorff dimension of its graph $\mathcal{G}(f)$. He introduces a two-parameter family of norms $\Delta^{\alpha,\beta}$ to measure oscillatory behavior. As an example, he computes these values for the Takagi function and its relatives ([82, Cor. 6.5, Cor. 6.7])

§ 12. Level Sets of the Takagi Function

We next consider properties of level sets of the Takagi function. For $y \in [0, \frac{2}{3}]$ we denote the *level set* at level y by

$$L(y) := \{x : \ \tau(x) = y, \quad 0 \le x \le 1\}.$$

The level sets have a complicated and interesting structure.

Theorem 12.1. *(1) The Takagi function has level sets $L(y)$ that take a finite number of values, resp. a countably infinite set of values, resp. an uncountably infinite set of values. Specificially, the level set $L(0) = \{0, 1\}$ takes two values, the level set $L(\frac{1}{2})$ takes a countably infinite set of values, and the level set $L(\frac{2}{3})$ takes an uncountably infinite set of values.*

(2) The set of levels y such that $L(y)$ is finite, resp. countably infinite, resp uncountably infinite, are each dense in $[0, \frac{2}{3}]$.

Proof. (1) It is clear that $L(0) = \{0, 1\}$ is a finite level set. A finite example given by Knuth [51] is $L(\frac{1}{5}) = \{x, 1-x\}$ with $x = \frac{83581}{87040}$. Knuth [51, pp. 20-21, 32–33] also showed that certain dyadic rationals $x = \frac{m}{2^k}$ have countably infinite level sets; these include $x = \frac{1}{2}$. The author and Maddock [56, Theorem 7.1] also show that $L(\frac{1}{2})$ is countably infinite. For uncountably infinite level sets, Baba [9] showed that $L(\frac{2}{3})$ has positive Hausdorff dimension, which implies it is uncountable.

(2) For uncountably infinite sets, this follows from the self-similarity relation in Theorem 4.2. For countably infinite case it follows from Knuth's results, see also Allaart [3]. For the finite case, it follows from Theorem 12.4 below. $\square$.

We next consider bounds on the Hausdorff dimension of level sets.

Theorem 12.2. (Baba 1985; de Amo et al 2011)
(1) The maximal level set $L(\frac{2}{3})$ has Hausdorff dimension $\frac{1}{2}$. The set of levels having level set of Hausdorff dimension $\frac{1}{2}$ is dense in $[0, \frac{2}{3}]$.

(2) Every level set $L(y)$ of the Takagi function has Hausdorff dimension at most $\frac{1}{2}$.

Proof. (1) In 1984 Baba [9] showed that $L(\frac{2}{3})$ has Hausdorff dimension $\frac{1}{2}$. Using the self-similarity Theorem 4.2 for balanced dyadic rationals, the same holds for $y = \tau(x)$ such that x has a binary expansion whose purely periodic part is $(01)^\infty$, and whose preperiodic part has an equal number of zeros and ones. The set of such x is dense in $[0, 1]$, so their image values y are dense in $[0, \frac{2}{3}]$.

(2) An upper bound on the Hausdorff dimension of 0.699 was established in 2010 by Maddock [60]. Recently de Amo, Bhouri, Díaz Carrillo and Fernández-Sánchez [7] improved on his argument to establish the optimal upper bound of $\frac{1}{2}$ on the Hausdorff dimension. $\square$.

We next consider the nature of "generic level sets", considered in two different senses. The first is, to draw a abscissa value x at random in $[0, 1]$ with respect to Lebesgue measure, and to ask about the nature of the level set $L(\tau(x))$. The second is, to draw an ordinate value y at random in $[0, \frac{2}{3}]$, and ask what is the nature of the level set $L(y)$. Figure 4 illustrates the two senses.

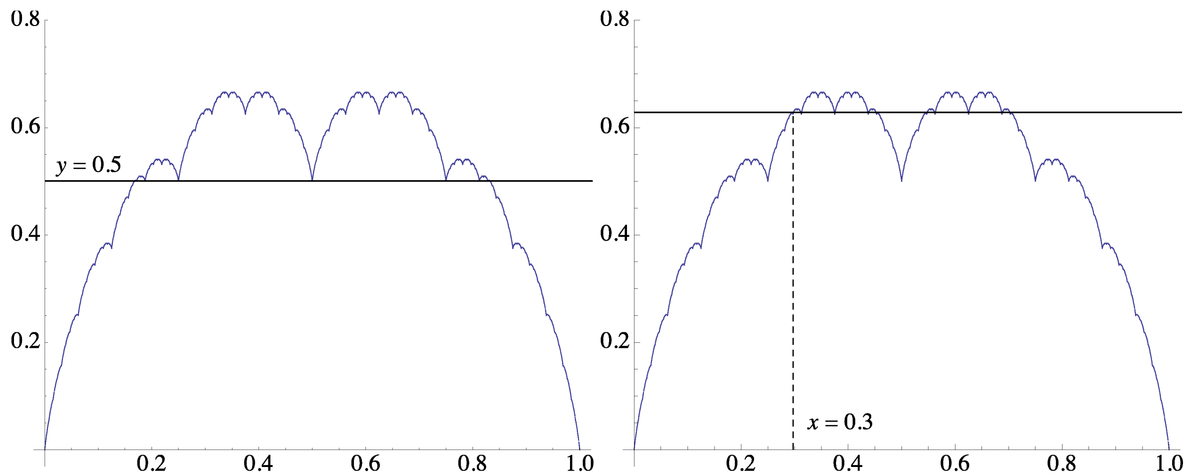

Figure 4. Ordinate level set $L(y)$ at $y = 0.5$ and abscissa level set $L(\tau(x))$ at $x = 0.3$.

For sampling random abscissa values the result is as follows.

Theorem 12.3. *For a full Lebesgue measure set of abscissa points $x \in [0, 1]$ the level set $L(\tau(x))$ contains an uncountable Cantor set.*

Proof. This is a corollary of a result of the author and Maddock [56, Theorem 1.4], The latter result concerns local level sets, discussed below in Section 13. □.

For sampling random ordinate values the corresponding result is quite different.

Theorem 12.4. (Buczolich 2008) *For a full Lebesgue measure set of ordinate points $y \in [0, \frac{2}{3}]$ the level set $L(y)$ is a finite set.*

Proof. This result was proved by Buczolich [19] in 2008. □

A refinement of the ordinates result is as follows.

Theorem 12.5. *Letting $|L(y)|$ denote the number of elements in $L(y)$, if finite, and setting $|L(y)| = 0$ otherwise, one has $\int_0^{\frac{2}{3}} |L(y)|dy = +\infty$. That is, the expected number of elements in $|L(y)|$ for $y \in [0, \frac{2}{3}]$ drawn uniformly, is infinite.*

Proof. This result was obtained in [56, Theorem 6.3]. A simplified proof was given by Allaart [2]. □

The ordinate and abscissa results on the size of level sets are not contradictory; sampling a point x on the abscissa will tend to pick level sets which are "large". In order for these two results to hold it appears necessary that the Takagi function must (in some imprecise sense) have "infinite slope" over part of its domain.

The following result goes some way towards reconciling these two results, by showing that the set of large "ordinate" level sets has full Hausdorff dimension.

Theorem 12.6. *Let Γ_H^{ord} be set of ordinates $y \in [0, \frac{2}{3}]$ such that the Takagi function level set $L(y)$ has positive Hausdorff dimension, i.e.*

$$\Gamma_H^{ord} := \{y : \ \dim_H(L(y)) > 0\}.$$

Then Γ_H^{ord} has full Hausdorff dimension, i.e.

$$\dim_H(\Gamma_H^{ord}) = 1. \tag{12.1}$$

It also has Lebesgue measure zero.

Proof. The Hausdorff dimension 1 property is shown by the author and Maddock [57, Theorem 1.5]). The Lebesgue measure zero property follows from Theorem 12.4(1). □.

There is potentially a multifractal formalism connected to the Hausdoff dimensions of the sets $\Gamma_H^{ord}(\alpha) := \{y : \ \dim_H(L(y)) > \alpha\}$, see [57, Sec. 1.2], and for multifractal formalism see Jaffard [42], [43].

Theorem 12.4 asserts that almost all level sets in the ordinate sense are finite. The finite level sets have an intricate structure, which is analyzed by Allaart [3]. Finite level sets must have even cardinality, and all even values occur.

§ 13. Local Level Sets of the Takagi Function

The notion of *local level set* of the Takagi function was recently introduced by the author and Maddock [56]. These sets are closed subsets of level sets that are directly constructible from the binary expansions of any one of their members.

Local level sets are defined by equivalence relation on elements $x \in [0,1]$ based on properties of their binary expansion: $x = .b_1 b_2 b_3 \cdots$. Recall that in Section 2 we defined the deficient digit function

$$D_j(x) := j - 2(b_1 + b_2 + \cdots + b_j).$$

The quantity $D_j(x)$ counts the excess of binary digits $b_k = 0$ over those with $b_k = 1$ in the first j digits. We then associate to any x the sequence of "breakpoints" j at which tie-values $D_j(x) = 0$ occur, setting

$$Z(x) := \{c_m : \quad D_{c_m}(x) = 0\}.$$

where we define $c_0 = c_0(x) = 0$ and set $c_0(x) < c_1(x) < c_2(x) < \ldots.$ This sequence of tie-values may be finite or infinite, and if it is finite, ending in $c_n(x)$, we make the convention to adjoin a final "breakpoint" $c_{n+1}(x) = +\infty$. We call a "block" a set of digits between two consecutive tie-values,

$$B_k(x) := \{b_j : \ c_k(x) < j \le c_{k+1}(x)\}.$$

Two blocks are called equivalent, written $B_k(x) \sim B_{k'}(x')$, if their endpoints agree ($c_k(x) = c_{k'}(x')$ and $c_{k+1}(x) = c_{k'+1}(x')$) and either $B_k(x) = B_{k'}(x')$ or $B_k(x) = \bar{B}_{k'}(x')$, where the bar operation flips all the digits in the block, i.e.

$$b_j \mapsto b_j^{'} := 1 - b_j, \qquad c_k < j \le c_{k+1}.$$

Finally, we define the equivalence relation on binary expansions $x \sim x'$ to mean that $Z(x) \equiv Z(x')$, and furthermore every block $B_k(x) \sim B_k(x')$ for $k \ge 0$. We define the *local level set* L_x^{loc} associated to x to be the set of equivalent points,

$$L_x^{loc} := \{x' : \quad x' \sim x\}.$$

It is easy to show using Takagi's formula (Theorem 3.2) that the relation $x \sim x'$ implies that $\tau(x) = \tau(x')$ so that x and x' are in the same level set of the Takagi function.

Each local level set L_x^{loc} is a closed set, and is either a finite set if $Z(x)$ is finite, or else is a perfect totally disconnected set (Cantor set) if $Z(x)$ is infinite. The case of dyadic rationals $x = \frac{k}{2^n}$ is exceptional, since they have two binary expansions, and we remove this ambiguity by taking the binary expansion for x that ends in zeros.

The definition implies that each level set $L(y)$ partitions into a disjoint union of local level sets L_x^{loc}. A priori this union may be finite, countable or uncountable. Finite and countable examples are given by the author and Maddock [57]. For the uncountable case, see Theorem 13.5 below. Note that all countably infinite level sets necessarily are a countable union of finite local level sets. The only level sets currently known to be countably infinite are certain dyadic rational levels, including $x = \frac{1}{2}$.

The paper [56] characterizes the size of a "random" local level set sampled by randomly drawing an abscissa value x, as follows; this result immediately implies Theorem 12.1.

Theorem 13.1. *For a full Lebesgue measure set of abscissa points $x \in [0,1]$ the local level set L_x^{loc} is a Cantor set of Hausdorff dimension* 0.

Proof. This is shown in [56, Theorem 1.4]. $\square$.

In order to analyze local level sets in the ordinate space $0 \le y \le \frac{2}{3}$, we label each local level set by its leftmost endpoint. We define the *deficient digit set* Ω^L by the condition

$$\Omega^L := \{x \in [0,1] : \; x = 0.b_1b_2b_3... \text{such that} \;\; D_j(x) \ge 0, \; j = 1,2,3,...\} \tag{13.1}$$

This set turns out to be quite important for understanding the Takagi function.

Theorem 13.2. *(1) The deficient digit set Ω^L is the set of leftmost endpoints of all local level sets.*

(2) The set Ω^L is a closed, perfect set (Cantor set). It has Lebesgue measure 0.

(3) The set Ω^L has Hausdorff dimension 1.

Proof. (1) and (2) are shown in [56, Theorem 4.6].

(3) This was shown in [57, Theorem 6.1]. $\square$

We show that the Takagi function restricted to the set Ω^L is quite nicely behaved, as given in the following result.

Theorem 13.3. *The function $\tau^S(x)$ defined by $\tau^S(x) = \tau(x) + x$ for $x \in \Omega^L$ is a nondecreasing function on Ω^L. Define its extension to all $x \in [0,1]$ by*

$$\tau^S(x) := \sup\{\tau^S(x_1) : x_1 \le x \quad \text{with} \;\; x_1 \in \Omega^L\}.$$

Then the function $\tau^S(x)$ is a monotone singular function. That is, it is a nondecreasing continuous function having $\tau^S(0) = 0, \tau^S(1) = 1$, which has derivative zero at (Lebesgue) almost all points of $[0, 1]$. The closure of the set of points of increase of $\tau^S(x)$ is the deficient digit set Ω^L.

Proof. This is shown in [56, Theorem 1.5]. $\square$.

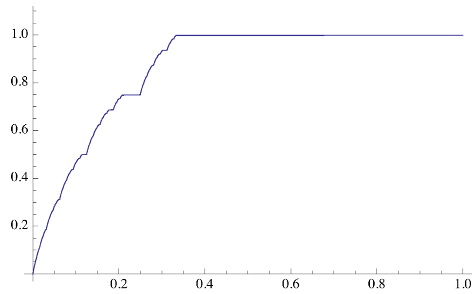

Figure 5. Graph of Takagi singular function $\tau^S(x)$

We call the function $\tau^S(x) : [0, 1] \to [0, 1]$ the *Takagi singular function*; it is pictured in Figure 5. Using the functional equation $\tau(\frac{1}{2}x) = \frac{1}{2}(\tau(x)+x)$ given in Theorem 4.1(1), we deduce that the function $\frac{1}{2}\tau^S(x)$ agrees with $\tau(x)$ on the set $\frac{1}{2}\Omega^L$. This shows that the Takagi function is strictly increasing when restricted to the domain $\frac{1}{2}\Omega^L$.

Associated to the Takagi singular function is a nonnegative Radon measure $d\mu_S$, which we call the *Takagi singular measure*, such that

$$\tau^S(x) = \int_0^x d\mu_S. \tag{13.2}$$

It is singular with respect to Lebesgue measure and defines an interesting probability measure on $[0, 1]$. The Takagi singular measure is not translation-invariant, but it has certain self-similarity properties under dyadic rescalings, compatible with the functional equations of the Takagi function. These are useful in explicitly computing the measure of various interesting subsets of Ω^L, see [57]. One may compare its properties with those of the Cantor function, see Dovghoshey et al [27, Sect. 5]. A major difference is that its support Ω^L has full Hausdorff dimension, while the Cantor set has Hausdorff dimension $\log_3(2) \approx 0.63092$.

By an application of the co-area formula of geometric measure theory for functions of bounded variation (in the version in Leoni [58, Theorem 7.2 and Theorem 13.25]) to a relative of the Takagi singular function, the author with Maddock [56] determined the average number of local level sets on a random level.

Theorem 13.4. *For a full Lebesgue measure set of ordinate points $y \in [0, \frac{2}{3}]$ the number $N^{loc}(y)$ of local level sets at level y is finite. Furthermore*

$$\int_0^{\frac{2}{3}} N^{loc}(y)dy = 1.$$

That is, the expected number of local level sets at a uniformly drawn random level in $[0, \frac{2}{3}]$ is exactly $\frac{3}{2}$.

Proof. This is shown in [56, Theorem 6.3]. A different derivation is given in [2], that avoids the co-area formula. □

One can easily derive both parts of Theorem 12.4 from this result. This result fails to give any information about the multiplicity of local level sets on those levels having an uncountable level set, because the set of such levels y has Lebesgue measure 0.

Recently Allaart [2] established the following result about the multiplicities of local level sets in a level set.

Theorem 13.5. (Allaart 2011) *There exist levels y such that the level set $L(y)$ contains uncountably many distinct local level sets. The set of such levels is dense in $[0, \frac{2}{3}]$.*

Allaart ([2] , [3]) has obtained further information on the structure of level sets and local level sets, classifying the values giving different types of level sets in terms of the Borel hierarchy of descriptive set theory. Some of these sets are complicated enough that they apparently live in the third level of the Borel hierarchy.

§ 14. Level Sets at Rational Levels

In 2005 Knuth [51, Problem 83, p. 32] raised the question of determining which rational levels $y = \frac{r}{s}$ have level sets have $L(\frac{r}{s})$ that are uncountable.

The author with Maddock [56] answered the much easier question of determining when certain rational numbers x give uncountable local level sets $L_x^{loc} \subset L(\tau(x))$.

Theorem 14.1. *(1) A rational number $x = \frac{p}{q} \in [0, 1]$ has an uncountable local level set $L_{\frac{p}{q}}^{loc}$ if and only if its binary expansion has a pre-periodic part with an equal number of zeros and ones, and if also its purely periodic part has an equal number of zeros and ones.*

(2) If a rational x has L_x^{loc} uncountable, then L_x^{loc} contains a countably infinite set of rational numbers.

This criterion implies that dyadic rationals $x = \frac{m}{2^n}$ must have finite local level sets. For such values $y = \tau(\frac{k}{2^l})$ is also a dyadic rational. Concerning dyadic rational levels, Allaart [3] obtains the following much stronger result.

Theorem 14.2. (Allaart 2011) *Let* $y = \frac{k}{2^m}$ *be a dyadic rational with* $0 \le y \le \frac{2}{3}$. *Then the level set* $L(y)$ *is either finite or countably infinite.*

The examples $y = 0$ and $y = \frac{1}{2}$ given above show that both alternatives in this result occur. Furthermore all elements on the given dyadic rational level are rational. However if the set is countably infinite then these elements need not all be dyadic rationals. For example $\tau(\frac{1}{6}) = \frac{1}{2}$, cf. [56, Theorem 7.1].

§ 15. Open Problems

There remain many open questions about the Takagi function. We mention a few of them.

1. Determine analytic and other properties of the conjugate Takagi function $Y(x)$ defined in Theorem 6.2.

2. Consider the set of all x such that the abscissa level set $L(\tau(x))$ has Hausdorff dimension zero. Does this set have full Lebesgue measure in $[0, 1]$?

3. Is there a continuous function on the interval for which the Takagi singular measure is a "natural" invariant probability measure?

4. Determine the dimension spectrum function

$$f_\tau(\alpha) := \dim_H(\{y : \quad \dim_H(L(y)) \ge \alpha\}.$$

   (Some bounds on $f_\tau(\alpha)$ are given in [57].)

5. Knuth's problem ([51, 7.2.1.3, Prob. 83]): Find a direct characterization of the set of rational y which have an uncountable level set.

Acknowledgments. I thank the following people for help. D. E. Knuth originally brought problems on the Takagi function to my attention via postcard. S.T. Kuroda supplied helpful remarks on Takagi's original work, and J.-P. Allouche supplied some corrections. P. Allaart made many helpful comments and supplied additional references. The figures in this paper are reproduced from a joint paper [56] with Z. Maddock, excluding Figure 3. G. Alkauskas provided a plot in Figure 3 of the conjugate Takagi function. I gratefully thank the reviewer for detailed comments and corrections.

References

[1] Allaart, P. C., On a flexible class of continuous functions with uniform local structure, *J. Math. Soc. Japan*, **61** (2009), No. 1, 237–262.

[2] Allaart, P. C., How large are the level sets of the Takagi function? *Monatshefte für Math.,* in press.

[3] Allaart, P. C., The finite cardinalites of the level sets of the Takagi function, *J. Math. Anal. Appl.* **388** (2012), no. 2, 1117–1129.

[4] Allaart, P. C. and Kawamura, K., Extreme values of some continuous nowhere differentiable functions, *Math. Proc. Camb. Phil. Soc.*, **140** (2006), No. 2, 269–295.

[5] Allaart, P. C. and Kawamura, K., The improper infinite derivatives of Takagi's nowhere-differentiable function, *J. Math. Anal. Appl.*, **372** (2010), 656–665.

[6] Allaart, P. C. and Kawamura, K., The Takagi function: a survey, *Real Analysis Exchange* **37** (2011/2012), no.1, 1–54.

[7] de Amo, E., Bhouri, I., Díaz Carrillo, M., and Fernández-Sánchez, J., Hausdorff dimension of Takagi's function level sets, *Nonlinear Analysis*, **74** (2011), No. 15, 5081–5087.

[8] Anderson, J. M. and Pitt, L. D., Probabilistic behavior of functions in the Zygmund spaces Λ^* and λ^*, *Proc. London Math. Soc.*, **59** (1989), No. 3, 558–592.

[9] Baba, Y., On maxima of Takagi-van der Waerden functions, *Proceedings of the American Mathematical Society*, **91** (1984), No.3, pp. 373-376.

[10] Balasubramanian, R., Kanemitsu, S., and Yoshimoto, M., Euler products, Farey series and the Riemann hypothesis II, *Publ. Math. Debrecen*, **69** (2006), no. 1-2, 1–16.

[11] Barat, G., Berthé, V., Liardet, P., and Thuswaldner, J., Dynamical directions in numeration, *Ann. Inst. Fourier*, **56** (2006), no. 7, 1987–2092.

[12] Berg, L. and Krüppel, M., de Rham's singular function, two-scale difference equations and Appell polynomials, *Results Math.*, **38** (2000), 18–47.

[13] Behrend, F. A., Some remarks on the construction of continuous non-differentiable functions, *Proc. London Math. Soc. Series 2*, (1948), 463–481.

[14] Billingsley, P. Van der Waerden's continuous nowhere differentiable function, *Amer. Math. Monthly*, **91** (1984), 373–376.

[15] du Bois-Reymond, P., Versuch einer Classification der willkürlichen Functionen reeller Argumente nach ihren Aenderungen in den kleinsten Intervallen, *J. Reine Angew. Math.*, **79** (1875), 21–37.

[16] Boros, Z., An inequality for the Takagi function, *Math. Inequalities & Applications*, **11** (2008), No. 4, 757–765.

[17] Brown, J. B. and Kozlowski, G., Smooth Interpolation, Hölder continuity and the Takagi-van der Waerden function, *Amer. Math. Monthly*, **110** (2003), No.2, 142–147.

[18] Buczolich, Z., Micro tangent sets of continuous functions, *Math. Bohemica*, **128** (2003), No. 2, 147–167.

[19] Buczolich, Z., Irregular 1-sets on the graphs of continuous functions, *Acta Math. Hungar.*, **121** (2008), 371–393.

[20] Cater, F. S., On van der Waerden's nowhere differentiable function, *Amer. Math. Monthly*, **91** (1984), 307–308.

[21] Coquet, J. Power sums of digital sums, *J. Number Theory* **22** (1986), 161–176.

[22] Daubechies, I., Ten Lectures on Wavelets, *CBMS-NSF Regional Conference Series in Applied Mathematics, 61.* Society for Industrial and Applied Mathematics (SIAM), Philadelpha, PA 1992.

[23] Daubechies, I. and Lagarias, J. C., Two-scale difference equations. I. Existence and global regularity of solutions, *SIAM J. Math. Anal.*, **22** (1991), no. 5, 1388–1410.

[24] Daubechies, I. and Lagarias, J. C., Two-scale difference equations. II. Local regularity, infinite products of matrices and fractals, *SIAM J. Math. Anal.*, **23** (1992), no. 4, 1031–1079.

[25] Daubechies, I. and Lagarias, J. C., Sets of matrices all infinite products of which converge, *Lin. Alg. Appl.*, **161** (1992), 227–263. [Corrigendum/addendum, Lin. Alg. Appl. **327** (2001), 69–83.]

[26] Delange, H., Sur la fonction sommatoire de la fonction "somme des chiffres", *Enseign. Math.*, **21** (1975), 31–47.

[27] Dovgoshey, O., Martio, O., Ryazanov, V., and Vuorinen, M., The Cantor function, *Expo. Math.*, **24** (2006), 1–37.

[28] Faber, G., Über stetige Funktionen, *Math. Annalen*, **66** (1908), 81–94.

[29] Faber, G., Über stetige Funktionen (Zweite Abhandlung), *Math. Annalen*, **69** (1910), No. 3, 372–443.

[30] Faber, G. Über die Orthogonalfunktionen des Herrn Haar, *Jahrseberícht Deutsch. Math. Verein.*, **19** (1910), pp. 104–112.

[31] Flajolet, P., Grabner, P., Kirschenhofer, P., Prodinger H., and Tichy, R. F., Mellin transforms and asymptotics: digital sums, *Theor. Comput. Sci.*, *123* (1994), 191–214.

[32] Frankl, P., Matsumoto, M., Rusza I. Z., and Tokushige, N., Minimum shadows in uniform hypergraphs and a generalization of the Takagi function, *J. Comb. Theory A*, **69** (1995), 125–148.

[33] Franel, J., Les suites de Farey et les problémes des nombres premiers, *Nachr. Ges. Wiss. Göttingen*, 1924, 198–201.

[34] Gamkrelidze, N. G. On a probabilistic property of Takagi's function, *J. Math. Kyoto Univ.*, **30** (1990), No. 2, 227–229.

[35] Gaspard, P. and Klages, R., Chaotic and fractal properties of deterministic diffusion-reaction processes, *Chaos*, **8** (1998), No. 2, 409–423.

[36] Hardy, G. H., Weierstrass's non-differentiable function, *Trans. Amer. Math. Soc.*, **17** (1916), no. 3, 301–325.

[37] Házy, A. and Páles, Z., On approximately midconvex functions, *Bull. London Math. Soc.*, **36** (2004), no. 3, 339–350.

[38] Hata, M. and Yamaguti, M., The Takagi function and its generalization, *Japan J. Appl. Math.*, **1** (1984), 183–199.

[39] Hilberdink, T., Some connections between Bernoulli convolutions and analytic number theory, in: *Fractal gometry and applications: a jubilee of Benoît Mandelbrot, Part I,* pp. 233–271, Proc. Symp. Pure Math., No. 72, Part 1, AMS: Providence, RI 2004.

[40] Hildebrandt, T. H., A simple continuous function with a finite derivative at no point, *Amer. Math. Monthly*, **40** (1933), 547–548.

[41] Huxley, M., *The Distribution of Prime Numbers. Large sieves and zero-density theorems.* Oxford University Press, Oxford 1972.

[42] Jaffard, S., Multifractal formalism for functions part I: results valid for all functions, *SIAM J. Math. Anal.*, **28** (1997), No. 4, 944-970.

[43] Jaffard, S., Multifractal formalism for functions part II: Self-similar functions, *SIAM J. Math. Anal.*, **28** (1997), No. 4, 971-998.

[44] Jessen, B. and Wintner, A., Distribution functions and the Riemann zeta-function, *Trans. Amer. Math. Soc.*, **38** (1935), 48–88.

[45] Kac, M., *Statistical Independence in Probability, Analysis and Number Theory,* Carus Mathematical Monograph No. 12. Math. Assoc. of America: New York, 1959.

[46] Kahane, J.-P., Sur L'exemple donné par M. de Rham, d'une fonction continue sans dérivée, *Enseign. Math.*, **5** (1959), 53–57.

[47] Kairies, H.-H., Takagi's function and its functional equations, *Rocz. Nauk. Dydakt. Pr. Mat.*, **15** (1998), 73–83.

[48] Kairies, H.-H., Darslow, W. F., and Frank, M. J., Functional equations for a function of van der Waerden type, *Rad. Mat.*, **4** (1988), no. 2, 361–374. (Errata: *Rad Math.*, **5** (1989), no. 1, 179–180.)

[49] Kanemitsu, S. and Yoshimoto, M., Euler products, Farey series and the Riemann hypothesis, *Publ. Math. Debrecen*, **56** (2000), No. 3-4, 431–449.

[50] Knopp, K., Ein einfaches Verfahren zur Bildung stetiger nirgends differenzierbarer Funktionen, *Math. Z.*, **2** (1918), 1–26.

[51] Knuth, D. E. *The Art of Computer Programming, Vol. 4, Fascicle 3. Generating all Combinations and Partitions*, Addison-Wesley: Upper Saddle River, NJ 2005.

[52] Kôno, N., On generalized Takagi functions, *Acta Math. Hung.*, **49** (1987), No 3-4, 315–324.

[53] Krüppel, M., On the extrema and improper derivatives of Takagi's continuous nowhere differentiable function, *Rostock Math. Kolloq.*, **62** (2007), 41–59.

[54] Krüppel, M., Takagi's continuous nowhere differentiable functions and binary digit sums, *Rostock Math. Kolloq.* **63** (2008), 37–54.

[55] Krüppel, M. De Rham's singular function, its partial derivatives with respect to the parameter and binary digital sums, *Rostock Math. Kolloq.* **64** (2009), 57–74.

[56] Lagarias, J. C. and Maddock, Z., Level sets of the Takagi function: Local level sets, *Monatshefte für Math.*, **166** (2012), no. 2, 201–238.

[57] Lagarias, J. C. and Maddock, Z., Level sets of the Takagi function: Generic level sets, eprint: `arXiv: 1011.3183 v4`, 26 July 2011

[58] Leoni, G., *A First Course in Sobolev Spaces,* Graduate Studies in Mathematics 105, AMS: Providence, RI 2009.

[59] Lomnicki, E. and Ulam, S. M., Sur la théorie de la mesure dans les espaces combinatoires et son application au calcul des probabilités I. Variables indépendantes, *Fund. Math.*, **23** (1934), 237–278.

[60] Maddock, Z., Level sets of the Takagi function: Hausdorff dimension, *Monatshefte für Math.*, **160** (2010), 167–186.

[61] Mauldin, R. D. and Williams, S. C., On the Hausdorff dimension of some graphs, *Trans. Amer. Math. Soc.*, **298**, No. 2 (1986), 793–803.

[62] Megginson, R. E., *An Introduction to Banach Space Theory,* Graduate Texts in Math. No. 183, Springer-Verlag: New York 1998.

[63] Mikolas, M., Farey series and their connection with the prime number problem I. *Acta Univ. Szeged. Sect. Sci. Math.*, **13** (1949), 93–117.

[64] Mikolas, M., Un Théorème d'équivlence et ses applications, *Norske. Vid. Selsk. Forh. Trondheim*, **22** (1950), no. 28, 128–131.

[65] Mikolas, M., Farey series and their connection with the prime number problem II. *Acta Univ. Szeged. Sect. Sci. Math.*, **14** (1951), 5–21.

[66] Mirsky, L., A theorem on representation of integers in the scale of r, *Scripta Math.*, **15** (1949), 11-12.

[67] Okada, T., Sekiguchi, T., and Shiota, Y., Applications of binomial measures to power sums of digital sums, *J. Number Theory*, **52** (1995), no. 2, 256–266.

[68] Okada, T., Sekiguchi, T., and Shiota, Y., A generalization of Hata-Yamaguti's results on the Takagi function II. Multinomial case, *Japan J. Indust. Appl. Math.*, **13** (1996), No. 3, 435–463.

[69] Peres, Y., Schlag, W., and Solomyak, B., Sixty years of Bernoulli convolutions, in: *Fractal geometry and Stochastics II (Grefswald/Koserov 1998)*, 39–65, Prog. Prob. Vol. 46, Birkhäuser: Basel 2000.

[70] Pinkus, A., Weierstrass and Approximation Theory, *J. Approximation Theory*, **107**, 1–66.

[71] de Rham, G., Sur une courbe plane, *J. Math. Pures Appl. Ser. 9*, **35** (1956), 25–42.

[72] de Rham, G., Sur quelques courbes définies par des équations fonctionnelles, *Univ. e Politec. Torino. Rend. Sem. Mat.*, **16** (1956/57), 101–113.

[73] de Rham, G., Sur un exemple de fonction continue sans dériveée, *Enseign. Math.*, **3** (1957), 71–72.

[74] Salem, R., On some singular monotonic functions which are strictly increasing, *Trans. Amer. Math. Soc.*, **53** (1943), 427–439.

[75] Schauder, J., Zur Theorie stetiger Abbildungen in Funktionalräumen, *Math. Z.*, **26** (1927), 47–65.

[76] Sekiguchi, T. and Shiota, Y., A generalization of Hata-Yamaguti's results on the Takagi function, *Japan J. Indust. Appl. Math.*, **8** (1991), 203–219.

[77] Shidfar, A. and Sabetfakhri, K., On the continuity of van der Waerden's function in the Hölder sense, *Amer. Math. Monthly*, **93** (1986), 375–376.

[78] Shidfar, A. and Sabetfakhri, K., *On the Hölder continuity of certain functions,* Exposition. Math. **8** (1990), 365–369.

[79] Shiota, Y., Sekiguchi, T., and Okada, T., Hata-Yamaguti's result on Takagi functions and its applications to digital sum problems, In: *(Analytic Number Theory , Kyoto 1995)* (Japanese) , No. 961, Aug. 1996, 73–80.

[80] Stolarsky, K. B., Power and exponential sums of digital sums related to binomial coefficient parity, *SIAM J. Appl. Math.*, **32** (1977), 717–730.

[81] Takagi, T., A simple example of the continuous function without derivative, from *Proceedings of the Physico-Mathematical Society of Japan, ser II*, Vol **1**. (1903), pp 176-177. [Collected Papers of Teiji Takagi (S. Iyanaga, Ed), Springer Verlag, New York 1990].

[82] Tricot, C., Function norms and fractal dimension, *SIAM J. Math. Anal.*, **28** (1997), no. 1, 189–212.

[83] Trollope, J. R., An explicit expression for binary digital sums, *Math. Mag.*, **41** (1968), 21–25.

[84] de Vito, L., Su un esempio di funzione continua senza derivata, *Enseign. Math.* **4** (1958), 53–57.

[85] van der Waerden, B. L., Ein einfaches Beispiel einer nichtdifferenzierbaren stetigen Funktion, *Math. Z.*, **32** (1930), 474–475.

[86] Wintner, A., *Asymptotic distributions and infinite convolutions,* Lecture notes by Ralph P. Boas, Frank Smithies and John W. Tukey with sympathetic encouragement from Cyrus C. Mac Duffee. The Institute for Advanced Study., Princeton NJ 1938.

[87] Yamaguchi, Y., Tanikawa, K., and Mishima, N., Fractal basin boundary in dynamical systems and the Weierstrass-Takagi functions, *Physics Letters A*, **128** (1988), No. 9, 470–478.

[88] Yamaguti, M. and Hata, M., Weierstrass's function and chaos, *Hokkaido Math. J.*, **12** (1983), 333-342.

[89] Yamaguti, M. Schauder expansion by some quadratic base function, *J. Fac. Sci. Univ.*

Tokyl, Sect IA Math., **36** (1989), no. 2, 187–191.

[90] Yamaguti, M., Hata, M., and Kigami, J., *Mathematics of Fractals*, Translations of Mathematical Monographs, No. 167, American Mathematical Society: Providence, RI 1997. (Translation from Japanese of 1993 original.)

[91] Yamaguti, M. and Kigami, J., Some remarks on Dirichlet problems of Poisson equation, *Analyse mathématique et applications (J.-L. Lion's 50th birthday)*, Gauthier-Villars, Montrouge 1988, pp. 465–471.