RIMS Kôkyûroku Bessatsu
B34 (2012), 235–246

On joint universality for derivatives of the Riemann zeta function and automorphic L-functions

By

Hidehiko MISHOU*

Abstract

In this paper we establish two results. The first result looks like a joint universality theorem for a set of derivatives of the Riemann zeta function. The second result is a joint universality theorem for a pair of the Riemann zeta function and an automorphic L-function in different strips.

§ 1. Introduction

As usual, let $s = \sigma + it$ be a complex variable and $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of all natural numbers, non-negative integers, integers, rational numbers, real numbers and complex numbers respectively. Here and henceforth the letter p denotes a prime number.

In order to state our results, we define some symbols. Let D be the strip $\{s \in \mathbb{C} \mid \frac{1}{2} < \sigma < 1\}$. Let μ be the Lebesgue measure on $\mathbb{R}$ and for $T > 0$

$$\nu_T(\cdots) = \frac{1}{T}\mu\{\tau \in [0,T] \ : \cdots\},$$

where in place of dots we write some conditions satisfied by τ.

Let $\zeta(s)$ be the Riemann zeta function. In 1975 Voronin [15] established the remarkable universality theorem for $\zeta(s)$.

Theorem 1.1 (Voronin, 1975). *Let K be a compact subset of the strip D with connected complement. Let $f(s)$ be a non-vanishing and continuous function on K which*

Received May 29, 2011. Revised October 4, 2011. Revised November 18, 2011. Accepted December 12, 2011.
2000 Mathematics Subject Classification(s): 11M06.
Key Words: *zeta functions, Selberg class, value distribution, universality*
*Ube National College of Technology, 2-14-1 Tokiwadai, Ube-city, Yamaguchi, 755-8555, Japan
email: mishou@ube-k.ac.jp

is analytic in the interior of K. Then for any small positive number ε we have

$$\liminf_{T\to\infty} \nu_T(\max_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon) > 0.$$

Roughly speaking this theorem asserts that any analytic function can be uniformly approximated by a suitable shift of $\zeta(s)$. Such universality theorems have been established for many arithmetic zeta functions and some non-arithmetic zeta functions such as Hurwitz zeta functions. For more details, refer to Steuding [14]. For each Dirichlet character χ, the attached Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \quad (\sigma > 1)$$

also has universality property. Moreover Bagchi [2], Gonek [4] and Voronin [15] independently proved that for a set of Dirichlet L-functions universality properties hold simultaneously.

Theorem 1.2 (Bagchi, Gonek, Voronin). *Let χ_j $(1 \le j \le r)$ be pairwise inequivalent Dirichlet characters. For each $1 \le j \le r$, let K_j be a compact subset of D with connected complement and $f_j(s)$ be a non-vanishing and continuous function on K_j which is analytic in the interior of K_j. Then for any $\varepsilon > 0$ we have*

$$\liminf_{T\to\infty} \nu_T \left(\max_{1\le j\le r} \max_{s\in K_j} |L(s+i\tau,\chi_j) - f_j(s)| < \varepsilon \right) > 0.$$

Such joint universality theorems have been established for the following sets of arithmetic zeta functions.

(A) Laurinčikas and Matsumoto [7] obtained the joint universality theorem for a set of twisted automorphic L-functions $\{L(s,f,\chi_j)\}$ where f is a holomorphic normalized Hecke eigen cusp form for $SL_2(\mathbb{Z})$ and χ_j are inequivalent Dirichlet characters.

(B) Bauer [3] established the joint universality for a set of Artin L-functions $\{L(s,\chi_j,K/\mathbb{Q})\}$ when χ_j are linearly independent characters of $Gal(K/\mathbb{Q})$.

(C) Sander and Steuding [12] investigated the joint universality for products and sums of Dirichlet L-functions. As corollaries, they obtained the joint universality theorems for

  (a) a set of Dedekind zeta functions $\{\zeta_{K_j}(s)\}$ where K_j are abelian extensions of $\mathbb{Q}$ satisfying some algebraic conditions.

  (b) a set of Hurwitz zeta functions $\{\zeta(s,\alpha_j)\}$ where α_j are rational numbers.

Now we will explain the reason why there are not many results on joint universality compared to results on ordinary universality.

First of all, the joint universality does not hold for every sets of zeta functions with universality property. We give one example. For each $j \geq 0$, the j-th derivative $\zeta^{(j)}(s)$ of $\zeta(s)$ has universality property in the strip D. Now we prove that the joint universality theorem does not hold for a set of derivatives of $\zeta(s)$. Let K_1 and K_2 be simply connected compact subsets of D such that K_1 is included in the interior of K_2. Let $f(s)$ be a non-vanishing and continuous function on K_2 which is analytic in the interior of K_2. Let $\varepsilon > 0$. Assume that for a real number τ the inequality

$$\max_{s \in K_2} |\zeta(s + i\tau) - f(s)| < \varepsilon$$

holds. Then the Cauchy's integral formula

$$g^{(n)}(s) = \frac{n!}{2\pi i} \int_{|z-s|=r} \frac{g(z)}{(z-s)^{n+1}} dz$$

implies that

$$|\zeta'(s + i\tau) - f'(z)| < C(K_1, K_2) \cdot \varepsilon$$

holds for any $s \in K_1$. Therefore the joint universality does not hold for a pair of $\zeta(s)$ and $\zeta'(s)$.

Secondly, in the current method to prove the joint universality theorem, the following two properties of Dirichlet characters play crucial roles:

- Periodicity

  Let χ be a Dirichlet character modulo q, then

$$\chi(n) = \chi(a) \quad \text{if } n \equiv a \pmod{q}.$$

- Orthogonality

  Let χ_1 and χ_2 be Dirichlet characters modulo q, then

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} \chi_1(p)\overline{\chi_2(p)} = \begin{cases} \frac{\phi(q)}{q} & (\chi_1 = \chi_2), \\ 0 & (\text{otherwise}), \end{cases}$$

  where $\pi(x) = \sum_{p \leq x} 1$ and $\phi(q)$ is the Euler's totient function.

In fact, as shown in Theorem 1.2 and results (A) – (C), all known families of arithmetic zeta functions with joint universality are concerned with characters of finite order.

Which is the more important property, the periodicity or the orthogonality? Steuding [14] gave a conjecture on this problem. Let $\mathcal{S}$ denote the Selberg class, which is the set of Dirichlet series satisfying (i) Ramanujan conjecture (estimation for Dirichlet coefficients) , (ii) Analytic continuation to $\mathbb{C}$, (iii) Functional equation, and (iv) Euler product. A function $L(s) \in \mathcal{S}$ is called primitive if it cannot be factored as a product of two elements in $\mathcal{S}$ non-trivially.

Conjecture (Steuding). *For each $j = 1, 2$, let $L_j(s) = \sum_n a_j(n) n^{-s}$ be a primitive function in $\mathcal{S}$. Assume the Selberg conjecture*

$$\sum_{p \leq x} \frac{a_1(p)\overline{a_2(p)}}{p} = \begin{cases} \log\log x + O(1) & (L_1 = L_2), \\ O(1) & (\textit{otherwise}). \end{cases} \tag{1.1}$$

Then $L_1(s)$ and $L_2(s)$ are joint universal.

The condition (1) is regarded as a generalization of the orthogonality of Dirichlet characters.

Recently Nagoshi [10] obtained a remarkable result on automorphic L-functions which confirms the Steuding's conjecture. For an even positive integer k let $\mathcal{F}_k$ be the set of all holomorphic normalized Hecke eigen cusp forms of weight k with respect to $SL_2(\mathbb{Z})$. For $f \in \mathcal{F}_k$ let $\tilde{\lambda}_f(n) = \lambda_f(n) n^{\frac{k-1}{2}}$ denote the n-th Fourier coefficient of f. Then the associated automorphic L-function $L(s, f)$ is given by

$$L(s, f) = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1} = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1} \quad (\sigma > 1),$$

where $\alpha(p), \beta(p) \in \mathbb{C}$ with $|\alpha(p)| = |\beta(p)| = 1$ and $\lambda_f(p) = \alpha(p) + \beta(p)$. For the coefficients $\lambda_f(n)$, we have

$$\sum_{p \leq x} \lambda_f(p) = o(\pi(x)),$$

and

$$\sum_{p \leq x} |\lambda_f(p)|^2 \sim \pi(x),$$

where the latter estimate is due to Rankin [11]. These estimates imply that the orthogonal relation (1) holds for a pair of $L_1(s) = \zeta(s)$ and $L_2(s) = L(s, f)$. For each prime p, define an angle $\theta_p \in [0, \pi]$ by $\lambda_f(p) = 2\cos\theta_p$. Then, for a positive integer m, the symmetric m-th power L-function $L(s, f, m)$ is given by

$$L(s, f, m) = \prod_p \prod_{j=0}^{m} \left(1 - \frac{e^{(m-2j)\theta_p}}{p^s}\right)^{-1} \quad (\sigma > 1).$$

Murty [9] showed that the famous Sato - Tate conjecture is valid if all $L(s, f, m)$ have analytic continuation to $\sigma \geq 1$. When $m \leq 4$, Shahidi [13] obtained the analytic continuation of $L(s, f, m)$ to $\sigma \geq 1$. However the case $m > 4$ is still open.

Theorem 1.3 (Nagoshi, 2005). *Let $f \in \mathcal{F}_k$. Assume that for each positive integers m, the attached L-function $L(s, f, m)$ admits analytic continuation to $\mathbb{C}$ and satisfies the Grand Riemann hypothesis. Let σ_1 and σ_2 be real numbers satisfying $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$ and $1 + \sigma_1 - 2\sigma_2 > 0$. For each $j = 1, 2$, let K_j be a compact subset in the strip $\sigma_1 < \sigma < \sigma_2$ with connected complement and $f_j(s)$ be a non-vanishing and continuous function on K_j which is analytic in the interior of K_j. Then for any $\varepsilon > 0$ we have*

$$\liminf_{T\to\infty} \nu_T \begin{pmatrix} \max_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon \\ \max_{s\in K_2} |L(s+i\tau, f) - f_2(s)| < \varepsilon \end{pmatrix} > 0.$$

In the above theorem, the assumption on the L-functions $L(s, f, m)$ is probably correct, but very strong. Theorem 1.3 also implies the difficulty of the Steuding's conjecture.

§ 2. Results

In the previous section we saw that the joint universality does not hold for a set of derivatives of $\zeta(s)$ in the strict sense. However we can obtain the following result, which is similar to the joint universality theorem for the set.

Theorem 2.1. *Let K_1 and K_2 be simply connected compact subsets of D. Suppose that K_1 is included in the interior of K_2. Let $f(s)$ be a non-vanishing and continuous function on K_2 which is analytic in the interior of K_2. Let $f_j(s)$ $(1 \leq j \leq l)$ be continuous functions on K_2 which are analytic in the interior of K_2. Then for any $\varepsilon > 0$ there exist positive integers N_j $(1 \leq j \leq l)$ such that*

$$\liminf_{T\to\infty} \nu_T \begin{pmatrix} \max_{s\in K_1} |\zeta(s+i\tau) - f(s)| < \varepsilon \\ \max_{1\leq j\leq l} \max_{s\in K_1} |\zeta^{(N_j)}(s+i\tau) - f_j(s)| < \varepsilon \end{pmatrix} > 0.$$

Also we saw that the joint universality theorem has not been yet proved for a pair of $\zeta(s)$ and $L(s, f)$ unconditionally. However, if we restrict compact subsets K_1 and K_2, we can prove the joint universality theorem for this pair.

Theorem 2.2. *Let $f \in \mathcal{F}_k$. Let $D_1 = \{s \in \mathbb{C} \mid \frac{1}{2} < \sigma < \frac{3}{4}\}$ and $D_2 = \{s \in \mathbb{C} \mid \frac{3}{4} < \sigma < 1\}$. For each $j = 1, 2$, let K_j be a compact subset in D_j with connected complement and $f_j(s)$ be a non-vanishing and continuous function on K_j which is analytic in the interior of K_j. Then for any $\varepsilon > 0$ we have*

$$\liminf_{T\to\infty} \nu_T \begin{pmatrix} \max_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon \\ \max_{s\in K_2} |L(s+i\tau, f) - f_2(s)| < \varepsilon \end{pmatrix} > 0.$$

In the next section we will prove Theorem 2.1. In §4, we will sketch the proof of Theorem 2.2.

§ 3. Proof of Theorem 2.1

In 1952 Maclane [8] showed that the following theorem, which is the key of the proof of Theorem 2.1.

Lemma 3.1 (Maclane, 1952). *There exists an entire function $F(s)$ such that to any entire function $g(s)$ there is an increasing sequence $n_k \in \mathbb{Z}_{\geq 0}$ for which*

$$F^{(n_k)}(s) \to g(s) \quad \text{as} \quad k \to \infty$$

holds locally uniformly in $\mathbb{C}$.

Lemma 3.2 (Mergelyan, 1952). *Let K be a compact subset of $\mathbb{C}$ with connected complement. Then any continuous function on K which is analytic in the interior of K is approximable uniformly on K by the polynomials of s.*

Proof of Theorem 2.1. Let $f(s)$ and $f_j(s)$ be the same functions as in Theorem 2.1. First we consider the case that all $f(s)$ and $f_j(s)$ are entire functions. Let $F(s)$ be the same entire function as in Lemma 3.1. There exist non-negative integers $n_0 < n_1 < \cdots < n_l$ for which inequalities

$$\max_{s \in K_2} |F^{(n_0)} - f(s)| < \frac{\varepsilon}{2} \quad \text{and} \quad \max_{1 \leq j \leq l} \max_{s \in K_2} |F^{(n_j)}(s) - f_j(s)| < \frac{\varepsilon}{2} \tag{3.1}$$

hold. We put

$$N_j = n_j - n_0 \ (1 \leq j \leq l) \quad \text{and} \quad \delta = \min\{|z_1 - z_2| \mid z_1 \in \partial K_1, \ z_2 \in \partial K_2\}.$$

According to the classical Rouche's theorem, we may assume that $F^{(n_0)}(s)$ is non-vanishing on K_2. Then, by the universality of $\zeta(s)$, the set A_T of real numbers $\tau \in [0, T]$ for which

$$\max_{s \in K_2} \left| \zeta(s + i\tau) - F^{(n_0)}(s) \right| < \frac{\delta^{N_l+1}}{2N_l!} \varepsilon \tag{3.2}$$

hold has a positive lower density as $T \to \infty$. Let $1 \leq j \leq l$ and $s \in K_1$ be fixed. From the Cauchy's integral formula and (3), it follows that for any $\tau \in A_T$

$$\begin{aligned} &\left|\zeta^{(N_j)}(s+i\tau) - F^{(n_j)}(s)\right| \\ &= \left|\frac{N_j!}{2\pi i}\int_{|z-s|=\delta} \frac{\zeta(z+i\tau) - F^{(n_0)}(z)}{(z-s)^{N_j+1}} dz\right| \\ &\le \frac{N_j!}{\delta^{N_j+1}} \max_{z\in K_2} |\zeta(z+i\tau) - F\ (n_0)(z)| \\ &< \frac{\varepsilon}{2}. \end{aligned} \tag{3.3}$$

Combining (2) and (4), we complete the proof in the first case.

Now we consider the general case. By Lemma 3.2 there exist polynomials $p(s)$ and $q_j(s)$ which satisfy

$$\max_{s\in K_2} |f(s) - e^{p(s)}| < \varepsilon \quad \text{and} \quad \max_{s\in K_2} |f_j(s) - q_j(s)| < \varepsilon.$$

These $e^{p(s)}$ and $q_j(s)$ are entire functions, therefore the proof of the theorem is completed. ☐

§ 4. Outline of the proof of Theorem 2.2

In this section we sketch the proof of Theorem 2.2, which is equivalent to the joint universality for $\zeta(s)$ and $L(s+\frac{1}{4}, f)$ in the strip $D_1 = \{s \in \mathbb{C} \mid \frac{1}{2} < \sigma < \frac{3}{4}\}$. In these days the joint universality property for zeta functions is mainly obtained as an application of the joint limit theorem on the weak convergence of probability measure associated with the zeta functions. This method was devised by Bagchi [1]. In order to state the joint limit theorem, we need some notation. For space S, $\mathcal{B}(S)$ will stand for the family of Borel subsets of S. Let $H(D_1)$ be the space of analytic functions on the strip D_1 equipped with the topology of uniform convergence on compact subsets. Let $H^2(D_1) = H(D_1) \times H(D_1)$. Denote by γ the unit circle $\{s \in \mathbb{C} \mid |s| = 1\}$ and set

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. The infinite-dimensional torus Ω is a compact Abelian groups with respect to the product topology and pointwise multiplication. Therefore the probability Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ can be defined. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p for any p. For $\sigma > \frac{1}{2}$ and $\omega \in \Omega$ we define

$$\zeta(s,\omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}$$

and

$$L(s,f,\omega)=\prod_p\left(1-\frac{\alpha(p)\omega(p)}{p^s}\right)^{-1}\left(1-\frac{\beta(p)\omega(p)}{p^s}\right)^{-1}.$$

Since for almost all $\omega\in\Omega$ these products are convergent uniformly on compact subsets of D_1, a pair of these products become a $H^2(D_1)$-valued random element on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. For $A\in\mathcal{B}(H^2(D_1))$, define two probability measures

$$P_T(A)=\nu_T((\zeta(s+i\tau),L(s+\frac{1}{4},f))\in A),$$

where $T>0$ and

$$P(A)=m_H((\zeta(s,\omega),L(s+\frac{1}{4},f,\omega))\in A).$$

Lemma 4.1 (Joint limit theorem). *The probability measure P_T converges weakly to the probability measure P as $T\to\infty$.*

Proof. The lemma is easily obtained by combining the limit theorem for $\zeta(s)$ (Theorem 5.1.8 in Laurinčikas [5]) and the limit theorem for $L(s,f)$ (Lemma 1 in Laurinčikas and Matsumoto [6]). □

For $\sigma>\frac{1}{2}$ and $\omega\in\Omega$ we define functions g_p and h_p by

$$\log\left(1-\frac{\omega(p)}{p^s}\right)^{-1}=\frac{\omega(p)}{p^s}+g_p(s)$$

and

$$\log\left(1-\frac{\alpha(p)\omega(p)}{p^s}\right)^{-1}+\log\left(1-\frac{\beta(p)\omega(p)}{p^s}\right)^{-1}=\frac{\lambda_f(p)\omega(p)}{p^s}+h_p(s).$$

Then for all $s\in D_1$ and almost all $\omega\in\Omega$

$$\left(\log\zeta(s,\omega),\log L(s+\frac{1}{4},f,\omega)\right)=\sum_p\left(\frac{\omega(p)}{p^s},\frac{\lambda_f(p)\omega(p)}{p^s}\right)+\sum_p\left(g_p(s),h_p(s)\right),$$

where the sum is taken over all prime numbers. Remark that the series $\sum_p(g_p(s),h_p(s))$ converges uniformly for $\omega\in\Omega$ and on any compact subset of D_1. For each prime p we set

$$f_p(s)=\left(\frac{1}{p^s},\frac{\lambda_f(p)}{p^{s+\frac{1}{4}}}\right)\in H^2(D_1).$$

Lemma 4.2 (Joint denseness lemma). *The set of convergent series*

$$\left\{\sum_p\omega(p)f_p(s)\in H^2(D_1)\;\middle|\;\omega\in\Omega\right\}$$

is dense in $H^2(D_1)$.

The lemma implies that the set $\{(\zeta(s,\omega), L(s+\frac{1}{4}, f, \omega)) \in H^2(D_1) \mid \omega \in \Omega\}$ is also dense in the space $H^2(D_1)$. From Lemma 4.1 and Lemma 4.2, the joint universality follows immediately.

Proof of Lemma 4.2. Let U be a bounded simply connected region in D_1. Let $\mathcal{H}$ be the Hardy space on U, which is the set of analytic and second integrable functions on U. Let $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$. The space $\mathcal{H}$ becomes a complex Hilbert space with the inner product

$$\langle g_1, g_2 \rangle = \iint_U g_1(s)\overline{g_2(s)} d\sigma dt.$$

We will prove that the set $\{\sum_p a_p f_p(s) \in \mathcal{H}^2 \mid |a_p| = 1\}$ is dense in $\mathcal{H}^2$ by using the following general denseness lemma, which was essentially obtained by Voronin [15].

Lemma 4.3. *Let H be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Suppose that a sequence $\{u_n\} \subset H$ satisfies*

(i) $\displaystyle\sum_n \|u_n\| < \infty,$

(ii) for any non-zero element $u \in H$

$$\sum_n |\langle u_n, u \rangle| = \infty.$$

Then for any $m > 0$ the set

$$\left\{ \sum_{n \geq m} a_n u_n \in H \,\middle|\, |a_n| = 1 \right\}$$

is dense in H.

We return to the proof of Lemma 4.2. Let $\sigma_0 = \min\{\Re s \mid s \in \overline{U}\} > \frac{1}{2}$. Then

$$\sum_p \|f_p(s)\|^2 = \sum_p \iint_U \frac{1 + |\lambda_f(p)|^2 p^{-\frac{1}{2}}}{p^{2\sigma}} d\sigma dt \ll_U \sum_p \frac{1}{p^{2\sigma_0}} < \infty.$$

Therefore the sequence $\{f_p(s)\}$ satisfies the condition (i) in Lemma 4.3. For $g(s) = (g_1(s), g_2(s)) \in \mathcal{H}^2$ we have

$$\begin{aligned}\langle f_p(s), g(s) \rangle &= \iint_U \frac{1}{p^s} \overline{g_1(s)} d\sigma dt + \iint_U \frac{\lambda_f(p)}{p^{s+\frac{1}{4}}} \overline{g_2(s)} d\sigma dt \\ &= \Delta_1(\log p) + \frac{\lambda_f(p)}{p^{\frac{1}{4}}} \Delta_2(\log p),\end{aligned}$$

where we set

$$\Delta_j(z) = \iint_U e^{-sz}\overline{g_j(s)}d\sigma dt$$

for $z \in \mathbb{C}$ and $j = 1, 2$. To prove the condition (ii) in Lemma 4.3, we quote the following lemma from Nagoshi [10].

Lemma 4.4. *Let $\frac{1}{2} < \sigma_1 < \sigma_2 < \frac{3}{4}$ such that U in included in the strip $\sigma_1 < \Re s < \sigma_2$. Let $h(s)$ be a function in $\mathcal{H}$. Define*

$$\Delta_h(z) = \iint_U e^{-sz}\overline{h(s)}d\sigma dt.$$

Then $\Delta_h(z)$ is entire, and satisfies the following property:

(I) there exists a positive constant $c = c(U, h)$ such that

$$|\Delta_h(x)| \le ce^{-\sigma_1 x}$$

holds for all $x \ge 0$,

(II) if $h(s)$ is a non-zero element, there exists a divergent positive sequence $x_n \to \infty$ $(n \to \infty)$ and a sequence of intervals $I_n = [\alpha_n, \alpha_n + \beta_n] \subset [x_n - 1, x_n + 1]$ such that

$$|\Delta_h(x)| \ge \frac{1}{4}e^{-\sigma_2 x_n} \quad (x \in I_n).$$

and such that

$$\beta_n \sim x_n^{-4}. \tag{4.1}$$

The statement (I) easily follows from the Cauchy-Schwarz inequality. The statement (II) was essentially obtained by Voronin [15]. Now we prove

$$\sum_p |\langle f_p(s), g(s)\rangle| = \infty$$

for any non-zero $g(s) = (g_1(s), g_2(s)) \in \mathcal{H}^2$. When $g_1 = 0$, the divergence of the series was established in [6] essentially. Therefore we may assume that g_1 is non-zero. By the statement (II) in Lemma 4.4 there exists a sequence $x_n \to \infty$ and a sequence of intervals $I_n = [\alpha_n, \alpha_n + \beta_n]$ such that

$$|\Delta_1(x)| \ge \frac{1}{4}e^{-\sigma_2 x_n} \quad (x \in I_n). \tag{4.2}$$

Also, by the statement (I) in Lemma 4.4 there exists a positive constant c_2 such that

$$|\Delta_2(x)| \le c_2 e^{-\sigma_1 x} \quad (x \in I_n). \tag{4.3}$$

From (6) and (7), for any p with $\log p \in I_n$ we have

$$\begin{aligned} |\langle f_p(s), g(s)\rangle| &\geq |\Delta_1(\log p)| - \frac{|\lambda_f(p)|}{p^{\frac{1}{4}}} |\Delta_2(\log p)| \\ &\geq \frac{1}{4} e^{-\sigma_2 x_n} - 2ce^{-(\sigma_1+\frac{1}{4})(x_n-1)} \\ &\geq \left(\frac{1}{4} - 2ce^{(\sigma_1+\frac{1}{4})} e^{-(\frac{1}{4}+\sigma_1-\sigma_2)x_n}\right) e^{-\sigma_2 x_n}. \end{aligned}$$

Since $\sigma_2 - \sigma_1 < \frac{1}{4}$, there exists a positive constant c_3 such that

$$|\langle f_p(s), g(s)\rangle| \geq c_3 e^{-\sigma_2 x_n}$$

for sufficiently large n and all p with $\log p \in I_n$. By the prime number theorem and (5), we have

$$\sum_{\log p \in I_n} |\langle f_p(s), g(s)\rangle| \geq (\pi(e^{\alpha_n+\beta_n}) - \pi(e^{\alpha_n}))\cdot c_3 e^{-\sigma_2 x_n} \gg \frac{1}{x_n^8} e^{(1-\sigma_2)x_n} \to \infty \quad (n \to \infty).$$

Therefore the sequence $\{f_p(s)\}$ satisfies the condition (ii) in Lemma 4.3. This completes the proof of Theorem 2.2.

References

[1] Bagchi, B., *The statistical behavior and universality properties of the Riemann zeta-function and other allied Dirichlet series*, Ph. D. Thesis. Calcutta, Indian Statistical Institute, 1981.

[2] Bagchi, B., *A joint universality theorem for Dirichlet L-functions*, Math. Zeitschrift, 181(3), 319 - 334, 1982.

[3] Bauer, H., *The value distribution of Artin L-series and zeros of zeta-functions*, J. Number Theory, **98(2)**, 254-279, 2003.

[4] Gonek, S. M., *Analytic properties of zeta and L-functions*, Thesis, Univ. of Michigan, 1979.

[5] Laurinčikas, A., *Limit theorems for the Riemann zeta-function*, Mathematics and its Applications, 352. Kluwer Academic Publishers Group, Dordrecht, 1996.

[6] Laurinčikas, A. and Matsumoto, K., *The universality of zeta-functions attached to certain cusp forms*, Acta Arith., **98**, (2001), 345-359, 2001.

[7] Laurinčikas, A. and Matsumoto, K., *The joint universality of twisted automorphic L-functions*, J. Math. Soc. Japan, **56**, 923-939, 2004.

[8] Maclane, G. R., *Sequence of derivatives and normal families*, J. Analyse Math., **2**, 72-87, 1952.

[9] Murty, V. K., *On the Sato-Tate conjecture*, in Number theory related to Fermat's last theorem, 195 - 205, Progr. Math., 26, Birkhauser, Boston, Mass., 1982.

[10] Nagoshi, H., *Discrepancy of Hecke eigenvalues and joint universality of automorphic L-functions*, preprint.

[11] Rankin, R. A., *An Ω-results for the coefficients of cusp forms*, Math. Ann., **203**, 239-250, 1973.

[12] Sander, J. and Steuding, J., *Joint universality for sums and products of Dirichlet L-functions*, Analysis, **26**, 295-312, 2006.

[13] Shahidi, F., *Automorphic L-functions and functoriality*, Proc. of the International Congress of Math., Vol. II (Beijing, 2002), 655 - 666, Higher Ed. Press, Beijing, 2002.

[14] Steuding, J., *Value-distribution of L-functions*, Lecture Notes in Math., vol.1877, Springer, 2007.

[15] Voronin, S. M., *Analytic properties of Dirichlet generating functions of arithmetic objects*, Math. Notes, **24(6)**, 966 - 969, 1978.