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Universality of composite functions

By

Antanas LAURINČIKAS*

Abstract

We modify and extend some results of [10] on the universality of composite functions of the Riemann zeta-function.

§ 1. Introduction

The famous Mergelyan theorem [15], see also [18], asserts that every function $f(s)$ of complex variable $s = \sigma + it$ continuous on a compact subset $K \subset \mathbb{C}$ and analytic in the interior of K can be approximated uniformly on K by polynomials in s. Thus,for every $\varepsilon > 0$, there exists a polynomial $P(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \varepsilon.$$

Examples show that the hypotheses on K and $f(s)$ can't be weakened. For example, K can't be a closed rectangle with an excluded disc. Thus, really the Mergelyan theorem gives necessary and sufficient conditions for the approximation of analytic functions by polynomials. The theorem is very important not only theoretically but also practically because polynomials are rather simple entire functions.

Also, it is known that there exist so called universal functions whose shifts approximate any analytic function. The first result in this direction belongs to Birkhoff [3]. From his theorem, it follows that there exists an entire function $g(s)$ such that, for every entire function $f(s)$, a compact subset $K \subset \mathbb{C}$ and arbitrary $\varepsilon > 0$, there exists a number $a \in \mathbb{C}$ such that

$$\sup_{s \in K} |g(s + a) - f(s)| < \varepsilon.$$

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*Faculty of Mathematics and Informatics, Vilnius University, LT-03225 Vilnius, Lithuania.
e-mail: `antanas.laurincikas@mif.vu.lt`

Unfortunately, the Birkhoff theorem is entirely non-effective. It gives only the existence of the function $g(s)$, and any example of $g(s)$ is not given. The number a which depends, obviously, on $f(s)$, K and ε also can't be effectively evaluated.

Only in 1975, the first example of functions $g(s)$ such that their shifts $g(s+i\tau)$, $\tau \in \mathbb{R}$, approximate uniformly on some compact subsets every analytic function was found. It turned out that the above property is due to the famous Riemann zeta-function $\zeta(s)$ which is defined, for $\sigma > 1$, by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

and by analytic continuation elsewhere, except for a simple pole at $s = 1$ with residue 1. Generalizing Bohr's [5] and Bohr-Courant's [6] denseness results on value-distribution of $\zeta(s)$, Voronin discovered [17] a remarkable property of $\zeta(s)$ which now is called universality.

Theorem 1.1 ([17],[7]). *Let $0 < r < \frac{1}{4}$. Suppose that $f(s)$ is a continuous non-vanishing function on the disc $|s| \leq r$, and analytic for $|s| < r$. Then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that*

$$\max_{|s|\leq r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon. \tag{1.1}$$

As an important result, the Voronin theorem was observed by the mathematical community. Many number theorists, among them Reich, Gonek, Bagchi, Kohji Matsumoto, Garunkštis, J. Steuding, R. Steuding, Sander, Schwarz, Mishou, Bauer, Nakamura, Nagoshi, Kačinskaitė, Macaitienė, Genys, Šiaučiūnas, Kaczorowski, Pańkowski, Lee, the author and others improved and generalized Theorem 1.1 for other zeta and L-functions, see survey papers [9],[14] and the monograph [16]. At the moment, we know the following version of the Voronin theorem. Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$, and $\mathrm{meas}\{A\}$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

Theorem 1.2. *Suppose that K is a compact subset of the strip D with connected complement, and $f(s)$ is a continuous and non-vanishing function on K, and analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

Proof of Theorem 1.2 is given, for example, in [8]. Thus, Theorem 1.2 once more shows that the set of the values of $\zeta(s)$ is very dense, there are infinitely many shifts $\zeta(s + i\tau)$ which approximate a given analytic function. This fact was already known to Voronin. In the proof of Theorem 1.1, he obtains that the set of τ satisfying inequality

(1.1) has a positive lower density, however, this was not fixed in the statement of the theorem.

From hypotheses of Theorem 1.2, one can easily see a relation with the Mergelyan theorem.

The original Voronin's proof of Theorem 1.1 is based on the approximation of $\zeta(s)$ in the mean by a finite Euler's product over primes, it also uses a version of Riemann's theorem on rearrangement of series in Hilbert space as well as the Kronecker approximation theorem.

There exists another, probabilistic proof, of Theorem 1.2 proposed by Bagchi [1]. We recall shortly this proof because we will apply it later.

1. A limit theorem in the space of analytic functions for the function $\zeta(s)$.

Denote by $H(G)$ the space of analytic functions in the region $G \subset \mathbb{C}$ equipped with the topology of uniform convergence on compacta. Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S. Moreover, define

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for each prime p. By the Tikhonov theorem, the torus Ω with the product topology and pointwise multiplication is a compact topological Abelian group. Thus, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H exists, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(p)$ denote the projection of $\omega \in \Omega$ to the coordinate space γ_p. Then, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s, \omega)$ by

$$\zeta(s, \omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}.$$

Note that the latter infinite product over primes, for almost all $\omega \in \Omega$, converges uniformly on compact subsets of the strip D, and define there a $H(D)$-valued random element. Denote by P_ζ the distribution of the random element $\zeta(s, \omega)$, i.e., a probability measure defined by

$$P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s, \omega) \in A), \quad A \in \mathcal{B}(H(D)).$$

Proposition 1.3. *The probability measure*

$$P_T(A) \overset{def}{=} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \zeta(s + i\tau) \in A\right\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P_ζ as $T \to \infty$.

2. The support of the measure P_ζ (or of the random element $\zeta(s, \omega)$).

The space $H(D)$ is separable. Therefore, the support of the measure P_ζ is a minimal closed set S_P such that $P_\zeta(S_{P_\zeta}) = 1$. The set S_{P_ζ} consists of all elements $g \in H(D)$ such that, for each open neighbourhood G of g, the inequality $P_\zeta(G) > 0$ is satisfied.

An application of elements from the theory of Hilbert spaces and of entire functions of exponential type leads to the following assertion.

Proposition 1.4. *The support of the measure P_ζ is the set*

$$S \stackrel{def}{=} \left\{ g \in H(D) : g^{-1}(s) \in H(D) \text{ or } g(s) \equiv 0 \right\}.$$

Proof of Theorem 1.2. Theorem 1.2 is a direct consequence of the Mergelyan theorem, and Propositions 1.3 and 1.4. By the Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2}. \tag{1.2}$$

Define

$$G = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2} \right\}.$$

Since G is an open neighbourhood of the function $\mathrm{e}^{p(s)}$ and, by Proposition 1.4, $\mathrm{e}^{p(s)}$ is an element of the support of the measure P_ζ, we obtain that $P_\zeta(G) > 0$. Using Proposition 1.4 and an equivalent of the weak convergence of probability measures in terms of open sets, see Theorem 2.1 of [2], we find that

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - \mathrm{e}^{p(s)}| < \frac{\varepsilon}{2} \right\} \geq P_\zeta(G) > 0.$$

This together with (1.2) proves the theorem. ☐

Universality is a very important property of zeta and L-functions, it has deep theoretical and practical applications. For example, it is known [1] that the Riemann hypothesis is equivalent to the assertion that, for every compact subset $K \subset D$ with connected complement and any $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - \zeta(s)| < \varepsilon \right\} > 0.$$

This result was generalized by Steuding in [16]. Also, universality implies the functional independence for zeta-functions, can be used for estimation of the number of zeros in some regions as well as in the moment problem of zeta-functions. An example of practical applications is given in [4]. Therefore, it is an important problem to extend the class of universal functions.

There exists the Linnik-Ibragimov conjecture that all functions in some half-plane given by Dirichlet series, analytically continuable to the left of the absolute convergence half-plane and satisfying certain natural growth conditions are universal. It is a bit strange because Voronin published his universality theorem in 1975 while Linnik died in 1972. However, Voronin discovered the universality of $\zeta(s)$ earlier, his candidate degree thesis (1972) contains the results close to the universality theorem, and Linnik knew this. The theorem of 1975 has only a simplified short enough proof. Ibragimov informed the author on Linnik's conjecture in 1990, and said that he also supports this conjecture. Thus, it is reasonable to call it the Linnik-Ibragimov conjecture.

On the other hand, there exist non-universal functions given by Dirichlet series. For example, if

$$a_m = \begin{cases} 1 \text{ if } & m = m_0^k, \ k \in \mathbb{N}, \\ 0 \text{ if } & m \neq m_0^k, \end{cases}$$

where $m_0 \in \mathbb{N} \setminus \{1\}$, then we have that, for $\sigma > 0$,

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s} = \sum_{k=1}^{\infty} \frac{1}{m_0^{ks}} = \frac{1}{m_0^s - 1}.$$

Then function $(m_0^s - 1)^{-1}$ is analytic in the whole complex plane, except for simple poles lying on the line $\sigma = 0$, however, obviously, is non-universal.

§ 2. Universality of the logarithm

Define $\log \zeta(s)$ in the strip D from $\log \zeta(2) \in \mathbb{R}$ by continuous variation along the line segments $[2, 2+it]$ and $[2+it, \sigma+it]$ provided that the path does not pass a possible zero of $\zeta(s)$ or pole $s = 1$. If this does, then we take

$$\log \zeta(\sigma + it) = \lim_{\varepsilon \to +0} \log \zeta(\sigma + i(t + \varepsilon)).$$

In [7], it is proved that the function $\log \zeta(s)$ is also universal.

Theorem 2.1 ([7]). *Let $0 < r < \frac{1}{4}$. Suppose that $f(s)$ is a function continuous on the disc $|s| \leq r$ and analytic for $|s| < r$. Then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that*

$$\max_{|s| \leq r} \left| \log \zeta(s + \frac{3}{4} + i\tau) - f(s) \right| < \varepsilon.$$

In [7], Theorem 1.1 is deduced from Theorem 2.1. However, we do not know does Theorem 1.2 imply the universality of $\log \zeta(s)$. Indeed, let $K \subset D$ be a compact subset with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Clearly, $\mathrm{e}^{f(s)} \neq 0$ on K. Denote by $M_K = \max_{s\in K} \left|\mathrm{e}^{-f(s)}\right|$. Then Theorem 1.2 implies that, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{s\in K}\left|\zeta(s+i\tau) - \mathrm{e}^{f(s)}\right| < \frac{\varepsilon}{2M_K}\right\} > 0. \tag{2.1}$$

Suppose that $\tau \in \mathbb{R}$ satisfies the inequality

$$\sup_{s\in K}\left|\zeta(s+i\tau) - \mathrm{e}^{f(s)}\right| < \frac{\varepsilon}{2M_K}.$$

Then we have that, for such τ and all $s \in K$,

$$\mathrm{e}^{\log\zeta(s+i\tau)-f(s)} = 1 + \varepsilon(s),$$

where $\sup_{s\in K} |\varepsilon(s)| < \frac{\varepsilon}{2}$. However, from this, it does not follow that the difference $|\log\zeta(s+i\tau) - f(s)|$ is small for all $s \in K$, therefore (2.1) does not imply the inequality

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\log\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

I thank Professor Kohji Matsumoto who pointed out the above problem.

Theorem 2.1 shows that the composite function $F(\zeta(s)) = \log\zeta(s)$ is universal. Also, it is known [1], that the derivative $\zeta'(s)$ is universal, too. Thus, a problem arises to describe classes of functions F such that the composite function $F(\zeta(s))$ preserve the universality property. The first results in this direction were obtained in [10], and our aim is to present them in a more precise and convenient form.

§ 3. Lipschitz class

A sufficiently wide class of functions $F : H(D) \to H(D)$ with the universality property for $F(\zeta(s))$ can be described as follows. We say that a function $F : H(D) \to H(D)$ belongs to the class $\mathrm{Lip}(\alpha)$ if the following hypotheses are satisfied:

1^0 For every polynomial $p = p(s)$ and every compact subset $K \subset D$ with connected complement, there exists an element $g \in F^{-1}\{p\} \subset H(D)$ such that $g(s) \neq 0$ on K;

2^0 For every compact subset $K \subset D$ with connected complement, there exist constants $c > 0$ and $\alpha > 0$, and a compact subset $K_1 \subset D$ with connected complement such that

$$\sup_{s\in K} |F(g_1(s)) - F(g_2(s))| \leq c \sup_{s\in K_1} |g_1(s) - g_2(s)|^{\alpha}$$

for all $g_1, g_2 \in H(D)$.

Hypothesis 2^0 is an analogue of the classical Lipschitz condition with exponent α.

We note that the function $F : H(D) \to H(D)$, $F(g) = g'$, is in the class $\mathrm{Lip}(1)$. Obviously, hypothesis 1^0 is satisfied. It remains to check hypothesis 2^0. Let $K \subset D$ be a compact subset with connected complement, $G \supset K$ be an open set, and $K_1 \subset D$ be a compact subset with connected complement such that $G \subset K_1$. We take a simple closed contour l lying in $K_1 \setminus G$ and enclosing the set K. Then, by the Cauchy integral formula, for $g_1, g_2 \in H(D)$,

$$F(g_1(s)) - F(g_2(s)) = \frac{1}{2\pi i} \int\limits_l \frac{g_1(z) - g_2(z)}{(s-z)^2} \mathrm{d}z.$$

Thus, for all $s \in K$,

$$|F(g_1(s)) - F(g_2(s))| \le c \sup_{z \in l} |g_1(z) - g_2(z)| \le c \sup_{z \in K_1} |g_1(s) - g_2(s)|$$

with some $c > 0$, and we obtain hypothesis 2^0 with $\alpha = 1$.

Theorem 3.1. *Suppose that $F \in \mathrm{Lip}(\alpha)$. Let $K \subset D$ be a compact subset with connected complement, and $f(s)$ be a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |F(\zeta(s + i\tau)) - f(s)| < \varepsilon \right\} > 0.$$

Proof. By the Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{3.1}$$

In view of hypothesis 1^0 of the class $\mathrm{Lip}(\alpha)$, we have that there exists $g \in F^{-1}\{p\} \subset H(D)$ and, moreover, $g(s) \neq 0$ on K. Let $\tau \in \mathbb{R}$ be such that

$$\sup_{s \in K_1} |\zeta(s + i\tau) - g(s)| < c^{-\frac{1}{\alpha}} \left(\frac{\varepsilon}{2} \right)^{\frac{1}{\alpha}}, \tag{3.2}$$

where $K_1 \subset D$ is a compact subset with connected complement corresponding the set K in hypothesis 2^0 of the class $\mathrm{Lip}(\alpha)$. By hypothesis 2^0, for τ satisfying (3.2),

$$\sup_{s \in K} |F(\zeta(s + i\tau)) - p(s)| \le c \sup_{s \in K_1} |\zeta(s + i\tau) - g(s)|^\alpha < \frac{\varepsilon}{2}.$$

This and Theorem 1.2 show that

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |F(\zeta(s + i\tau)) - p(s)| < \frac{\varepsilon}{2} \right\} > 0,$$

and by use of (3.1) the proof is complete. $\square$

In [10], Theorem 3.1 in a bit different form was only mentioned without proof.

§ 4. Other universality classes

In [10], three classes of functions F such that the function $F(\zeta(s))$ is universal are presented. We will remind and complement them. The set S is defined in Proposition 1.4.

Theorem 4.1 ([10]). *Suppose that $F : H(D) \to H(D)$ is a continuous function such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \bigcap S$ is non-empty. Let K and $f(s)$ be the same as in Theorem 3.1. Then the same assertion as in Theorem 3.1 is true.*

The hypothesis of Theorem 4.1 that the set $(F^{-1}G) \bigcap S$ is non-empty for every open set $G \subset H(D)$ is very general, it is difficult to check it.

In the space $H(G)$, we can use the metric

$$\rho(g_1, g_2) = \sum_{m=1}^{\infty} 2^{-m} \frac{\sup\limits_{s \in K_m} |g_1(s) - g_2(s)|}{1 + \sup\limits_{s \in K_m} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(G),$$

which induces the topology of uniform convergence on compacta. Here $\{K_m : m \in \mathbb{N}\}$ is a sequence of compact subsets of G such that

$$G = \bigcup_{m=1}^{\infty} K_m,$$

$K_m \subset K_{m+1}$ for all $m \in \mathbb{N}$, and if $K \subset G$ is a compact subset, then $K \subset K_m$ for some $m \in \mathbb{N}$. Therefore, it is easily seen that the approximation in the space $H(G)$ reduces to that on the sets K_m with large enough m. If we consider the space $H(D)$, it is possible to choose the sets K_m to be with connected complements. Thus, in view of the Mergelyan theorem, we can involve polynomials in the approximation process. This leads to the following version of Theorem 4.1.

Theorem 4.2. *Suppose that $F : H(D) \to H(D)$ is a continuous function such that, for each polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \bigcap S$ is non-empty. Let K and $f(s)$ be the same as in Theorem 3.1. Then the same assertion as in Theorem 3.1 is true.*

Proof. Theorem 4.2 is not contained in [10], therefore we give its proof. First we observe that Proposition 1.3, the continuity of F and Theorem 5.1 of [2] imply the weak convergence for

$$P_{T,F}(A) \stackrel{def}{=} \frac{1}{T} \text{meas} \{\tau \in [0, T] : F(\zeta(s + i\tau)) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

to the measure $P_{\zeta,F}$ as $T \to \infty$, where $P_{\zeta,F}$ is the distribution of the random element $F(\zeta(s,\omega))$.

In the next step, we find the support of the measure $P_{\zeta,F}$. Let g be an arbitrary element of $H(D)$, and G be an any open neighbourhood of g. Since the function F is continuous, the set $F^{-1}G$ is open, too. Let $\hat{K} \subset D$ be an arbitrary compact subset with connected complement. Then, by the Mergelyan theorem, for every $\delta > 0$, there exists a polynomial $p = p(s)$ such that

$$\sup_{s\in\hat{K}} |g(s) - p(s)| < \delta.$$

Therefore, taking into account the above remark on the approximation in the space $H(D)$, we may assume that $p \in G$, too. This and the hypothesis $(F^{-1}\{p\}) \bigcap S \neq \emptyset$ show that the set $(F^{-1}G) \bigcap S$ is non-empty. So, we obtained the hypothesis of Theorem 4.1, and we might finish the proof, however, for fullness, we continue it. From Proposition 1.4 and properties of the support, we deduce that

$$m_H(\omega \in \Omega : F(\zeta(s,\omega)) \in G) = m_H(\omega \in \Omega : \zeta(s,\omega) \in F^{-1}G) > 0.$$

Since g and G are arbitrary, this shows that the support of the measure $P_{\zeta,F}$ is the whole of $H(D)$.

Now it is easy to complete the proof. By the Mergelyan theorem again, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{4.1}$$

Define

$$G = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

Since the polynomial $p(s)$ is an element of the support of $P_{\zeta,F}$, and G is an open neighbourhood of $p(s)$, we have that $P_{\zeta,F}(G) > 0$. Therefore, using an equivalent of the weak convergence of probability measures in terms of open sets, Theorem 2.1 of [2], we obtain from the weak convergence of $P_{T,F}$ that

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : F(\zeta(s+i\tau)) \in G\right\} \geqslant P_{\zeta,F}(G) > 0,$$

or, by the definition of G,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |F(\zeta(s+i\tau)) - p(s)| < \frac{\varepsilon}{2}\right\} > 0.$$

Combining this with (4.1) gives the theorem. □

The main property of the set S is a non-vanishing of functions $g \in H(D)$. As Theorem 4.2 shows, in the definition of the function F, we may use polynomials. However, in the infinite strip D, it is difficult to derive some information on the non-vanishing for the functions from the pre-image $F^{-1}\{p\}$ of a given polynomial $p = p(s)$. Therefore, it is more convenient to consider the space $H(G)$, where G is some bounded region. This is discussed in Theorem 6 of [10].

For $V > 0$, let $D_V = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1,\ |t| < V\}$, and

$$S_V = \left\{g \in H(D_V) : g^{-1}(s) \in H(D_V) \,\text{or}\, g(s) \equiv 0\right\}.$$

Theorem 4.3 ([10]). *Let K and $f(s)$ be the same as in Theorem 3.1. Suppose that V is such that $K \subset D_V$, and that $F : H(D_V) \to H(D_V)$ is a continuous function such that, for each polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \bigcap S_V$ is non-empty. Then the same assertion as in Theorem 3.1 is true.*

It is not difficult to show that, for some functions F, for each polynomial $p = p(s)$, there exists a polynomial $q = q(s)$, $q \in F^{-1}\{p\}$, and $q(s) \neq 0$ for $s \in D_V$. For example, this holds for the function

$$F(g) = c_1 g' + ... + c_r g^{(r)}, \quad g \in H(D_V), \quad c_1, ..., c_r \in \mathbb{C}\backslash\{0\}.$$

Thus, $F(\zeta(s))$ is universal in the sense of Theorem 3.1.

In Theorems 3.1–4.3, the shifts $F(\zeta(s + i\tau))$ approximate any analytic function. If to approximate analytic functions from some subset of $H(D)$, it is possible to extend the class of universal functions $F(\zeta(s))$. This is implemented in the next theorem which is a corrected and extended version of Theorem 7 from [10].

For $a_1, ..., a_r \in \mathbb{C}$, let

$$H_{a_1,...,a_r}(D) = \left\{g \in H(D) : (g(s) - a_j)^{-1} \in H(D),\, j = 1, ..., r\right\} \cup \{F(0)\}.$$

Theorem 4.4. *Suppose that $F : H(D) \to H(D)$ is a continuous function such that $F(S) \supset H_{a_1,...,a_r}(D)$. If $r = 1$, let $K \subset D$ be a compact subset with connected complement, and let $f(s)$ be a continuous and $\neq a_1$ function on K which is analytic in the interior of K. If $r \geq 2$, let $K \subset D$ be an arbitrary compact subset, and $f(s) \in H_{a_1,...,a_r}(D)$. Then the same assertion as in Theorem 3.1 is true.*

Proof. The proof in the case $r = 1$ is the same as in [10]. However, the case $r = 2$ in [10] is not correct because the function $h_{a,b}(s)$ in p. 2330 can take the value b.

In the case $r \geq 2$, the proof is very short. Let g be an arbitrary element of $H_{a_1,...,a_r}(D)$. Then there exists an element $\hat{g} \in S$ such that $F(\hat{g}) = g$. This, the continuity of F and Proposition 1.4 show that every open neighbourhood G of the element g has positive $P_{\zeta,F}$ - measure: $P_{\zeta,F}(G) > 0$. Hence, g is an element of the support of the measure $P_{\zeta,F}$. Therefore, the closure of $H_{a_1,\dots,a_r}(D)$ is a subset of the support of $P_{\zeta,F}$.

Define

$$G_2 = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \frac{\varepsilon}{2} \right\}.$$

Since the function $f(s) \in H_{a_1,\dots,a_r}(D)$, it is an element of the support of $P_{\zeta,F}$. Thus, $P_{\zeta,F}(G_2) > 0$, and, using Proposition 1.3, we obtain that

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} |F(\zeta(s+i\tau)) - f(s)| < \varepsilon \right\} \geq P_{\zeta,F}(G_2) > 0.$$

□

For example, if $r = 1$ and $a_1 = 0$, we obtain the universality for the function $\zeta^N(s)$, $N \in \mathbb{N}$. If $r = 2$ and $a_1 = -1$, $a_2 = 1$, Theorem 4.4 gives the universality for the functions $\sin\zeta(s)$, $\cos\zeta(s)$, $\sinh\zeta(s)$ and $\cosh\zeta(s)$. For the later functions, we have to check the inclusion $F(S) \supset H_{-1,1}(D)$. Let $F(g) = \cosh(g)$. We consider the equation

$$\frac{\mathrm{e}^{g(s)} + \mathrm{e}^{-g(s)}}{2} = f(s),$$

which implies that

$$g(s) = \log\left(f(s) \pm \sqrt{f^2(s) - 1} \right).$$

Therefore, if $f(s) \neq -1$ or 1, then there exists $g(s) \in S$ if we choose a suitable branch of the logarithm.

Really, the following general theorem is valid.

Theorem 4.5. *Suppose that $F : H(D) \to H(D)$ is a continuous function, $K \subset D$ is a compact subset, and $f(s) \in F(S)$. Then the same assertion as in Theorem 3.1 is true.*

Proof. First we observe that $f(s)$ is an element of the support of the measure $P_{\zeta,F}$. Indeed, if g is an arbitrary element of $F(S)$ and G is its any open neighbourhood, then

$$m_H\left(\omega \in \Omega : \zeta(s,\omega) \in F^{-1}G\right) > 0.$$

Hence,

$$m_H\left(\omega \in \Omega : F\left(\zeta(s,\omega)\right) \in G\right) > 0. \tag{4.2}$$

Moreover, by Proposition 1.4,

$$m_H\left(\omega \in \Omega : F\left(\zeta(s,\omega)\right) \in F(S)\right) = m_H\left(\omega \in \Omega : \zeta(s,\omega) \in S\right) = 1.$$

This, and (4.2) show that the closure of $F(S)$ is a support of $P_{\zeta,F}$.

The remainder part of the proof is the same as that in the case of Theorem 4.4, the case $r \geq 2$. $\square$

So, if we want to detect what analytic functions $f(s)$ can be approximated by shifts $F(\zeta(s+i\tau))$, we have to solve the equation

$$F(g(s)) = f(s) \tag{4.3}$$

in $g(s) \in S$. If, for a given $f(s)$, equation (4.3) has a solution, then $f(s)$ uniformly on compact subsets of the strip D is approximated by $F(\zeta(s+i\tau))$.

Consider some examples. Suppose that

$$F(g(s)) = (g'(s))^2 + g(s)g''(s).$$

We have the equation

$$(g(s)g'(s))' = f(s),$$

or

$$\left(g^2(s)\right)' = f_1(s),$$

where $f_1(s)$ is an analytic function. Hence, $g(s) = \sqrt{f_2(s)}$, where

$$f_2(s) = \int_{s_0}^{s} f_1(z)\mathrm{d}z$$

with some $s_0 \in D$. So, if $f_2(s) \neq 0$ on D, then the function $f(s) \in F(S)$ and can be approximated by $F(\zeta(s+i\tau))$.

Now let

$$F(g(s)) = g^2(s) + 4g(s) + 2.$$

Then from the equation

$$g^2(s) + 4g(s) + 2 = f(s)$$

we find that

$$g(s) = -2 \pm \sqrt{2 + f(s)}.$$

Thus, if $f(s) \neq \pm 2$ on D, then the function $g(s)$ is analytic and non-vanishing on D, hence $F(S) \supset H_{-2,2}(D)$ with $F(0) = 2$. Therefore, we have by Theorem 4.4 that any function

$$f(s) \in \left\{g \in H(D) : (g(s) \pm 2)^{-1} \in H(D) \text{ or } g(s) \equiv 2\right\}$$

can be approximated by shifts $F(\zeta(s+i\tau))$.

§ 5. Generalizations

It is possible to prove analogies of the above theorems for other zeta and L-functions. Analogous results for $F(\zeta(s, \mathcal{F}))$, where $\zeta(s, \mathcal{F})$ is the zeta-function of normalized Hecke eigen cusp form $\mathcal{F}$, have been shown in [12]. In [13], universality theorems for the function $F(L(s, \chi_1), ..., L(s, \chi_r))$, where $L(s, \chi_1), ..., L(s, \chi_r)$ are Dirichlet L-functions with pairwise non-equivalent characters $\chi_1, ..., \chi_r$, are discussed. Composite universal functions $F(\zeta(s, \alpha))$, where $\zeta(s, \alpha)$ is the Hurwitz zeta-function with transcendental parameter α, are considered in [11]. Also, generalizations of Theorems 3.1–4.5 for other zeta-functions are possible.

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