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On joint universality for the zeta-functions of newforms and periodic Hurwitz zeta-functions

By

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Abstract

In the paper, a short survey on universality for zeta-functions both having and not having Euler's product is given. Also, a joint universality theorem for the zeta-function of newforms and periodic Hurwitz zeta-functions is proved.

§ 1. Introduction

Let Δ be a vertical strip on the complex plane $\mathbb{C}$. Denote by $\mathcal{K}(\Delta)$ the class of compact subsets of the strip Δ with connected complements, for a compact set K, denote by $\mathcal{H}(K)$ the class of continuous functions on K which are analytic in the interior of K, and by $\mathcal{H}_0(K)$ the subclass of $\mathcal{H}(K)$ consisting of functions which are non-vanishing on K.

It is well known, see [1], [5], [12], [14], [29], [36], [37], [38], that the Riemann zeta-function $\zeta(s)$, $s=\sigma+it$, which is defined, for $\sigma>1$, by

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}=\prod_p\left(1-\frac{1}{p^s}\right)^{-1},$$

and is analytically continued to the whole complex plane, except for a simple pole at $s=1$, is universal in the sense that if $K\in\mathcal{K}(D)$, $D=\left\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\right\}$, and $f\in\mathcal{H}_0(K)$, then, for every $\epsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\epsilon\right\}>0.$$

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Here and in the sequel, p denotes a prime number, and $\mathrm{meas}\{A\}$ stands for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

Now let α, $0 < \alpha \le 1$, be a fixed parameter. Then the Hurwitz zeta-function $\zeta(s, \alpha)$ which is defined, for $\sigma > 1$, by

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

and is analytically continued to the whole complex plane, except for a simple pole at $s = 1$, is also in a similar sense universal. Namely, [1], [5], [21], [35] if α is a transcendental or rational number $\neq 1, \frac{1}{2}$, $K \in \mathcal{K}(D)$ and $f \in \mathcal{H}(K)$, then, for every $\epsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \epsilon \right\} > 0.$$

Since the cases $\alpha = 1$ $(\zeta(s, 1) = \zeta(s))$ and $\alpha = \frac{1}{2}$ $\big(\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)\big)$ in the later statement are excluded, the function $\zeta(s, \alpha)$ has no Euler's product over primes, and this is reflected in its universality: the shifts $\zeta(s + i\tau, \alpha)$ approximate every function $f \in \mathcal{H}(K)$, the restriction of the class $\mathcal{H}_0(K)$ is removed. Thus the universality of $\zeta(s, \alpha)$ is more general than that of $\zeta(s)$, and is called a strong universality.

Note that the universality of $\zeta(s, \alpha)$ with algebraic irrational parameter α remains an open problem.

Mishou in [31] obtained a very interesting joint universality theorem for the functions $\zeta(s)$ and $\zeta(s, \alpha)$.

Theorem 1.1 ([31]). *Suppose that α is a transcendental number, $K_1, K_2 \in \mathcal{K}(D)$, $f_1 \in \mathcal{H}_0(K)$ and $f_2 \in \mathcal{H}(D)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \Big\{ \tau \in [0, T] : \sup_{s \in K_1} |\zeta(s + i\tau) - f_1(s)| < \epsilon, \\ \sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \epsilon \Big\} > 0.$$

Theorem 1.1 joins the universality and strong universality. We will call this type of the joint universality a mixed universality.

The functions $\zeta(s)$ and $\zeta(s, \alpha)$ have their generalizations. Suppose that $\mathfrak{a} = \{a_m : m \in \mathbb{N}\}$ and $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}\}$ are two periodic sequences of complex numbers with minimal periods $k_1 \in \mathbb{N}$ and $k_2 \in \mathbb{N}$, respectively. Then the functions

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} \quad \text{and} \quad \zeta(s, \alpha; \mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m + \alpha)^s}, \quad \sigma > 1,$$

are called the periodic zeta and periodic Hurwitz zeta-function, respectively. The equalities

$$\zeta(s;\mathfrak{a}) = \frac{1}{k_1^s}\sum_{k=1}^{k_1} a_k \zeta\Big(s, \frac{k}{k_1}\Big), \quad \zeta(s,\alpha;\mathfrak{b}) = \frac{1}{k_2^s}\sum_{k=0}^{k_2-1} b_k \zeta\Big(s, \frac{k+\alpha}{k_2}\Big)$$

give for the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ meromorphic continuation to the whole complex plane with possible simple pole at $s = 1$.

The universality of the function $\zeta(s;\mathfrak{a})$ with multiplicative sequence $\mathfrak{a}$, has been studied by Steuding [36], and Laurinčikas and Šiaučiūnas [28]. In a general case, the problem was solved by Kaczorowski [10].

The strong universality of the function $\zeta(s,\alpha;\mathfrak{b})$ with transcendental parameter α has been obtained by Javtokas and Laurinčikas [7], [8]. Nakamura [34] studied $\zeta(s,\alpha;\mathfrak{b})$ with a special bounded sequence.

A generalization of Theorem 1.1 for the functions $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ with multiplicative sequence $\mathfrak{a}$ has been obtained in [11]. A joint universality theorem for periodic zeta-functions with multiplicative coefficients satisfying a certain "independence" condition has been proved in [22]. The joint universality of Hurwitz zeta-functions by different methods has been considered in [33] and [19]. A series of works [15]-[18] and [9], [26], [27] are devoted to joint universality of periodic Hurwitz zeta-functions. A mixed universality theorem for zeta-functions with periodic coefficients can be found in [20].

In [24], Laurinčikas and Matsumoto observed that, for Lerch zeta-functions

$$L(\lambda_j, \alpha_j, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m}}{(m+\alpha_j)^s}, \quad \sigma > 1, \quad j = 1, ..., r,$$

a more general setting of joint universality is possible. To each parameter α_j, they attached a collection of the parameters λ_j. For periodic Hurwitz zeta-functions, the latter idea was applied by Laurinčikas [18], and Laurinčikas and Skerstonaitė [27]. We will state the latter result. For $j = 1, ..., r$, let $l_j \in \mathbb{N}$, and, for $j = 1, ..., r$ and $l = 1, ..., l_j$, let $\mathfrak{b}_{jl} = \{b_{mjl} : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k_{jl} \in \mathbb{N}$, and $\zeta(s, \alpha_j; \mathfrak{b}_{jl})$ denotes the corresponding periodic Hurwitz zeta-function. Moreover, let

$$L(\alpha_1, ..., \alpha_r) = \{\log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, ..., r\},$$

k_j be the least common multiple of the periods $k_{j1}, ..., k_{jl_j}$, and

$$B_j = \begin{pmatrix} b_{1j1} & b_{1j2} & \dots & b_{1jl_j} \\ b_{2j1} & b_{2j2} & \dots & b_{2jl_j} \\ \dots & \dots & \dots & \dots \\ b_{k_j j1} & b_{k_j j2} & \dots & b_{k_j j l_j} \end{pmatrix}, \quad j = 1, ..., r.$$

Theorem 1.2 ([27]). *Suppose that the set $L(\alpha_1, ..., \alpha_r)$ is linearly independent over the field of rational numbers $\mathbb{Q}$, and that* $\operatorname{rank}(B_j) = l_j$, $j = 1, ..., r$. *For $j = 1, ..., r$ and $l = 1, ..., l_j$, let $K_{jl} \in \mathcal{K}(D)$, and let $f_{jl}(s) \in \mathcal{H}(K_{jl})$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas \left\{ \tau \in [0,T] : \sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} |\zeta(s+i\tau, \alpha_j; \mathfrak{b}_{jl}) - f_{jl}(s)| < \epsilon \right\} > 0.$$

Note that in the latter theorem the information on the values of b_{mjl} related only to α_j is used.

In [4], a mixed universality theorem for the Riemann zeta-function and periodic Hurwitz zeta-functions in the frame of Theorem 1.2 has been proved.

Theorem 1.3 ([4]). *Suppose that $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that all hypotheses of Theorem 1.2 for $\mathfrak{b}_{jl}$, K_{jl} and f_{jl} are satisfied. Moreover, let $K \in \mathcal{K}(D)$ and $f(s) \in \mathcal{H}_0(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \Big\{ \tau \in [0;T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \epsilon,$$
$$\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} |\zeta(s+i\tau, \alpha_j; \mathfrak{b}_{jl}) - f_{jl}(s)| < \epsilon \Big\} > 0.$$

The aim of this paper is to replace the function $\zeta(s)$ in Theorem 1.3 by zeta-functions of newforms. To state our result, we need some definitions and notation. Let

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a\ b \\ c\ d \end{pmatrix} : \ a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

be the full modular group. For $N \in \mathbb{N}$, the subgroup of $SL_2(\mathbb{Z})$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a\ b \\ c\ d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 (\operatorname{mod} N) \right\}$$

is called the Hecke subgroup or the congruence subgroup mod N. Suppose that $F(z)$ is a holomorphic function in the upper half-plane $\Im z > 0$, and $\kappa \in 2\mathbb{N}$. The function $F(z)$ is called a cusp form of weight κ and level N if

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^\kappa F(z) \quad \text{for all} \quad \begin{pmatrix} a\ b \\ c\ d \end{pmatrix} \in \Gamma_0(N),$$

and $F(z)$ is holomorphic and vanishing at the cusps. In this case, $F(z)$ has the following Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m) \mathrm{e}^{2\pi i m z}.$$

Denote by $S_\kappa(\Gamma_0(N))$ the space of all cusp forms of weight κ and level N. For every $d \mid N$, the elements of $S_\kappa(\Gamma_0(d))$ also belong to $S_\kappa(\Gamma_0(N))$. $F \in S_\kappa(\Gamma_0(N))$ is called a newform if F is not a cusp form of level less than N, and F is an eigenfunction of all Hecke operators. Then we have that $c(1) \neq 0$, so we may assume that $c(1) = 1$, i.e., F is a normalized newform. To each cusp form we can attach the zeta-function

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$

In view of the Deligne estimate [3]

$$|c(m)| \leq m^{\frac{\kappa-1}{2}} d(m), \tag{1.1}$$

where $d(m)$ denotes the divisor function, the series for $\zeta(s, F)$ converges absolutely for $\sigma > \frac{\kappa+1}{2}$. In this region, $\zeta(s, F)$ also has a representation by Euler's product. If F is a newform, then this representation is of the form

$$\zeta(s, F) = \prod_{p \mid N} \left(1 - \frac{c(p)}{p^s}\right)^{-1} \prod_{p \nmid N} \left(1 - \frac{c(p)}{p^s} + \frac{1}{p^{2s+1-\kappa}}\right)^{-1}.$$

Moreover, the function $\zeta(s, F)$ can by analytically continued to an entire function. These and other facts of the theory of modular forms can be found, for example, in [6] and [32].

The universality for zeta-functions of normalized Hecke eigen cusp forms was obtained by Laurinčikas and Matsumoto [23], and for zeta-functions of newforms by Laurinčikas, Matsumoto and Steuding [25].

Denote $D_\kappa = \left\{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\right\}$.

Theorem 1.4 ([25]). *Suppose that F is a normalized newform of weight κ and level N, $K \in \mathcal{K}(D_\kappa)$ and $f(s) \in \mathcal{H}_0(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} meas \left\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, F) - f(s)| < \epsilon\right\} > 0.$$

Our main result is the following theorem. It joins Theorem 1.2 with algebraically independent numbers $\alpha_1, ..., \alpha_r$ over $\mathbb{Q}$ with Theorem 1.4.

Theorem 1.5. *Suppose that the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that $\operatorname{rank}(B_j) = l_j$, $j = 1, ..., r$. Let K and $f(s)$ be the same as in Theorem 1.4, and K_{jl} and f_{jl} be the same as in Theorem 1.2. Then, for every $\epsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas}\Big\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, F) - f(s)| < \epsilon,$$
$$\sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l_j} \sup_{s \in K_{jl}} |\zeta(s + i\tau, \alpha_j; \mathfrak{b}_{jl}) - f_{jl}(s)| < \epsilon\Big\} > 0.$$

For the proof of Theorem 1.5, we will apply a modification of the probabilistic method used in [4] which is based on a joint limit theorem in the space of analytic functions. The case of Theorem 1.5 is more complicated because we deal with two strips D_κ and D.

§ 2. Joint functional limit theorems

Let G be a region on $\mathbb{C}$. Denote by $H(G)$ the space of analytic functions on G equipped with the topology of uniform convergence on compacta. For $u = l_1 + \cdots + l_r$ and $v = u + 1$, let

$$H^v(D_\kappa, D) = H(D_\kappa) \times \underbrace{H(D) \times \cdots \times H(D)}_{u}.$$

Moreover, we set $\underline{\alpha} = (\alpha_1, ..., \alpha_r)$ and $\underline{\mathfrak{b}} = (\mathfrak{b}_{11}, ..., \mathfrak{b}_{1l_1}, ..., \mathfrak{b}_{r1}, ..., \mathfrak{b}_{rl_r})$. This section is devoted to a limit theorem in the space $H^v(D_\kappa, D)$ for the vector

$$\begin{aligned}\underline{\zeta}(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{b}}, F) = \big(\zeta(\hat{s}, F), \zeta(s, \alpha_1; \mathfrak{b}_{11}), ..., \zeta(s, \alpha_1; \mathfrak{b}_{1l_1}), ..., \\ \zeta(s, \alpha_r; \mathfrak{b}_{r1}), ..., \zeta(s, \alpha_r; \mathfrak{b}_{rl_r})\big).\end{aligned}$$

Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and define

$$\hat{\Omega} = \prod_p \gamma_p \quad \text{and} \quad \Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ for all primes p, and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem, the tori $\hat{\Omega}$ and Ω with the product topology and pointwise multiplication are compact topological Abelian groups. Therefore, on $\big(\hat{\Omega}, \mathcal{B}(\hat{\Omega})\big)$ and $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measures $\hat{m}_H$ and m_H, respectively, exist, and we have two probability spaces $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \hat{m}_H)$ and $(\Omega, \mathcal{B}(\Omega), m_H)$. Moreover, let

$$\underline{\Omega} = \hat{\Omega} \times \prod_{j=1}^{r} \Omega_j,$$

where $\Omega_j = \Omega$ for all $j = 1, ..., r$. Similarly as above, we obtain the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$, where $\underline{m}_H$ is the probability Haar measure on $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}))$. Denote by $\hat{\omega}(p)$ the projection of $\hat{\omega} \in \hat{\Omega}$ to γ_p, and by $\omega_j(m)$ the projection of $\omega_j \in \Omega_j$ to γ_m. Let $\underline{\omega} = (\hat{\omega}, \omega_1, ..., \omega_r)$ be the elements of $\underline{\Omega}$. On the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$, define the $H^v(D_\kappa, D)$-valued random element $\underline{\zeta}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{b}}, F)$ by the formula

$$\begin{aligned}\underline{\zeta}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{b}}, F) = \big(\zeta(\hat{s}, \hat{\omega}, F), \zeta(s, \alpha_1, \omega_1; \mathfrak{b}_{11}), ..., \zeta(s, \alpha_1, \omega_1; \mathfrak{b}_{1l_1}), ..., \\ \zeta(s, \alpha_r, \omega_r; \mathfrak{b}_{r1}), ..., \zeta(s, \alpha_r, \omega_r; \mathfrak{b}_{rl_r})\big),\end{aligned}$$

where

$$\zeta(\hat{s}, \hat{\omega}, F) = \prod_{p|N} \left(1 - \frac{c(p)\hat{\omega}(p)}{p^s}\right)^{-1} \prod_{p\nmid N} \left(1 - \frac{c(p)\hat{\omega}(p)}{p^s} + \frac{\hat{\omega}^2(p)}{p^{2s-1+\kappa}}\right)^{-1}$$

and

$$\zeta(s, \alpha_j, \omega_j; \mathfrak{b}_{jl}) = \sum_{m=0}^{\infty} \frac{b_{mjl}\omega_j(m)}{(m+\alpha_j)^s}, \quad j = 1, ..., r, \quad l = 1, ..., l_j.$$

Denote by $P_{\underline{\zeta}}$ the distribution of the random element $\underline{\zeta}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{b}}, F)$, i.e., for $A \in \mathcal{B}(H^v(D_\kappa, D))$,

$$P_{\underline{\zeta}}(A) = \underline{m}_H \left(\underline{\omega} \in \underline{\Omega} : \underline{\zeta}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{b}}, F) \in A\right).$$

Theorem 2.1. *Suppose that the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then*

$$P_T(A) \stackrel{def}{=} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \underline{\zeta}(\hat{s} + i\tau, s + i\tau, \underline{\alpha}; \underline{\mathfrak{b}}, F) \in A\right\}, \quad A \in \mathcal{B}(H^v(D_\kappa, D)),$$

converges weakly to $P_{\underline{\zeta}}$ as $T \to \infty$.

The proof of Theorem 2.1 is similar to that Theorem 4 of [4], therefore, we will give only its sketch.

Let $\sigma_1 > \frac{1}{2}$ be a fixed number, and

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}, \quad m, n \in \mathbb{N},$$

$$v_n(m, \alpha_j) = \exp\left\{-\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^{\sigma_1}\right\}, \quad m, n \in \mathbb{N}_0, \quad j = 1, ..., r.$$

Define

$$\zeta_n(\hat{s}, F) = \sum_{m=1}^{\infty} \frac{c(m)v_n(m)}{m^s}$$

and

$$\zeta_n(s, \alpha_j; \mathfrak{b}_{jl}) = \sum_{m=0}^{\infty} \frac{b_{mjl}v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, ..., r, \quad l = 1, ..., l_j.$$

By a standard method involving an application of the Mellin formula can be proved that the series for $\zeta_n(\hat{s}, F)$ and $\zeta_n(s, \alpha_j; \mathfrak{b}_{jl})$ are absolutely convergent for $\Re\hat{s} > \frac{\kappa}{2}$ and $\sigma > \frac{1}{2}$, respectively.

The formula

$$\hat{\omega}(m) = \prod_{p^l \| m} \omega^l(p), \quad m \in \mathbb{N},$$

where $p^l \parallel m$ means that a power p^l occurs precisely in the canonical representation of m, extends the functions $\hat{\omega}(p)$ to the set $\mathbb{N}$. Define

$$\zeta_n(\hat{s}, \hat{\omega}, F) = \sum_{m=1}^{\infty} \frac{c(m)\hat{\omega}(m)v_n(m)}{m^s},$$

and

$$\zeta_n(s, \alpha_j, \omega_j; \mathfrak{b}_{jl}) = \sum_{m=0}^{\infty} \frac{b_{mjl}\omega_j(m)v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, ..., r, \quad l = 1, ..., l_j,$$

the series being absolutely convergent for $\Re\hat{s} > \frac{\kappa}{2}$ and $\sigma > \frac{1}{2}$, respectively. Moreover, we set

$$\underline{\zeta}_n(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{b}}, F) = \big(\zeta_n(\hat{s}, F), \zeta_n(s, \alpha_1; \mathfrak{b}_{11}), ..., \zeta_n(s, \alpha_1; \mathfrak{b}_{1l_1}), ..., \\ \zeta_n(s, \alpha_r; \mathfrak{b}_{r1}), ..., \zeta_n(s, \alpha_r; \mathfrak{b}_{rl_r})\big)$$

and

$$\underline{\zeta}_n(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{b}}, F) = \big(\zeta_n(\hat{s}, \hat{\omega}, F), \zeta_n(s, \alpha_1, \omega_1; \mathfrak{b}_{11}), ..., \zeta_n(s, \alpha_1, \omega_1; \mathfrak{b}_{1l_1}), ..., \\ \zeta_n(s, \alpha_r, \omega_r; \mathfrak{b}_{r1}), ..., \zeta_n(s, \alpha_r, \omega_r; \mathfrak{b}_{rl_r})\big).$$

The first step in the proof of Theorem 2.1 is the following statement.

Lemma 2.2. *Suppose that the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the probability measures*

$$P_{T,n}(A) \overset{def}{=} \frac{1}{T}\text{meas}\Big\{\tau \in [0, T] : \underline{\zeta}_n(\hat{s} + i\tau, s + i\tau, \underline{\alpha}; \underline{\mathfrak{b}}, F) \in A\Big\}$$

and

$$P_{T,n,\underline{\omega}}(A) \overset{def}{=} \frac{1}{T}\text{meas}\Big\{\tau \in [0, T] : \underline{\zeta}_n(\hat{s} + i\tau, s + i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{b}}, F) \in A\Big\},$$

$A \in \mathcal{B}(H^v(D_\kappa, D))$, both converge weakly, for any fixed $\underline{\omega} \in \underline{\Omega}$, to the same probability measure P_n on $(H^v(D_\kappa, D), \mathcal{B}(H^v(D_\kappa, D)))$ as $T \to \infty$.

Lemma 2.2 is a result of the application of Theorem 5.1 from [2] and a limit theorem on the torus $\underline{\Omega}$ which is contained in the next lemma obtained in [4], Lemma 1. Let $\mathcal{P}$ be the set of all prime numbers.

Lemma 2.3. *Suppose that the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then*

$$\frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \big((p^{-i\tau} : p \in \mathcal{P}), \big((m+\alpha_j)^{-i\tau} : m \in \mathbb{N}_0, j = 1, ..., r\big)\big) \in A\right\}, \\ A \in \mathcal{B}(\underline{\Omega}),$$

converges weakly to the Haar measure $\underline{m}_H$ as $T \to \infty$.

The next step of the proof of Theorem 2.1 contains the results which allow to pass from the vector $\underline{\zeta}_n(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{b}}, F)$ to $\underline{\zeta}(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{b}}, F)$. For this, we need a metric on $H^v(D_\kappa, D)$.

It is well known that there exist a sequence $\{\hat{K}_m : m \in \mathbb{N}\}$ of compact subsets of D_κ, and a sequence $\{K_m : m \in \mathbb{N}\}$ of D such that

$$D_\kappa = \bigcup_{m=1}^{\infty} \hat{K}_m \quad \text{and} \quad D = \bigcup_{m=1}^{\infty} K_m.$$

Moreover, the sets $\hat{K}_m$ and K_m can be chosen to satisfy $\hat{K}_m \subset \hat{K}_{m+1}$, $K_m \subset K_{m+1}$ for all $m \in \mathbb{N}$, and, for every compact subsets $\hat{K} \subset D_\kappa$ and $K \subset D$, there exists $\hat{m}, m \in \mathbb{N}$ such that $\hat{K} \subset \hat{K}_{\hat{m}}$ and $K \subset K_m$. For $\hat{g}_1, \hat{g}_2 \in H(D_\kappa)$ and $g_1, g_2 \in H(D)$, define

$$\hat{\rho}(\hat{g}_1, \hat{g}_2) = \sum_{m=1}^{\infty} 2^{-m} \frac{\sup\limits_{s \in \hat{K}_m} |\hat{g}_1(s) - \hat{g}_2(s)|}{1 + \sup\limits_{s \in \hat{K}_m} |\hat{g}_1(s) - \hat{g}_2(s)|}$$

and

$$\rho(g_1, g_2) = \sum_{m=1}^{\infty} 2^{-m} \frac{\sup\limits_{s \in K_m} |g_1(s) - g_2(s)|}{1 + \sup\limits_{s \in K_m} |g_1(s) - g_2(s)|}.$$

Then $\hat{\rho}$ and ρ are the metrics on $H(D_\kappa)$ and $H(D)$, respectively, inducing the topology of uniform convergence on compacta. For

$$\underline{f} = (\hat{f}, f_{11}, ..., f_{1l_1}, ..., f_{r1}, ..., f_{rl_r}),\ \underline{g} = (\hat{g}, g_{11}, ..., g_{1l_1}, ..., g_{r1}, ..., g_{rl_r}) \in H^v(D_\kappa, D),$$

let

$$\rho_v(\underline{f}, \underline{g}) = \max\left(\hat{\rho}(\hat{f}, \hat{g}), \max_{1 \le j \le r} \max_{1 \le l \le l_j} \rho(f_{jl}, g_{jl})\right).$$

Then ρ_v is a metric on the space $H^v(D_\kappa, D)$ which induces its topology.

Now we are able to approximate $\underline{\zeta}(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{b}}, F)$ by $\underline{\zeta}_n(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{b}}, F)$ in the mean.

Lemma 2.4. *We have*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho_v\left(\underline{\zeta}(\hat{s}+i\tau, s+i\tau, \underline{\alpha}; \underline{\mathfrak{b}}, F), \underline{\zeta}_n(\hat{s}+i\tau, s+i\tau, \underline{\alpha}; \underline{\mathfrak{b}}, F)\right) \mathrm{d}\tau = 0.$$

As it was observed in [25], the zeta-functions associated to newforms constitute a subclass of Matsumoto zeta-functions. Therefore, the lemma follows from the relation

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \hat{\rho}\left(\zeta(\hat{s}+i\tau, F), \zeta_n(\hat{s}+i\tau, F)\right) \mathrm{d}\tau = 0$$

which is a corollary of Lemma 8 from [13], and from the equalities

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho\left(\zeta(s+i\tau,\alpha_j;\mathfrak{b}_{jl}),\zeta_n(s+i\tau,\alpha_j;\mathfrak{b}_{jl})\right)\mathrm{d}\tau=0,$$
$$j=1,...,r,\quad l=1,...,l_j,$$

which are deduced from formula (3) of [18].

An analogue of Lemma 2.4 is also true for $\underline{\zeta}(\hat{s},s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{b}},F)$ and $\underline{\zeta}_n(\hat{s},s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{b}},F)$, where

$$\underline{\zeta}(\hat{s},s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{b}},F)=\big(\zeta(\hat{s},\hat{\omega},F),\zeta(s,\alpha_1,\omega_1;\mathfrak{b}_{11}),...,\zeta(s,\alpha_1,\omega_1;\mathfrak{b}_{1l_1}),...,$$
$$\zeta(s,\alpha_r,\omega_r;\mathfrak{b}_{r1}),...,\zeta(s,\alpha_r,\omega_r;\mathfrak{b}_{rl_r})\big).$$

Lemma 2.5. *Suppose that the numbers $\alpha_1,...,\alpha_r$ are algebraically independent over $\mathbb{Q}$. Then, for almost all $\underline{\omega}\in\Omega$, we have*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho_v\left(\underline{\zeta}(\hat{s}+i\tau,s+i\tau,\underline{\alpha},\underline{\omega};\underline{\mathfrak{b}},F),\underline{\zeta}_n(\hat{s}+i\tau,s+i\tau,\underline{\alpha},\underline{\omega};\underline{\mathfrak{b}},F)\right)\mathrm{d}\tau=0.$$

Proof. Lemma 11 of [13], for almost all $\hat{\omega}\in\hat{\Omega}$, implies the relation

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \hat{\rho}\left(\zeta(\hat{s}+i\tau,\hat{\omega}),\zeta_n(\hat{s}+i\tau,\hat{\omega})\right)\mathrm{d}\tau=0. \tag{2.1}$$

Let

$$\rho_u(\underline{\underline{f}},\underline{\underline{g}})=\max_{1\le j\le r}\max_{1\le l\le l_j}\rho(f_{jl},g_{jl}).$$

Denote by $\underline{\underline{m}}_H$ the Haar measure on $\big(\underline{\underline{\Omega}},\mathcal{B}(\underline{\underline{\Omega}})\big)$, where $\underline{\underline{\Omega}}=\Omega_1\times\cdots\times\Omega_r$. Then, for almost all $\underline{\underline{\omega}}\in\Omega$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho_u\left(\underline{\underline{\zeta}}(s+i\tau,\underline{\alpha},\underline{\underline{\omega}};\underline{\mathfrak{b}}),\underline{\underline{\zeta}}_n(s+i\tau,\underline{\alpha},\underline{\underline{\omega}};\underline{\mathfrak{b}})\right)\mathrm{d}\tau=0, \tag{2.2}$$

see formula (2.5) of [4]. Here $\underline{\underline{\zeta}}(s,\underline{\alpha},\underline{\underline{\omega}};\underline{\mathfrak{b}})$ and $\underline{\underline{\zeta}}_n(s,\underline{\alpha},\underline{\underline{\omega}};\underline{\mathfrak{b}})$ are obtained from $\underline{\zeta}(\hat{s},s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{b}},F)$ and $\underline{\zeta}_n(\hat{s},s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{b}},F)$ by removing $\zeta(\hat{s},\hat{\omega},F)$ and $\zeta_n(\hat{s},\hat{\omega},F)$, respectively. Since the measure $\underline{m}_H$ is the product of the measures $\hat{m}_H$ and $\underline{\underline{m}}_H$, the lemma follows from (2.1), (2.2), and the definition of ρ_v. □

We can deduce from Lemmas 2.2 and 2.4 the weak convergence for the measure P_T, as $T\to\infty$. However, the identification of the limit measure requires the next lemma.

Lemma 2.6. *Suppose that the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the probability measures P_T and*

$$\frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \underline{\zeta}(\hat{s}+i\tau, s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{b}}, F) \in A\right\}, \quad A \in \mathcal{B}(H^v(D_\kappa, D)),$$

both converge weakly, for almost all $\underline{\omega} \in \underline{\Omega}$, to the same probability measure P on $(H^v(D_\kappa, D), \mathcal{B}(H^v(D_\kappa, D)))$ as $T \to \infty$.

Proof. We omit the details which are similar to those of [4]. Let θ be a random variable defined on a certain probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$ and uniformly distributed on $[0,1]$. On the later probability space, define the $H^v(D_\kappa, D)$-valued random element $\underline{X}_{T,n}$ by the formula

$$\begin{aligned} \underline{X}_{T,n}(\hat{s}, s) &= \big(X_{T,n}(\hat{s}), X_{T,n,1,1}(s), ..., X_{T,n,1,l_1}(s), ..., X_{T,n,r,1}(s), ..., X_{T,n,r,l_r}(s)\big) \\ &\stackrel{def}{=} \underline{\zeta}_n(\hat{s}+i\theta T, s+i\theta T, \underline{\alpha}; \underline{\mathfrak{b}}, F). \end{aligned}$$

Then, denoting by $\xrightarrow{\mathcal{D}}$ the convergence in distribution, we have, by Lemma 2.2, that

$$\underline{X}_{T,n}(\hat{s}, s) \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}_n(\hat{s}, s), \tag{2.3}$$

where $\underline{X}_n(\hat{s}, s)$ is the $H^v(D_\kappa, D)$-valued random element with the distribution P_n (P_n is the limit measure in Lemma 2.2). Our first task is to prove the tightness of the family $\{P_n : n \in \mathbb{N}\}$.

In view of the Deligne estimate (1.1), the well-known properties of the mean square of Dirichlet series and Cauchy integral formula show that, for all $n \in \mathbb{N}$,

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in\hat{K}_m} |\zeta_n(\hat{s}+i\tau, F)|\,\mathrm{d}\tau \le \hat{C}_m\left(\sum_{k=1}^{\infty}\frac{c^2(k)}{k^{2\hat{\sigma}_m}}\right)^{\frac{1}{2}}, \quad m \in \mathbb{N}, \tag{2.4}$$

with some $\hat{C}_m > 0$ and $\hat{\sigma}_m > \frac{\kappa}{2}$. Similarly, for all $n \in \mathbb{N}$,

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K_m} |\zeta_n(s+i\tau, \alpha_j; \mathfrak{b}_{jl})|\,\mathrm{d}\tau \le C_m\left(\sum_{k=0}^{\infty}\frac{|b_{kjl}|^2}{(k+\alpha_j)^{2\sigma_m}}\right)^{\frac{1}{2}}, \tag{2.5}$$

with some $C_m > 0$ and $\sigma_m > \frac{1}{2}$, $m \in \mathbb{N}$, $j = 1, ..., r$, $l = 1, ..., l_j$. The compact sets $\hat{K}_m$ and K_m come from the definition of the metric ρ_v.

Now let

$$\hat{R}_m = \hat{C}_m\left(\sum_{k=1}^{\infty}\frac{c^2(k)}{k^{2\hat{\sigma}_m}}\right)^{\frac{1}{2}}, \quad R_{jlm} = C_m\left(\sum_{k=0}^{\infty}\frac{|b_{kjl}|^2}{(k+\alpha_j)^{2\sigma_m}}\right)^{\frac{1}{2}}.$$

Taking $\hat{M}_m = \hat{R}_m 2^{m+1}\epsilon^{-1}$ and $M_{jlm} = R_{jlm}2^{u+m+1}\epsilon^{-1}$, where $m \in \mathbb{N}$ and $\epsilon > 0$ is an arbitrary number, we find by (2.4) and (2.5) that

$$\limsup_{T\to\infty} \mathbb{P}\left(\left(\sup_{\hat{s}\in\hat{K}_m} |X_{T,n}(\hat{s})| > \hat{M}_m\right) \vee \left(\exists j,l : \sup_{s\in K_m} |X_{T,n,j,l}(s)| > M_{jlm}\right)\right) \leq \frac{\epsilon}{2^m}.$$

This together with (2.3) implies

$$\mathbb{P}\left(\left(\sup_{\hat{s}\in\hat{K}_m} |X_n(\hat{s})| > \hat{M}_m\right) \vee \left(\exists j,l : \sup_{s\in K_m} |X_{n,j,l}(s)| > M_{jlm}\right)\right) \leq \frac{\epsilon}{2^m},$$

where $X_n(\hat{s})$, $X_{n,j,l}(s)$, $j = 1,...,r$, $l = 1,...,l_j$, are the elements of the random vector $\underline{X}_n(\hat{s}, s)$. From this, we obtain that

$$P_n(H^v_\epsilon) \geq 1 - \epsilon,$$

where

$$H^v_\epsilon = \Big\{ f \in H^v(D_\kappa, D) : \sup_{\hat{s}\in\hat{K}_m} |\hat{f}(\hat{s})| \leq \hat{M}_m,\ \sup_{s\in K_m} |f_{jl}(s)| \leq M_{jlm},$$
$$j = 1,...,r,\ l = 1,...,l_j,\ m \in \mathbb{N}\Big\}$$

is a compact subset of the space $H^v(D_\kappa, D)$. This proves the tightness of the family $\{P_n : n \in \mathbb{N}\}$.

By the Prokhorov theorem, the family $\{P_n : n \in \mathbb{N}\}$ is relatively compact. Hence, there exists a sequence $n_k \to \infty$ and a probability measure P on $(H^v(D_\kappa, D), \mathcal{B}(H^v(D_\kappa, D)))$ such that

$$\underline{X}_{n_k}(\hat{s}, s) \xrightarrow[k\to\infty]{\mathcal{D}} P. \tag{2.6}$$

On the probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$, define the $H^v(D_\kappa, D)$-valued random element $\underline{X}_T(\hat{s}, s)$ by the formula

$$\underline{X}_T(\hat{s}, s) = \underline{\zeta}(\hat{s} + i\theta T, s + i\theta T, \underline{\alpha}; \underline{\mathfrak{b}}, F).$$

Then Lemma 2.4 yields, for every $\epsilon > 0$, the relation

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\left(\rho_v\left(\underline{X}_T(\hat{s}, s), \underline{X}_{T,n}(\hat{s}, s)\right) \geq \epsilon\right) = 0.$$

This, (2.3), (2.6) and Theorem 4.2 of [2] show that $\underline{X}_T(\hat{s}, s) \xrightarrow[T\to\infty]{\mathcal{D}} P$, or P_T converges weakly to P as $T \to \infty$.

Using the random elements

$$\underline{\zeta}_n(\hat{s} + i\theta T, s + i\theta T, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{b}}, F)$$

and

$$\underline{\zeta}(\hat{s}+i\theta T, s+i\theta T, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{b}}, F),$$

as well as Lemma 2.5, we obtain in a similar way that the second measure of Lemma 2.6 also converges weakly to P as $T \to \infty$. □

The end of the proof of Theorem 2.1 is standard. We apply Lemma 2.6, the ergodicity of the one-parameter group $\{\underline{\Phi}_\tau : t \in \mathbb{R}\}$ of measurable and measure preserving transformations on $\underline{\Omega}$, where, for $\underline{\omega} \in \Omega$ and $\tau \in \mathbb{R}$,

$$\underline{\Phi}_\tau(\underline{\omega}) = \big((p^{-i\tau} : p \in \mathcal{P}), ((m+\alpha_j)^{-i\tau} : m \in \mathbb{N}_0, j = 1, ..., r)\big)\, \underline{\omega},$$

see Lemma 7 of [20], as well as the classical Birkhoff-Khintchine theorem.

§ 3. Support of the measure $P_{\underline{\zeta}}$

The space $H^v(D_\kappa, D)$ is separable, therefore the support of $P_{\underline{\zeta}}$ is the minimal closed set $S_{P_{\underline{\zeta}}} \subset H^v(D_\kappa, D)$ such that $P_{\underline{\zeta}}(S_{P_{\underline{\zeta}}}) = 1$. The set $S_{P_{\underline{\zeta}}}$ consists of all points $\underline{g} \in H^v(D_\kappa, D)$ such that, for every open neighbourhood G of $\underline{g}$, the inequality $P_{\underline{\zeta}}(G) > 0$ holds.

Define

$$S_\kappa = \{g \in H(D_\kappa) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Theorem 3.1. *Suppose that the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that* $\operatorname{rank}(B_j) = l_j$, $j = 1, ..., r$. *Then the support of the measure $P_{\underline{\zeta}}$ is the set $S_\kappa \times H^u(D)$.*

Proof. We have that

$$H^v(D_\kappa, D) = H(D_\kappa) \times H^u(D).$$

In view of separability of the above spaces, the equality

$$\mathcal{B}(H^v(D_\kappa, D)) = \mathcal{B}(H(D_\kappa)) \times \mathcal{B}(H^u(D))$$

is true [2]. Therefore, it suffices to investigate $P_{\underline{\zeta}}(A)$ for $A = B \times C$, where $B \in \mathcal{B}(H(D_\kappa))$ and $C \in \mathcal{B}(H^u(D))$. We already have mentioned that the measure $\underline{m}_H$ is the product of the measures $\hat{m}_H$ and $\underline{\underline{m}}_H$. Therefore, we have that

$$\begin{aligned} P_{\underline{\zeta}}(A) &= \underline{m}_H\left(\underline{\omega} \in \underline{\Omega} : \underline{\zeta}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{b}}, F) \in A\right) \\ &= \underline{m}_H\Big(\underline{\omega} \in \underline{\Omega} : \zeta(\hat{s}, \hat{\omega}, F) \in B,\ \underline{\underline{\zeta}}(s, \underline{\alpha}, \underline{\underline{\omega}}; \underline{\mathfrak{b}}) \in C\Big) \\ &= \hat{m}_H\Big(\hat{\omega} \in \hat{\Omega} : \zeta(\hat{s}, \hat{\omega}, F) \in B\Big)\, \underline{\underline{m}}_H\Big(\underline{\underline{\omega}} \in \underline{\underline{\Omega}} : \underline{\underline{\zeta}}(s, \underline{\alpha}, \underline{\underline{\omega}}; \mathfrak{b}) \in C\Big). \end{aligned} \tag{3.1}$$

In [25], Lemma 9, it was obtained that the support of the random element $\zeta(\hat{s}, \hat{\omega}, F)$ is the set S_κ, i.e., S_κ is a minimal closed subset of $H(D_\kappa)$ such that

$$\hat{m}_H\left(\hat{\omega} \in \hat{\Omega} : \zeta(\hat{s}, \hat{\omega}, F) \in S_\kappa\right) = 1. \tag{3.2}$$

To be precise, in [25] the space $H(D_{\kappa,M})$, where $D_{\kappa,M} = \{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}, |t| < M\}$, is considered, however, all arguments remain valid for the space $H(D_\kappa)$. Also, in [27], Theorem 3.1, it was proved that the support of the random element $\underline{\underline{\zeta}}(s, \alpha, \underline{\underline{\omega}}; \underline{\mathfrak{b}})$ is the whole of $H^u(D)$, i.e., $H^u(D)$ is a minimal closed set of $H^u(D)$ such that

$$\underline{\underline{m}}_H\left(\underline{\underline{\omega}} \in \underline{\underline{\Omega}} : \underline{\underline{\zeta}}(s, \underline{\alpha}, \underline{\underline{\omega}}; \underline{\mathfrak{b}}) \in H^u(D)\right) = 1.$$

From this and (3.1), (3.2), the theorem follows. □

§ 4. Proof of Theorem 1.5

We first recall the Mergelyan theorem on the approximation of analytic functions by polynomials.

Lemma 4.1. *Suppose that K is a compact subset on the complex plane with connected complement, and that $f(s)$ is a continuous function on K which is analytic in th interior of K. Then, for every $\epsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s \in K} |f(s) - p(s)| < \epsilon.$$

Proof of the lemma is given in [30], see also [39].

Proof. of Theorem 1.5. In view of Lemma 4.1, there exist polynomials $p(s)$ and $p_{jl}(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\epsilon}{4} \tag{4.1}$$

and

$$\sup_{1 \le j \le r} \sup_{1 \le l \le l_j} \sup_{s \in K_{jl}} |f_{jl}(s) - p_{jl}(s)| < \frac{\epsilon}{2}. \tag{4.2}$$

Since $f(s) \neq 0$ on K, we have that $p(s) \neq 0$ on K as well if ϵ is small enough. Therefore, we can define a continuous branch of $\log p(s)$ on K which will be analytic in the interior of K. By Lemma 4.1 again, there exists a polynomial $q(s)$ such that

$$\sup_{s \in K} \left| p(s) - \mathrm{e}^{q(s)} \right| < \frac{\epsilon}{4}.$$

From this and (4.1), we have that

$$\sup_{s\in K}\left|f(s)-\mathrm{e}^{q(s)}\right|<\frac{\epsilon}{2}. \tag{4.3}$$

Define

$$G=\left\{\underline{g}\in H^{v}(D_\kappa,D):\sup_{s\in K}\left|\hat{g}(s)-\mathrm{e}^{q(s)}\right|<\frac{\epsilon}{2},\sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}|g_{jl}(s)-p_{jl}(s)|<\frac{\epsilon}{2}\right\}.$$

In view of Theorem 3.1, the vector $(\mathrm{e}^{q(s)}, p_{jl}, j=1,...,r, l=1,...,l_j)$, is an element of the support of the measure $P_{\underline{\zeta}}$. Since G is an open set, this shows that $P_{\underline{\zeta}}(G)>0$. Therefore, Theorem 2.1 together with an equivalent of the weak convergence in terms of open sets yields the inequality

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\sup_{s\in K}\left|\zeta(s+i\tau,F)-\mathrm{e}^{q(s)}\right|<\frac{\epsilon}{2},\right.$$
$$\left.\sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}|\zeta(s+i\tau,\alpha_j;\mathfrak{b}_{jl})-p_{jl}(s)|<\frac{\epsilon}{2}\right\}>0.$$

From this, (4.2) and (4.3), the assertion of the theorem follows. □

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