On joint universality for the zeta-functions of newforms and periodic Hurwitz zeta-functions

By

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Abstract

In the paper, a short survey on universality for zeta-functions both having and not having Euler's product is given. Also, a joint universality theorem for the zeta-function of newforms and periodic Hurwitz zeta-functions is proved.

§1. Introduction

Let $\Delta$ be a vertical strip on the complex plane $\mathbb{C}$. Denote by $\mathcal{K}(\Delta)$ the class of compact subsets of the strip $\Delta$ with connected complements, for a compact set $K$, denote by $\mathcal{H}(K)$ the class of continuous functions on $K$ which are analytic in the interior of $K$, and by $\mathcal{H}_0(K)$ the subclass of $\mathcal{H}(K)$ consisting of functions which are non-vanishing on $K$.

It is well known, see [1], [5], [12], [14], [29], [36], [37], [38], that the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, which is defined, for $\sigma > 1$, by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

and is analytically continued to the whole complex plane, except for a simple pole at $s = 1$, is universal in the sense that if $K \in \mathcal{K}(D)$, $D = \{s \in \mathbb{C}: \frac{1}{2} < \sigma < 1\}$, and $f \in \mathcal{H}_0(K)$, then, for every $\epsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \epsilon \right\} > 0.$$
Here and in the sequel, \( p \) denotes a prime number, and \( \text{meas}\{A\} \) stands for the Lebesgue measure of a measurable set \( A \subset \mathbb{R} \).

Now let \( \alpha, 0 \leq \alpha \leq 1 \), be a fixed parameter. Then the Hurwitz zeta-function \( \zeta(s, \alpha) \) which is defined, for \( \sigma > 1 \), by

\[
\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},
\]

and is analytically continued to the whole complex plane, except for a simple pole at \( s = 1 \), is also in a similar sense universal. Namely, [1], [5], [21], [35] if \( \alpha \) is a transcendental or rational number \( \neq 1, \frac{1}{2}, K \in \mathcal{K}(D) \) and \( f \in \mathcal{H}(K) \), then, for every \( \epsilon > 0 \),

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s+\tau, \alpha) - f(s)| < \epsilon \right\} > 0.
\]

Since the cases \( \alpha = 1 \) (\( \zeta(s, 1) = \zeta(s) \)) and \( \alpha = \frac{1}{2} \left( \zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s) \right) \) in the later statement are excluded, the function \( \zeta(s, \alpha) \) has no Euler’s product over primes, and this is reflected in its universality: the shifts \( \zeta(s + \tau, \alpha) \) approximate every function \( f \in \mathcal{H}(K) \), the restriction of the class \( \mathcal{H}_0(K) \) is removed. Thus the universality of \( \zeta(s, \alpha) \) is more general than that of \( \zeta(s) \), and is called a strong universality.

Note that the universality of \( \zeta(s, \alpha) \) with algebraic irrational parameter \( \alpha \) remains an open problem.

Mishou in [31] obtained a very interesting joint universality theorem for the functions \( \zeta(s) \) and \( \zeta(s, \alpha) \).

**Theorem 1.1 ([31]).** Suppose that \( \alpha \) is a transcendental number, \( K_1, K_2 \in \mathcal{K}(D) \), \( f_1 \in \mathcal{H}_0(K) \) and \( f_2 \in \mathcal{H}(D) \). Then, for every \( \epsilon > 0 \),

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s \in K_1} |\zeta(s+\tau) - f_1(s)| < \epsilon, \right. \\
\left. \sup_{s \in K_2} |\zeta(s+\tau, \alpha) - f_2(s)| < \epsilon \right\} > 0.
\]

Theorem 1.1 joins the universality and strong universality. We will call this type of the joint universality a mixed universality.

The functions \( \zeta(s) \) and \( \zeta(s, \alpha) \) have their generalizations. Suppose that \( a = \{a_m : m \in \mathbb{N}\} \) and \( b = \{b_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \} \) are two periodic sequences of complex numbers with minimal periods \( k_1 \in \mathbb{N} \) and \( k_2 \in \mathbb{N} \), respectively. Then the functions

\[
\zeta(s; a) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} \quad \text{and} \quad \zeta(s, \alpha; b) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}, \quad \sigma > 1,
\]
are called the periodic zeta and periodic Hurwitz zeta-function, respectively. The equalities
\[
\zeta(s; a) = \frac{1}{k_1^s} \sum_{k=1}^{k_1} a_k \zeta\left(s, \frac{k}{k_1}\right), \quad \zeta(s, \alpha; b) = \frac{1}{k_2^s} \sum_{k=0}^{k_2-1} b_k \zeta\left(s, \frac{k+\alpha}{k_2}\right)
\]
give for the functions \(\zeta(s; a)\) and \(\zeta(s, \alpha; b)\) meromorphic continuation to the whole complex plane with possible simple pole at \(s = 1\).

The universality of the function \(\zeta(s; a)\) with multiplicative sequence \(a\), has been studied by Steuding [36], and Laurinčikas and Šiaučiūnas [28]. In a general case, the problem was solved by Kaczorowski [10].

The strong universality of the function \(\zeta(s, \alpha; b)\) with transcendental parameter \(\alpha\) has been obtained by Javtokas and Laurinčikas [7], [8]. Nakamura [34] studied \(\zeta(s, \alpha; b)\) with a special bounded sequence.

A generalization of Theorem 1.1 for the functions \(\zeta(s; a)\) and \(\zeta(s, \alpha; b)\) with multiplicative sequence \(a\) has been obtained in [11]. A joint universality theorem for periodic zeta-functions with multiplicative coefficients satisfying a certain "independence" condition has been proved in [22]. The joint universality of Hurwitz zeta-functions by different methods has been considered in [33] and [19]. A series of works [15]-[18] and [9], [26], [27] are devoted to joint universality of periodic Hurwitz zeta-functions. A mixed universality theorem for zeta-functions with periodic coefficients can be found in [20].

In [24], Laurinčikas and Matsumoto observed that, for Lerch zeta-functions
\[
L(\lambda_j, \alpha_j, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m}}{(m+\alpha_j)^s}, \quad \sigma > 1, \quad j = 1, \ldots, r,
\]
a more general setting of joint universality is possible. To each parameter \(\alpha_j\), they attached a collection of the parameters \(\lambda_j\). For periodic Hurwitz zeta-functions, the latter idea was applied by Laurinčikas [18], and Laurinčikas and Skerstonaite [27]. We will state the latter result. For \(j = 1, \ldots, r\), let \(l_j \in \mathbb{N}\), and, for \(j = 1, \ldots, r\) and \(l = 1, \ldots, l_j\), let \(b_{j\ell} = \{b_{mj\ell} : m \in \mathbb{N}_0\}\) be a periodic sequence of complex numbers with minimal period \(k_{j\ell} \in \mathbb{N}\), and \(\zeta(s, \alpha_j; b_{j\ell})\) denotes the corresponding periodic Hurwitz zeta-function. Moreover, let
\[
L(\alpha_1, \ldots, \alpha_r) = \{\log(m+\alpha_j) : m \in \mathbb{N}_0, j = 1, \ldots, r\},
\]
\(k_j\) be the least common multiple of the periods \(k_{j1}, \ldots, k_{jl_j}\), and
\[
B_j = \begin{pmatrix}
b_{1j1} & b_{1j2} & \cdots & b_{1jl_j} \\
b_{2j1} & b_{2j2} & \cdots & b_{2jl_j} \\
\cdots & \cdots & \cdots & \cdots \\
b_{kj_1} & b_{kj_2} & \cdots & b_{kj_{l_j}}
\end{pmatrix}, \quad j = 1, \ldots, r.
\]
Theorem 1.2 ([27]). Suppose that the set \( L(\alpha_1, \ldots, \alpha_r) \) is linearly independent over the field of rational numbers \( \mathbb{Q} \), and that \( \text{rank}(B_j) = l_j \), \( j = 1, \ldots, r \). For \( j = 1, \ldots, r \) and \( l = 1, \ldots, l_j \), let \( K_{jl} \in \mathcal{K}(D) \), and let \( f_{jl}(s) \in \mathcal{H}(K_{jl}) \). Then, for every \( \epsilon > 0 \),
\[
\lim_{T \to \infty} \inf \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} |\zeta(s + i\tau, \alpha_j; b_{jl}) - f_{jl}(s)| < \epsilon \right\} > 0.
\]

Note that in the latter theorem the information on the values of \( b_{m,j} \) related only to \( \alpha_j \) is used.

In [4], a mixed universality theorem for the Riemann zeta-function and periodic Hurwitz zeta-functions in the frame of Theorem 1.2 has been proved.

Theorem 1.3 ([4]). Suppose that \( \alpha_1, \ldots, \alpha_r \) are algebraically independent over \( \mathbb{Q} \), and that all hypotheses of Theorem 1.2 for \( b_{jl}, K_{jl} \) and \( f_{jl} \) are satisfied. Moreover, let \( K \in \mathcal{K}(D) \) and \( f(s) \in \mathcal{H}_0(K) \). Then, for every \( \epsilon > 0 \),
\[
\lim_{T \to \infty} \inf \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \epsilon, \quad \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} |\zeta(s + i\tau, \alpha_j; b_{jl}) - f_{jl}(s)| < \epsilon \right\} > 0.
\]

The aim of this paper is to replace the function \( \zeta(s) \) in Theorem 1.3 by zeta-functions of newforms. To state our result, we need some definitions and notation. Let
\[
SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}
\]
be the full modular group. For \( N \in \mathbb{N} \), the subgroup of \( SL_2(\mathbb{Z}) \)
\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0(\text{mod } N) \right\}
\]
is called the Hecke subgroup or the congruence subgroup mod \( N \). Suppose that \( F(z) \) is a holomorphic function in the upper half-plane \( \Im z > 0 \), and \( \kappa \in 2\mathbb{N} \). The function \( F(z) \) is called a cusp form of weight \( \kappa \) and level \( N \) if
\[
F \left( \frac{az + b}{cz + d} \right) = (cz + d)^\kappa F(z) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),
\]
and \( F(z) \) is holomorphic and vanishing at the cusps. In this case, \( F(z) \) has the following Fourier series expansion
\[
F(z) = \sum_{m=1}^\infty c(m)e^{2\pi imz}.
\]
Denote by $S_{\kappa}(\Gamma_0(N))$ the space of all cusp forms of weight $\kappa$ and level $N$. For every $d|N$, the elements of $S_{\kappa}(\Gamma_0(d))$ also belong to $S_{\kappa}(\Gamma_0(N))$. $F \in S_{\kappa}(\Gamma_0(N))$ is called a newform if $F$ is not a cusp form of level less than $N$, and $F$ is an eigenfunction of all Hecke operators. Then we have that $c(1) \neq 0$, so we may assume that $c(1) = 1$, i.e., $F$ is a normalized newform. To each cusp form we can attach the zeta-function

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}. $$

In view of the Deligne estimate [3]

$$|c(m)| \leq m^{\frac{\kappa-1}{2}}d(m),$$

where $d(m)$ denotes the divisor function, the series for $\zeta(s, F)$ converges absolutely for $\sigma > \frac{\kappa}{2} + 1$. In this region, $\zeta(s, F)$ also has a representation by Euler’s product. If $F$ is a newform, then this representation is of the form

$$\zeta(s, F) = \prod_{p|N} \left(1 - \frac{c(p)}{p^s}\right)^{-1} \prod_{p|N} \left(1 - \frac{c(p)}{p^s} + \frac{1}{p^{2s+1-\kappa}}\right)^{-1}.$$ 

Moreover, the function $\zeta(s, F)$ can be analytically continued to an entire function. These and other facts of the theory of modular forms can be found, for example, in [6] and [32].

The universality for zeta-functions of normalized Hecke eigen cusp forms was obtained by Laurinčikas and Matsumoto [23], and for zeta-functions of newforms by Laurinčikas, Matsumoto and Steuding [25].

Denote $D_{\kappa} = \{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\}$.

**Theorem 1.4** ([25]). Suppose that $F$ is a normalized newform of weight $\kappa$ and level $N$, $K \in \mathcal{K}(D_{\kappa})$ and $f(s) \in \mathcal{H}_0(K)$. Then, for every $\epsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, F) - f(s)| < \epsilon \right\} > 0.$$

Our main result is the following theorem. It joins Theorem 1.2 with algebraically independent numbers $\alpha_1, ..., \alpha_r$ over $\mathbb{Q}$ with Theorem 1.4.

**Theorem 1.5.** Suppose that the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that $\text{rank}(B_j) = l_j$, $j = 1, ..., r$. Let $K$ and $f(s)$ be the same as in Theorem 1.4, and $K_{jl}$ and $f_{jl}$ be the same as in Theorem 1.2. Then, for every $\epsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, \alpha_j; b_{jl}) - f_{jl}(s)| < \epsilon \right\} > 0.$$
For the proof of Theorem 1.5, we will apply a modification of the probabilistic method used in [4] which is based on a joint limit theorem in the space of analytic functions. The case of Theorem 1.5 is more complicated because we deal with two strips $D_\kappa$ and $D$.

§2. Joint functional limit theorems

Let $G$ be a region on $\mathbb{C}$. Denote by $H(G)$ the space of analytic functions on $G$ equipped with the topology of uniform convergence on compacta. For $u = l_1 + \cdots + l_r$ and $v = u + 1$, let

$$H^v(D_\kappa, D) = H(D_\kappa) \times H(D) \times \cdots \times H(D).$$

Moreover, we set $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\underline{b} = (b_{11}, \ldots, b_{l_1}, \ldots, b_{r1}, \ldots, b_{rl_r})$. This section is devoted to a limit theorem in the space $H^v(D_\kappa, D)$ for the vector

$$\zeta(\hat{s}, s, \alpha; \underline{b}, F) = (\zeta(\hat{s}, F), \zeta(s, \alpha_1; b_{11}), \ldots, \zeta(s, \alpha_r; b_{rl_r})).$$

Denote by $\mathcal{B}(S)$ the class of Borel sets of the space $S$, let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and define

$$\hat{\Omega} = \prod_p \gamma_p \quad \text{and} \quad \Omega = \prod_{m=0}^\infty \gamma_m,$$

where $\gamma_p = \gamma$ for all primes $p$, and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem, the tori $\hat{\Omega}$ and $\Omega$ with the product topology and pointwise multiplication are compact topological Abelian groups. Therefore, on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$ and $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measures $\hat{m}_H$ and $m_H$, respectively, exist, and we have two probability spaces $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \hat{m}_H)$ and $(\Omega, \mathcal{B}(\Omega), m_H)$. Moreover, let

$$\Omega = \hat{\Omega} \times \prod_{j=1}^r \Omega_j,$$

where $\Omega_j = \Omega$ for all $j = 1, \ldots, r$. Similarly as above, we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, where $m_H$ is the probability Haar measure on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$. Denote by $\hat{\omega}(p)$ the projection of $\hat{\omega} \in \hat{\Omega}$ to $\gamma_p$, and by $\omega_j(m)$ the projection of $\omega_j \in \Omega_j$ to $\gamma_m$. Let $\omega = (\hat{\omega}, \omega_1, \ldots, \omega_r)$ be the elements of $\Omega$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H^v(D_\kappa, D)$-valued random element $\zeta(\hat{s}, s, \alpha, \omega; \underline{b}, F)$ by the formula

$$\zeta(\hat{s}, s, \alpha, \omega; \underline{b}, F) = (\zeta(\hat{s}, \omega, F), \zeta(s, \alpha_1, \omega_1; b_{11}), \ldots, \zeta(s, \alpha_r, \omega_r; b_{rl_r})), $$

$$\zeta(s, \alpha_1, \omega_1; b_{11}), \ldots, \zeta(s, \alpha_r, \omega_r; b_{rl_r})), $$

$$\zeta(s, \alpha_r, \omega_r; b_{rl_r})).$$
where
\[ \zeta(\hat{s}, \hat{\omega}, F) = \prod_{p|N} \left(1 - \frac{c(p)\hat{\omega}(p)}{p^{s}}\right)^{-1} \prod_{p|N} \left(1 - \frac{c(p)\hat{\omega}(p)}{p^{s}} + \frac{\hat{\omega}^2(p)}{p^{2s-1+\kappa}}\right)^{-1} \]

and
\[ \zeta(s, \alpha_j, \omega_j; b_{jl}) = \sum_{m=0}^{\infty} \frac{b_{mjl}\omega_j(m)}{(m+\alpha_j)^s}, \quad j = 1, \ldots, r, \quad l = 1, \ldots, l_j. \]

Denote by \( P_\zeta \) the distribution of the random element \( \zeta(\hat{s}, s, \alpha, \omega; b, F) \), i.e., for \( A \in \mathcal{B}(H^v(D_\kappa, D)) \),
\[ P_\zeta(A) = m_H(\omega \in \Omega : \zeta(\hat{s}, s, \alpha, \omega; b, F) \in A). \]

**Theorem 2.1.** Suppose that the numbers \( \alpha_1, \ldots, \alpha_r \) are algebraically independent over \( \mathbb{Q} \). Then
\[ P_T(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas}\{\tau \in [0, T] : \zeta(\hat{s}+i\tau, s+i\tau, \alpha; b, F) \in A\}, \quad A \in \mathcal{B}(H^v(D_\kappa, D)), \]
converges weakly to \( P_\zeta \) as \( T \to \infty \).

The proof of Theorem 2.1 is similar to that Theorem 4 of [4], therefore, we will give only its sketch.

Let \( \sigma_1 > \frac{1}{2} \) be a fixed number, and
\[ v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}, \quad m, n \in \mathbb{N}, \]
\[ v_n(m, \alpha_j) = \exp\left\{-\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^{\sigma_1}\right\}, \quad m, n \in \mathbb{N}_0, \quad j = 1, \ldots, r. \]

Define
\[ \zeta_n(\hat{s}, F) = \sum_{m=1}^{\infty} \frac{c(m)v_n(m)}{m^s} \]
and
\[ \zeta_n(s, \alpha_j; b_{jl}) = \sum_{m=0}^{\infty} \frac{b_{mjl}v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \ldots, r, \quad l = 1, \ldots, l_j. \]

By a standard method involving an application of the Mellin formula can be proved that the series for \( \zeta_n(\hat{s}, F) \) and \( \zeta_n(s, \alpha_j; b_{jl}) \) are absolutely convergent for \( \Re \hat{s} > \frac{s}{2} \) and \( \sigma > \frac{1}{2} \), respectively.

The formula
\[ \hat{\omega}(m) = \prod_{p || m} \omega^l(p), \quad m \in \mathbb{N}, \]
where \( p^l \parallel m \) means that a power \( p^l \) occurs precisely in the canonical representation of \( m \), extends the functions \( \hat{\omega}(p) \) to the set \( \mathbb{N} \). Define

\[
\zeta_n(\hat{s}, \hat{\omega}, F) = \sum_{m=1}^{\infty} \frac{c(m)\hat{\omega}(m)v_n(m)}{m^s},
\]

and

\[
\zeta_n(s, \alpha_j, \omega_j; b_{jl}) = \sum_{m=0}^{\infty} \frac{b_{mjl}\omega_j(m)v_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r, \quad l = 1, \ldots, l_j,
\]

the series being absolutely convergent for \( \Re\hat{s} > \frac{\kappa}{2} \) and \( \sigma > \frac{1}{2} \), respectively. Moreover, we set

\[
\underline{\zeta}_n(\hat{s}, s, \underline{\alpha}; \underline{b}, F) = (\zeta_n(\hat{s}, F), \zeta_n(s, \alpha_1; b_{11}), \ldots, \zeta_n(s, \alpha_1; b_{1l_1}), \ldots, \zeta_n(s, \alpha_r; b_{rl_r}))
\]

and

\[
\underline{\zeta}_n(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{b}, F) = (\zeta_n(\hat{s}, \hat{\omega}, F), \zeta_n(s, \alpha_1, \omega_1; b_{11}), \ldots, \zeta_n(s, \alpha_1, \omega_1; b_{1l_1}), \ldots, \zeta_n(s, \alpha_r, \omega_r; b_{rl_r})).
\]

The first step in the proof of Theorem 2.1 is the following statement.

**Lemma 2.2.** Suppose that the numbers \( \alpha_1, \ldots, \alpha_r \) are algebraically independent over \( \mathbb{Q} \). Then the probability measures

\[
P_{T, n}(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \zeta_n(\hat{s} + i\tau, s + i\tau, \underline{\alpha}; \underline{b}, F) \in A \right\}
\]

and

\[
P_{T, n, \omega}(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \underline{\zeta}_n(\hat{s} + i\tau, s + i\tau, \underline{\alpha}; \underline{\omega}, \underline{b}, F) \in A \right\},
\]

\( A \in \mathcal{B}(H^v(D_\kappa, D)) \), both converge weakly, for any fixed \( \omega \in \Omega \), to the same probability measure \( P_n \) on \( (H^v(D_\kappa, D), \mathcal{B}(H^v(D_\kappa, D))) \) as \( T \to \infty \).

Lemma 2.2 is a result of the application of Theorem 5.1 from [2] and a limit theorem on the torus \( \underline{\Omega} \) which is contained in the next lemma obtained in [4], Lemma 1. Let \( \mathcal{P} \) be the set of all prime numbers.

**Lemma 2.3.** Suppose that the numbers \( \alpha_1, \ldots, \alpha_r \) are algebraically independent over \( \mathbb{Q} \). Then

\[
\frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \left( (p^{-i\tau} : p \in \mathcal{P}), (m + \alpha_j)^{-i\tau} : m \in \mathbb{N}_0, j = 1, \ldots, r \right) \in A \right\},
\]

\( A \in \mathcal{B}(\Omega) \),

converges weakly to the Haar measure \( m_H \) as \( T \to \infty \).
The next step of the proof of Theorem 2.1 contains the results which allow to pass from the vector $\zeta_n(\hat{s}, s, \underline{\alpha}; \underline{b}, F)$ to $\zeta(\hat{s}, s, \underline{\alpha}; \underline{b}, F)$. For this, we need a metric on $H^v(D_\kappa, D)$.

It is well known that there exist a sequence $\{\hat{K}_m : m \in \mathbb{N}\}$ of compact subsets of $D_\kappa$, and a sequence $\{K_m : m \in \mathbb{N}\}$ of $D$ such that

$$D_\kappa = \bigcup_{m=1}^\infty \hat{K}_m \quad \text{and} \quad D = \bigcup_{m=1}^\infty K_m.$$

Moreover, the sets $\hat{K}_m$ and $K_m$ can be chosen to satisfy $\hat{K}_m \subset \hat{K}_{m+1}$, $K_m \subset K_{m+1}$ for all $m \in \mathbb{N}$, and, for every compact subsets $\hat{K} \subset D_\kappa$ and $K \subset D$, there exists $\hat{m}, m \in \mathbb{N}$ such that $\hat{K} \subset \hat{K}_{\hat{m}}$ and $K \subset K_m$.

For $\hat{g}_1, \hat{g}_2 \in H(D_\kappa)$ and $g_1, g_2 \in H(D)$, define

$$\hat{\rho}(\hat{g}_1, \hat{g}_2) = \sum_{m=1}^\infty 2^{-m} \frac{\sup_{s \in \hat{K}_m} |\hat{g}_1(s) - \hat{g}_2(s)|}{1 + \sup_{s \in \hat{K}_m} |\hat{g}_1(s) - \hat{g}_2(s)|}$$

and

$$\rho(g_1, g_2) = \sum_{m=1}^\infty 2^{-m} \frac{\sup_{s \in K_m} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_m} |g_1(s) - g_2(s)|}.$$

Then $\hat{\rho}$ and $\rho$ are the metrics on $H(D_\kappa)$ and $H(D)$, respectively, inducing the topology of uniform convergence on compacta. For

$$\underline{f} = (\hat{f}, f_{11}, \ldots, f_{1l_1}, \ldots, f_{r1}, \ldots, f_{rl_r}), \quad \underline{g} = (\hat{g}, g_{11}, \ldots, g_{1l_1}, \ldots, g_{r1}, \ldots, g_{rl_r}) \in H^v(D_\kappa, D),$$

let

$$\rho_v(\underline{f}, \underline{g}) = \max(\hat{\rho}(\hat{f}, \hat{g}), \max_{1 \leq j \leq r} \max_{1 \leq i \leq l_i} \rho(f_{ji}, g_{ji})).$$

Then $\rho_v$ is a metric on the space $H^v(D_\kappa, D)$ which induces its topology.

Now we are able to approximate $\zeta(\hat{s}, s, \underline{\alpha}; \underline{b}, F)$ by $\zeta_n(\hat{s}, s, \underline{\alpha}; \underline{b}, F)$ in the mean.

**Lemma 2.4.** We have

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho_v(\zeta(\hat{s} + i\tau, s + i\tau, \underline{\alpha}; \underline{b}, F), \zeta_n(\hat{s} + i\tau, s + i\tau, \underline{\alpha}; \underline{b}, F)) \, d\tau = 0.$$

As it was observed in [25], the zeta-functions associated to newforms constitute a subclass of Matsumoto zeta-functions. Therefore, the lemma follows from the relation

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \hat{\rho}(\zeta(\hat{s} + i\tau, F), \zeta_n(\hat{s} + i\tau, F)) \, d\tau = 0.$$
which is a corollary of Lemma 8 from [13], and from the equalities

\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} \rho (\zeta(s + i\tau, \alpha_j; b_{jl}), \zeta_n(s + i\tau, \alpha_j; b_{jl})) \, d\tau = 0, \\
 \quad j = 1, ..., r, \quad l = 1, ..., l_j,
\]

which are deduced from formula (3) of [18].

An analogue of Lemma 2.4 is also true for \( \zeta(s, \underline{\alpha}, \underline{\omega}; \underline{b}, F) \) and \( \zeta_n(s, \underline{\alpha}, \underline{\omega}; \underline{b}, F) \), where

\[
\zeta(s, \underline{\alpha}, \underline{\omega}; \underline{b}, F) = (\zeta(s, \hat{\omega}, F), \zeta(s, \alpha_1, \omega_1; b_{11}), ..., \zeta(s, \alpha_r, \omega_r; b_{rl}), ..., \\
\zeta(s, \alpha_r, \omega_r; b_{rl}), \zeta(s, \omega_r; b_{r1}), ..., \zeta(s, \omega_r; b_{r1})).
\]

**Lemma 2.5.** Suppose that the numbers \( \alpha_1, ..., \alpha_r \) are algebraically independent over \( \mathbb{Q} \). Then, for almost all \( \underline{\omega} \in \Omega \), we have

\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} \rho_v(\zeta_n(s = +i\tau,  \underline{\alpha}, \underline{\omega}; \underline{b}), \zeta_n(s ==n+i\tau,  \underline{\alpha},  \underline{\omega}=;\underline{b})) \, d\tau = 0.
\]

**Proof.** Lemma 11 of [13], for almost all \( \hat{\omega} \in \hat{\Omega} \), implies the relation

(2.1) \[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} \rho (\zeta(s + i\tau, \hat{\omega}), \zeta_n(s + i\tau, \hat{\omega})) \, d\tau = 0.
\]

Let

\[
\rho_u(f, g) = \max_{1 \leq j \leq r} \max_{1 \leq l \leq l_j} \rho(f_{jl}, g_{jl}).
\]

Denote by \( m_H \) the Haar measure on \( (\Omega, \mathcal{B}(\Omega)) \), where \( \Omega = \Omega_1 \times \cdots \times \Omega_r \). Then, for almost all \( \underline{\omega} \in \Omega \),

(2.2) \[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} \rho_u\left(\zeta(s + i\tau, \underline{\alpha}, \underline{\omega}; \underline{b}), \zeta_n(s + i\tau, \underline{\alpha}, \underline{\omega}; \underline{b})\right) \, d\tau = 0,
\]

see formula (2.5) of [4]. Here \( \zeta(s, \alpha, \omega; b, F) \) and \( \zeta_n(s, \alpha, \omega; b, F) \) are obtained from \( \zeta(s, \alpha, \omega; b, F) \) and \( \zeta_n(s, \alpha, \omega; b, F) \) by removing \( \zeta(s, \omega, F) \) and \( \zeta_n(s, \omega, F) \), respectively. Since the measure \( m_H \) is the product of the measures \( \hat{m}_H \) and \( \underline{m}_H \), the lemma follows from (2.1), (2.2), and the definition of \( \rho_v \). \( \square \)

We can deduce from Lemmas 2.2 and 2.4 the weak convergence for the measure \( P_T \), as \( T \to \infty \). However, the identification of the limit measure requires the next lemma.
Lemma 2.6. Suppose that the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the probability measures $P_T$ and

$$
\frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \zeta(\hat{s} + i\tau, s + i\tau, \alpha; \omega; \mathfrak{b}, F) \in A \right\}, \quad A \in \mathcal{B}(H^v(D_K, D)),
$$

both converge weakly, for almost all $\omega \in \Omega$, to the same probability measure $P$ on $(H^v(D_K, D), \mathcal{B}(H^v(D_K, D)))$ as $T \to \infty$.

Proof. We omit the details which are similar to those of [4]. Let $\theta$ be a random variable defined on a certain probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$ and uniformly distributed on $[0, 1]$. On the later probability space, define the $H^v(D_K, D)$-valued random element $X_{T,n}$ by the formula

$$
X_{T,n}(\hat{s}, s) = \left( X_{T,n}(\hat{s}), X_{T,n,1,1}(s), ..., X_{T,n,1,l_j}(s), ..., X_{T,n,r,1}(s), ..., X_{T,n,r,1,l_r}(s) \right)
$$

$$
def \zeta_n(\hat{s} + i\theta T, s + i\theta T, \alpha; \mathfrak{b}, F).
$$

Then, denoting by $\overset{D}{\rightarrow}$ the convergence in distribution, we have, by Lemma 2.2, that

$$
(\ref{2.3}) \quad X_{T,n}(\hat{s}, s) \overset{D}{\rightarrow} X_n(\hat{s}, s),
$$

where $X_n(\hat{s}, s)$ is the $H^v(D_K, D)$-valued random element with the distribution $P_n$ ($P_n$ is the limit measure in Lemma 2.2). Our first task is to prove the tightness of the family $\{P_n : n \in \mathbb{N}\}$.

In view of the Deligne estimate (1.1), the well-known properties of the mean square of Dirichlet series and Cauchy integral formula show that, for all $n \in \mathbb{N},$

$$
(\ref{2.4}) \quad \limsup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in \hat{K}_m} |\zeta_n(\hat{s} + i\tau, s + i\tau, F)| d\tau \leq \hat{C}_m \left( \sum_{k=1}^{\infty} \frac{c^2(k)}{k^{2\hat{\sigma}_m}} \right)^{\frac{1}{2}}, \quad m \in \mathbb{N},
$$

with some $\hat{C}_m > 0$ and $\hat{\sigma}_m > \frac{\kappa}{2}$. Similarly, for all $n \in \mathbb{N},$

$$
(\ref{2.5}) \quad \limsup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in K_m} |\zeta_n(s + i\tau, \alpha_j; b_{jl})| d\tau \leq C_m \left( \sum_{k=0}^{\infty} \frac{|b_{kjl}|^2}{(k + \alpha_j)^{2\sigma_m}} \right)^{\frac{1}{2}},
$$

with some $C_m > 0$ and $\sigma_m > \frac{1}{2}$, $m \in \mathbb{N}$, $j = 1, ..., r$, $l = 1, ..., l_j$. The compact sets $\hat{K}_m$ and $K_m$ come from the definition of the metric $\rho_v$.

Now let

$$
\hat{R}_m = \hat{C}_m \left( \sum_{k=1}^{\infty} \frac{c^2(k)}{k^{2\hat{\sigma}_m}} \right)^{\frac{1}{2}}, \quad R_{jlm} = C_m \left( \sum_{k=0}^{\infty} \frac{|b_{kjl}|^2}{(k + \alpha_j)^{2\sigma_m}} \right)^{\frac{1}{2}}.
$$
Taking $\hat{M}_m = \hat{R}_m 2^{m+1} \epsilon^{-1}$ and $M_{jlm} = R_{jlm} 2^{u+m+1} \epsilon^{-1}$, where $m \in \mathbb{N}$ and $\epsilon > 0$ is an arbitrary number, we find by (2.4) and (2.5) that

$$\limsup_{T \rightarrow \infty} \mathbb{P} \left( \left( \sup_{\hat{s} \in \hat{K}_m} |X_{T,n}(\hat{s})| > \hat{M}_m \right) \vee \left( \exists j, l : \sup_{s \in K_m} |X_{T,n,j,l}(s)| > M_{jlm} \right) \right) \leq \frac{\epsilon}{2^m}.$$ 

This together with (2.3) implies

$$\mathbb{P} \left( \left( \sup_{\hat{s} \in \hat{K}_m} |X_n(\hat{s})| > \hat{M}_m \right) \vee \left( \exists j, l : \sup_{s \in K_m} |X_{n,j,l}(s)| > M_{jlm} \right) \right) \leq \frac{\epsilon}{2^m},$$

where $X_n(\hat{s})$, $X_{n,j,l}(s)$, $j = 1, \ldots, r$, $l = 1, \ldots, l_j$, are the elements of the random vector $X_n(\hat{s}, s)$. From this, we obtain that

$$P_n(H_\epsilon^{v}) \geq 1 - \epsilon,$$

where

$$H_\epsilon^{v} = \left\{ f \in H^{v}(D_{\kappa}, D) : \sup_{\hat{s} \in \hat{K}_m} |\hat{f}(\hat{s})| \leq \hat{M}_m, \sup_{s \in K_m} |f_{jl}(s)| \leq M_{jlm}, \quad \right.$$

$$j = 1, \ldots, r, l = 1, \ldots, l_j, m \in \mathbb{N} \right\}$$

is a compact subset of the space $H^{v}(D_{\kappa}, D)$. This proves the tightness of the family $\{P_n : n \in \mathbb{N}\}$.

By the Prokhorov theorem, the family $\{P_n : n \in \mathbb{N}\}$ is relatively compact. Hence, there exists a sequence $n_k \rightarrow \infty$ and a probability measure $P$ on $(H^{v}(D_{\kappa}, D), \mathcal{B}(H^{v}(D_{\kappa}, D)))$ such that

$$X_{n_k}(\hat{s}, s) \overset{D}{\rightarrow}_{k \rightarrow \infty} P.$$ 

(2.6)

On the probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$, define the $H^{v}(D_{\kappa}, D)$-valued random element $X_T(\hat{s}, s)$ by the formula

$$X_T(\hat{s}, s) = \zeta(\hat{s} + i \theta T, s + i \theta T, \alpha; \omega; \mathbf{b}, F).$$

Then Lemma 2.4 yields, for every $\epsilon > 0$, the relation

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \mathbb{P} \left( \rho_v \left( X_T(\hat{s}, s), X_{T,n}(\hat{s}, s) \right) \geq \epsilon \right) = 0.$$ 

This, (2.3), (2.6) and Theorem 4.2 of [2] show that $X_T(\hat{s}, s) \overset{D}{\rightarrow}_{T \rightarrow \infty} P$, or $P_T$ converges weakly to $P$ as $T \rightarrow \infty$.

Using the random elements

$$\zeta_n(\hat{s} + i \theta T, s + i \theta T, \alpha, \omega; \mathbf{b}, F)$$
and

\[ \zeta(s + i\theta T, s + i\theta T, \alpha, \omega; b, F), \]

as well as Lemma 2.5, we obtain in a similar way that the second measure of Lemma 2.6 also converges weakly to \( P \) as \( T \to \infty. \)

The end of the proof of Theorem 2.1 is standard. We apply Lemma 2.6, the ergodicity of the one-parameter group \( \{ \Phi_{\tau} : t \in \mathbb{R} \} \) of measurable and measure preserving transformations on \( \Omega \), where, for \( \omega \in \Omega \) and \( \tau \in \mathbb{R} \),

\[ \Phi_{\tau}(\omega) = ((p^{-i\tau} : p \in \mathcal{P}), ((m + \alpha_j)^{-i\tau} : m \in \mathbb{N}_0, j = 1, ..., r)) \omega, \]

see Lemma 7 of [20], as well as the classical Birkhoff-Khintchine theorem.

\section{3. Support of the measure \( P_{\underline{\zeta}} \)}

The space \( H^u(D_\kappa, D) \) is separable, therefore the support of \( P_{\underline{\zeta}} \) is the minimal closed set \( S_{P_{\underline{\zeta}}} \subset H^u(D_\kappa, D) \) such that \( P_{\underline{\zeta}}(S_{P_{\underline{\zeta}}}) = 1 \). The set \( S_{P_{\underline{\zeta}}} \) consists of all points \( g \in H^u(D_\kappa, D) \) such that, for every open neighbourhood \( G \) of \( g \), the inequality \( P_{\underline{\zeta}}(G) > 0 \) holds.

Define

\[ S_\kappa = \{ g \in H(D_\kappa) : g(s) \neq 0 \text{ or } g(s) \equiv 0 \}. \]

\textbf{Theorem 3.1.} Suppose that the numbers \( \alpha_1, ..., \alpha_r \) are algebraically independent over \( \mathbb{Q} \), and that \( \text{rank}(B_j) = l_j, j = 1, ..., r \). Then the support of the measure \( P_{\underline{\zeta}} \) is the set \( S_\kappa \times H^u(D) \).

\textbf{Proof.} We have that

\[ H^u(D_\kappa, D) = H(D_\kappa) \times H^u(D). \]

In view of separability of the above spaces, the equality

\[ \mathcal{B}(H^u(D_\kappa, D)) = \mathcal{B}(H(D_\kappa)) \times \mathcal{B}(H^u(D)) \]

is true [2]. Therefore, it suffices to investigate \( P_{\underline{\zeta}}(A) \) for \( A = B \times C \), where \( B \in \mathcal{B}(H(D_\kappa)) \) and \( C \in \mathcal{B}(H^u(D)) \). We already have mentioned that the measure \( m_H \) is the product of the measures \( \hat{m}_H \) and \( m_H \). Therefore, we have that

\begin{align*}
P_{\underline{\zeta}}(A) &= m_H \left( \omega \in \Omega : \zeta(\hat{s}, s, \alpha, \omega; b, F) \in A \right) \\
&= m_H \left( \omega \in \Omega : \zeta(\hat{s}, \hat{\omega}, F) \in B, \zeta(s, \alpha, \omega; b) \in C \right) \\
&= \hat{m}_H \left( \hat{\omega} \in \hat{\Omega} : \zeta(\hat{s}, \hat{\omega}, F) \in B \right) m_H \left( \omega \in \Omega : \zeta(s, \alpha, \omega; b) \in C \right). \tag{3.1}
\end{align*}
In [25], Lemma 9, it was obtained that the support of the random element \( \zeta(\hat{s}, \hat{\omega}, F) \) is the set \( S_\kappa \), i.e., \( S_\kappa \) is a minimal closed subset of \( H(D_\kappa) \) such that

\[
(3.2) \quad \hat{m}_H \left( \hat{\omega} \in \Omega : \zeta(\hat{s}, \hat{\omega}, F) \in S_\kappa \right) = 1.
\]

To be precise, in [25] the space \( H(D_{\kappa,M}) \), where \( D_{\kappa,M} = \{ s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}, |t| < M \} \), is considered, however, all arguments remain valid for the space \( H(D_\kappa) \). Also, in [27], Theorem 3.1, it was proved that the support of the random element \( \zeta(s, \alpha, \omega; \underline{b}) \) is the whole of \( H^u(D) \), i.e., \( H^u(D) \) is a minimal closed set of \( H^u(D) \) such that

\[
\hat{m}_H \left( \hat{\omega} \in \Omega : \zeta(s, \alpha, \omega; \underline{b}) \in H^u(D) \right) = 1.
\]

From this and (3.1), (3.2), the theorem follows.

\[\square\]

§ 4. Proof of Theorem 1.5

We first recall the Mergelyan theorem on the approximation of analytic functions by polynomials.

**Lemma 4.1.** Suppose that \( K \) is a compact subset on the complex plane with connected complement, and that \( f(s) \) is a continuous function on \( K \) which is analytic in the interior of \( K \). Then, for every \( \epsilon > 0 \), there exists a polynomial \( p(s) \) such that

\[
\sup_{s \in K} |f(s) - p(s)| < \epsilon.
\]

**Proof of the lemma** is given in [30], see also [39].

**Proof. of Theorem 1.5.** In view of Lemma 4.1, there exist polynomials \( p(s) \) and \( p_{jl}(s) \) such that

\[
(4.1) \quad \sup_{s \in K} |f(s) - p(s)| < \frac{\epsilon}{4}
\]

and

\[
(4.2) \quad \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} |f_{jl}(s) - p_{jl}(s)| < \frac{\epsilon}{2}.
\]

Since \( f(s) \neq 0 \) on \( K \), we have that \( p(s) \neq 0 \) on \( K \) as well if \( \epsilon \) is small enough. Therefore, we can define a continuous branch of \( \log p(s) \) on \( K \) which will be analytic in the interior of \( K \). By Lemma 4.1 again, there exists a polynomial \( q(s) \) such that

\[
\sup_{s \in K} \left| p(s) - e^{q(s)} \right| < \frac{\epsilon}{4}.
\]
From this and (4.1), we have that
\[
(4.3) \quad \sup_{s \in K} |f(s) - e^{q(s)}| < \frac{\epsilon}{2}.
\]
Define
\[
G = \left\{ g \in H^v(D_\kappa, D) : \sup_{s \in K} |\hat{g}(s) - e^{q(s)}| < \frac{\epsilon}{2}, \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} |g_{jl}(s) - P_{jl}(s)| < \frac{\epsilon}{2} \right\}.
\]
In view of Theorem 3.1, the vector \((e^{q(s)}, p_{jl}, j = 1, \ldots, r, l = 1, \ldots, l_j)\), is an element of the support of the measure \(P_{\zeta}\). Since \(G\) is an open set, this shows that \(P_{\zeta}(G) > 0\). Therefore, Theorem 2.1 together with an equivalent of the weak convergence in terms of open sets yields the inequality
\[
\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, F) - e^{q(s)}| < \frac{\epsilon}{2}, \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} |\zeta(s + i\tau, \alpha_j; b_{jl}) - p_{jl}(s)| < \frac{\epsilon}{2} \right\} > 0.
\]
From this, (4.2) and (4.3), the assertion of the theorem follows. \(\square\)

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**References**


