

The generalized strong recurrence and the Riemann hypothesis

By

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Abstract

In this article, we will give a survey on the theory of the Riemann hypothesis, strong recurrence and universality for the Riemann zeta-function. Then we state the main result and give a sketch the proof of it.

§ 1. Introduction

In this section we define the Riemann zeta-function. Next we explain some well-known properties of the Riemann zeta-function. We discuss the number of non-trivial zeros of the Riemann zeta function, the Riemann hypothesis and zero-free region. For details, we refer to for example, Ivić [14], Karatsuba and Voronin [18], and Titchmarsh [31]. In Section 1.3, we state two probabilistic arguments for the Riemann Hypothesis (see also Biane Pitman and Yor [4], a survey of the Riemann zeta-function from the view point of probability theory).

§ 1.1. The Riemann zeta-function

The Riemann zeta-function is defined by

$$(1.1) \quad \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma := \Re s > 1$$

where the letter p is a prime number, and the product of \prod_p is taken over all primes.

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The Dirichlet series and the Euler product of $\zeta(s)$ converges absolutely in the half-plane $\sigma > 1$ and uniformly in each compact subset of this half-plane. In view of the Euler product (1.1), it is seen easily that $\zeta(s)$ has no zeros in the half-plane $\sigma > 1$. For $s = 2k$, Euler established a simple, explicit formula which we state here as: If $k = 1, 2, \dots$ and B_n denotes the n -th Bernoulli number, then

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k} B_{2k}}{2(2k)!}.$$

The importance of $\zeta(s)$ comes from the fact that the variable s in (1.1) is a complex number, so that the function represented by the series $\sum_{n=1}^{\infty} n^{-s}$ may process an analytic continuation outside the region of absolute convergence. For $\sigma > 1$, we easily obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = (1 - 2^{1-s})\zeta(s).$$

The above formula and $2^x = 1 + x \log 2 + \dots$ give the analytic continuation for $\zeta(s)$ to the half-plane $\sigma > 0$ with a simple pole at $s = 1$ of residue 1. Moreover, this shows that $\zeta(s) \neq 0$ for all positive real s since $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} > 0$ when $s > 0$.

Riemann gave the functional equation

$$(1.2) \quad \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s),$$

where $\Gamma(s)$ denotes Euler's Gamma-function. Therefore we can continue $\zeta(s)$ analytically to the whole complex plane except for $s = 1$.

§ 1.2. Zeros of the Riemann zeta-function

It follows from the functional equation (1.2) and basic properties of the Gamma-function that $\zeta(s)$ vanishes in $\sigma < 0$ exactly at the so-called trivial zeros $s = -2n$, $n \in \mathbb{N}$. All other zeros of $\zeta(s)$ are said to be non-trivial, and we denote them by $\rho = \beta + i\gamma$. Obviously, they have to lie inside the strip $0 \leq \sigma \leq 1$. The identity $\zeta(\bar{s}) = \overline{\zeta(s)}$ and (1.2) shows that the non-trivial zeros of $\zeta(s)$ are distributed symmetrically with respect to the real axis and to the vertical line $\sigma = 1/2$.

In 1859, Riemann conjectured that the number $N(T)$ of non-trivial zeros $\rho = \beta + i\gamma$ with $0 < \gamma \leq T$ (counted with multiplicity) satisfies an asymptotic formula. This was proved by von Mangoldt in 1895 who found more precisely

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Riemann worked the function $t \mapsto \zeta(1/2 + it)$ and wrote that very likely all roots t are real, i.e., all non-trivial zeros lie on the so-called critical line $\sigma = 1/2$. This is the famous, yet unproved Riemann hypothesis which we rewrite equivalently as

$$\text{Riemann hypothesis.} \quad \zeta(s) \neq 0 \text{ for } \sigma > 1/2.$$

Classical density theorems state that almost all zeros of the Riemann zeta-function are clustered around the critical line. Denote by $N(\sigma, T)$ the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ for which $\beta > \sigma$ and $0 < \gamma \leq T$ (counted with multiplicity). Bohr and Landau proved $N(\sigma, T) = O(T)$ for any fixed $1/2 < \sigma < 1$. In 1920, Carlson showed the following density theorem: For any $\varepsilon > 0$ and $1/2 \leq \sigma \leq 1$, we have

$$(1.3) \quad N(\sigma, T) = O(T^{4\sigma(1-\sigma)+\varepsilon}).$$

Hardy (1914) was the first to prove that $\zeta(s)$ has infinitely many zeros on the critical line by considering moments of certain functions related to the zeta function. In 1942, Selberg proved that at least a (small) positive proportion of zeros lie on the critical line. Levinson (1974) improved this to one-third of the zeros by refining a mollifying technique of Selberg, and Conrey (1989) improved this further to two-fifths.

Finally we introduce information on the distribution of the non-trivial zeros. In 1896, de la Vallée-Poussin showed that $\zeta(s) \neq 0$, $\sigma \geq 1 - c/\log(|t| + 2)$, where c is some positive constant. The largest known zero-free region for $\zeta(s)$ was found by Vinogradov and Korobov (in 1958, independently). It should be noted that no one can improve this zero-free region in about 50 years. They proved

$$\zeta(s) \neq 0, \quad \sigma \geq 1 - c(\log(|t| + 2))^{-2/3}(\log \log(|t| + 3))^{-1/3}.$$

§ 1.3. Probabilistic arguments for the Riemann hypothesis

The Liouville function is defined by $\lambda(n) = (-1)^k$, where k is the number of, not necessarily distinct, prime factors of n , counted with multiplicity. The Möbius μ -function is defined by $\mu(1) = 1$, $\mu(n) = 0$ if n has a quadratic divisor $\neq 1$, and $\mu(n) = (-1)^r$ if n is the product of r distinct primes. For these functions, the following formulas are well-known:

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \quad \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \sigma > 1.$$

The following connection between the Riemann hypothesis and the Liouville function or the Möbius μ -function were explored by Landau (1899) or Littlewood (1912), respectively (see [31, Section 14.25] and [6, Theorem 1.2]). A necessary and sufficient condition for the Riemann hypothesis is

$$\sum_{n \leq x} \lambda(n) \ll x^{1/2+\varepsilon} \quad \text{or} \quad \sum_{n \leq x} \mu(n) \ll x^{1/2+\varepsilon}.$$

Denjoy [7] argued as follows (see also [30, Section 3.3]). Assume that $\{X_n\}$ is a sequence of independent random variables with distribution $P(X_n = 1) = P(X_n = -1) = 1/2$. Define $S_0 = 0$ and $S_n = \sum_{m=1}^n X_m$. By the central limit theorem, one has

$$\lim_{n \rightarrow \infty} \{ |S_n| < cn^{1/2} \} = \frac{1}{\sqrt{2\pi}} \int_{-c}^c \exp(-x^2/2) dx.$$

Hence we obtain $\lim_{n \rightarrow \infty} \mathbb{P}(|S_n| \ll n^{1/2+\varepsilon}) = 1$. In words of Edwards [8, Section 12.3]: ‘Thus these probabilistic assumptions about the values of $\mu(n)$ lead to the conclusion, ludicrous as it seems, that $\sum_{n \leq x} \mu(n) = O(x^{1/2+\varepsilon})$ with probability one and hence that the Riemann hypothesis is true with probability one!’.

To mention the second probabilistic argument, we prepare notation. Denote by γ the unit circle on \mathbb{C} , and let $\Omega := \prod_p \gamma_p$, where $\gamma_p = \gamma$ for all primes p . With product topology and pointwise multiplication this infinite-dimensional torus Ω is a compact topological abelian group; the compactness of Ω follows from Tikhonov’s theorem. Hence the normalized Haar measure m_H on $(\Omega, \mathfrak{B}(\Omega))$ exists. This gives a probability space $(\Omega, \mathfrak{B}(\Omega), m_H)$. Let $\omega(p)$ stand for the projection of $\omega \in \Omega$ to the coordinate space γ_p . Since the Haar measure m_H is the product of the Haar measures on the coordinate spaces $\gamma(p)$, $\{\omega(p)\}$ is a sequence of independent complex-valued random variables uniformly distributed on the circle γ . Let $\omega(1) := 1$ and $\omega(n) := \prod_p \omega(p)^{\theta(n;p)}$, where $\theta(n;p)$ is the exponent of p in the prime factorization of n . Helson [13] showed the following.

Theorem A. *For almost all ω , the series $\zeta_\omega(s) := \sum_{n=1}^{\infty} \omega(n)n^{-s}$ converges on the half-plane $\Re(s) > 1/2$ to an analytic function which has no zeros there.*

This theorem has a curious interpretation for the vertical limit functions $\zeta_\omega(s)$ of the Riemann zeta-function: the Riemann hypothesis holds for almost every ω , namely with probability one (see [12, p. 22]). It should be noted that similar results are found in [3, Lemma 4.10], [12, Theorem 4.6], [19, Theorem 5.1.7] and [30, Lemma 4.2].

§ 2. Universality and Strong recurrence

Firstly we briefly introduce the history of universality. For details, see Laurinćikas [19], Matsumoto [21] and Steuding [30]. Next, we present the notion of the strong recurrence which is equivalent to the Riemann hypothesis (see [30, Section 8]). Finally, we state the main theorem the generalized strong recurrence which should be compared with the probabilistic arguments for the Riemann hypothesis has been written in Section 1.3.

§ 2.1. Universality

The distribution of the values of the Riemann zeta-function $\zeta(\sigma + it)$ for fixed σ and variable $t > 0$ was investigated by H. Bohr. In 1914, he showed the following denseness theorem, as a joint work with Courant (see for example [31, Chapter 11]).

Theorem B. *For any fixed σ satisfying $1/2 < \sigma < 1$, the set $\{\zeta(\sigma + it) : t \in \mathbb{R}\}$ is dense in the whole complex plane.*

This fact should be compared with $0 < |\zeta(\sigma)|^{-1} \leq |\zeta(s)| \leq \zeta(\sigma)$, $\sigma > 1$. This theorem of Bohr was the first remarkable denseness result for the Riemann zeta-function and it was generalized by S. M. Voronin in 1972. He proved that if s_1, s_2, \dots, s_m are distinct points lying in the strip $1/2 < \sigma < 1$, and $h > 0$ is an arbitrary fixed number, then the sequence $(\zeta(s_1 + inh), \zeta(s_2 + inh), \dots, \zeta(s_m + inh))$, $n \in \mathbb{N}$, is dense in \mathbb{C}^m . He also obtained that the sequence $(\zeta(s_0 + inh), \zeta'(s_0 + inh), \dots, \zeta^{(m-1)}(s_0 + inh))$, $n \in \mathbb{N}$, is dense in \mathbb{C}^m for any fixed s_0 such that $1/2 < \Re(s_0) \leq 1$.

A natural next step is to study the situation on infinite dimensional spaces, namely on function spaces. Concerning this problem, in 1975, S. M. Voronin [32] showed the next theorem, which is now called the universality (see for instance [19, Theorem 6.5.1] or [30, Theorem 1.7]). To state it, we prepare some notation for universality. By $\text{meas}\{A\}$ we denote the Lebesgue measure of the set A , and, for $T > 0$, we use the notation $\nu_T\{\dots\} := T^{-1}\text{meas}\{\tau \in [0, T] : \dots\}$ where in place of dots some condition satisfied by τ is to be written. The original version of Voronin's theorem has the form; Let $0 < r < 1/4$ and suppose that $g(s)$ is a non-vanishing continuous function on the disk $|s| \leq r$ which is analytic in the interior. Then for any $\varepsilon > 0$, we have

$$\liminf_{T \rightarrow \infty} \nu_T \left\{ \max_{|s| \leq r} |\zeta(s + 3/4 + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

This theorem means that any non-vanishing analytic function can be uniformly approximated by certain purely imaginary shifts of the Riemann zeta-function $\zeta(s)$. Moreover, the set of approximating shifts has positive lower density. Reich [29] and Bagchi [2] improved Voronin's universality theorem significantly in replacing the disk by an arbitrary compact subset in the critical strip $D := \{s \in \mathbb{C} : 1/2 < \Re(s) < 1\}$ with connected complement, and by giving a lucid proof in the language of probability theory (see [19, Theorem 6.5.2] or [30, Theorem 1.9]). The modern version of universality theorem has the form.

Theorem C. *Let K be a compact subset of the strip D with connected complement, and $f(s)$ be a function analytic in the interior of K and continuous and non-vanishing on K . Then for every $\varepsilon > 0$, it holds that*

$$\liminf_{T \rightarrow \infty} \nu_T \left\{ \max_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

§ 2.2. Strong recurrence

The restriction non-vanishing for $f(s)$ in Theorem C can not be removed (see [30, Section 8.1]). This is closely related to the Riemann hypothesis and the strong recurrence. Bagchi in his Ph. D. Thesis [2], proved that the Riemann hypothesis is true if and only if the Riemann zeta-function can be approximated by itself in the sense of universality (see also [3, Theorem 3.7] and [30, Theorem 8.3]).

Theorem D. *The Riemann hypothesis is true if and only if, for any compact subset K in the strip D with connected complement and for any $\varepsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \nu_T \left\{ \max_{s \in K} |\zeta(s + i\tau) - \zeta(s)| < \varepsilon \right\} > 0.$$

This property is called strong recurrence (in the critical strip). We give a sketch of the proof (see [30, Theorem 8.3] for the details). If the Riemann hypothesis is true, we can apply universality Theorem C, which implies the strong recurrence. The idea for the proof of the other implication is that if there is at least one zero to the right of the critical line, then the strong recurrence (and Rouché's theorem) implies the existence of too many zeros with regard to the classic density theorem written by (1.3).

Note that Bohr [5] showed that $\zeta(s)$ has the strong recurrence not in the critical strip but in its half-plane of absolute convergence. Effective upper bounds for the strong recurrence in the region of absolute convergence of Dirichlet series with Euler products were considered by Gironde and Steuding [10] (see also [30, Sections 8 and 9]). Recently, Andersson [1] proved that $\zeta(s)$ has the strong recurrence for any compact set contained in D with connected complement and without interior points.

Another property similar to the strong recurrence follows from Kaczorowski, Laurinćikas and Steuding [16]. From this paper we have the following statement (see also [30, Section 10.6]): Let K be as in Theorem C and let $\lambda \in \mathbb{R}$ be such that sets K and $K + i\lambda := \{s + i\lambda : s \in K\}$ are disjoint. Then $\zeta(s + i\lambda + i\tau)$ approximates $\zeta(s + i\tau)$ in the sense of universality.

The main theorem in this article is as follows; *generalized strong recurrence*.

Theorem 2.1. *For any $0 \neq d \in \mathbb{R}$, $\varepsilon > 0$ and compact set K contained in D ,*

$$(2.1) \quad \liminf_{T \rightarrow \infty} \nu_T \left\{ \max_{s \in K} |\zeta(s + i\tau) - \zeta(s + id\tau)| < \varepsilon \right\} > 0.$$

This theorem is a kind of improvement of Denjoy's and Helson's probabilistic argument (see Section 1.3). Because the main theorem has only one exception when $d = 0$, and it is the Riemann hypothesis itself! The author proved the generalized strong recurrence for every algebraic irrational number and almost all real numbers in [23, Corollaries 1.2 and 1.4]. Afterwards, Pańkowski showed the generalized strong recurrence for any irrational number in [27, Theorem 1.1]. Garunkštis [9, Theorem 1] and the author [24, Theorem 1.1] independently proved the generalized strong recurrence for any non-zero rational number. Moreover, Garunkštis [9, Corollary 3] shows that 'liminf' in the inequality (2.1) often can be replaced by 'lim'.

§ 3. Proof of the generalized strong recurrence

Pańkowski [27] and Garunkštis [9] showed the generalized strong recurrence for any non-zero real number by Voronin’s method (see for example [32] and [18]). In Section 3.1, we give a sketch of the proof of the generalized strong recurrence for any irrational number d by using Bagchi’s method (see for instance [3], [19] and [30]). In Section 3.2, we show the generalized strong recurrence for much wider zeta- or L -functions.

§ 3.1. For the Riemann zeta-function

The main idea in the proof of the main theorem comes from not only Pańkowski [27] and Garunkštis [9] but also the joint universality for denominators proved by Laurinćikas and Matsumoto [20], Mishou [22], and the author [23], and the hybrid universality showed by Gonek [11], Kaczorowski and Kulas [15], and Pańkowski [28]. In order to prove the main theorem, we prepare a lemma and notations. The following lemma is proved by Pańkowski (see [27, Lemma 2.4]).

Lemma 3.1. *Let \mathcal{P} be the set of all primes. For arbitrary real irrational number d , there exists a finite set of primes A_d containing at most two elements such that the set $\{\log p\}_{\mathcal{P} \setminus A_d} \cup \{\log p^d\}_{\mathcal{P}}$ is linearly independent over \mathbb{Q} .*

Fix $d \in \mathbb{R} \setminus \mathbb{Q}$, and we can put $A := \{a_1, a_2\}$ in the above lemma without loss of generality. Let $\alpha_h \log a_h := \sum_{n=1}^m \beta_{h,n} \log p_n + \sum_{n=1}^m \gamma_{h,n} \log p_n^d$, where $h = 1, 2$ and $\alpha_h, \beta_{h,n}$ and $\gamma_{h,n}$ are integers. By Lemma 3.1, the set $\{\log p_n\}_{\mathcal{P} \setminus B} \cup \{\log p_n^d\}_{\mathcal{P}}$, $B := \{a_1, a_2, p_1, \dots, p_m\}$ is linearly independent over \mathbb{Q} . Denote by γ the unit circle and set

$$\Upsilon := \Phi \times \Psi \times \Omega, \quad \Phi := \prod_{p \in B \setminus A} \gamma_p, \quad \Psi := \prod_{p \in \mathcal{P} \setminus B} \gamma_p, \quad \Omega := \prod_{p \in \mathcal{P}} \gamma_p,$$

where $\gamma_p = p$ for all primes p . Since Υ is a compact Abelian group, the probability Haar measure m_H on $(\Upsilon, \mathfrak{B}(\Upsilon))$ can be defined. Denote respectively by $\phi(p), \psi(p)$ and $\omega(p)$ the projection of $\phi \in \Phi, \psi \in \Psi$ and $\omega \in \Omega$ to the coordinate space γ_p for any p . Put $\zeta_B(s) := \prod_{p \in B} (1 - p^{-s})^{-1}$,

$$b(\tau) := (a_1^{i\tau}, a_2^{i\tau}, p_1^{i\tau}, \dots, p_m^{i\tau}), \quad b(\phi) := (\phi(a_1), \phi(a_2), \phi(p_1), \dots, \phi(p_m))$$

$$\zeta_B(s, \psi) := \prod_{p \in \mathcal{P} \setminus B} \left(1 - \frac{\psi(p)}{p^s}\right)^{-1}, \quad \zeta(s, \omega) := \prod_{p \in \mathcal{P}} \left(1 - \frac{\omega(p)}{p^s}\right)^{-1},$$

$$Z(s, \tau) := (b(\tau), \zeta_B(s + i\tau), \zeta(s + id\tau)), \quad Z(s, \phi, \psi, \omega) := (b(\phi), \zeta_B(s, \psi), \zeta(s, \omega)).$$

In view of [30, Lemma 4.1], $Z(s, \phi, \psi, \omega)$ is a $J_m(D) := \mathbb{C}^{m+2} \times H^2(D)$ -valued random element on the probability space $(\Upsilon, \mathfrak{B}(\Upsilon), m_H)$. Hence we obtain the following propo-

sition by modifying the proofs of [20, Theorem 1], [22, Theorem 1] and [23, Proposition 2.3].

Proposition 3.2. *Define the probability measures*

$$P_T(A) := \nu_T\{Z(s, \tau) \in A\}, \quad P(A) := m_H\{Z(s, \phi, \psi, \omega) \in A\}$$

for $A \in \mathfrak{B}(J_m(D))$. Then the measure P_T converges weakly to P as $T \rightarrow \infty$.

By the above proposition and the method used in the proofs of [20, Theorem 2], [22, Theorem 2] and [23, Theorem 1.1], we have: Suppose θ_p , $p \in C := B \setminus A$ are arbitrary numbers with $|\theta_p| = 1$. Let K , $f(s)$ and $g(s)$ be as in Theorem C. Then for any $\varepsilon > 0$,

$$\liminf_{T \rightarrow \infty} \nu_T \left\{ \max_{s \in K} |\zeta_B(s + i\tau) - f(s)| < \varepsilon, \max_{s \in K} |\zeta(s + id\tau) - g(s)| < \varepsilon, \right. \\ \left. \max_{p \in C} |p^{-i\tau} - \theta_p| < \varepsilon \right\} > 0.$$

Take $f(s) \equiv g(s) \equiv 0$ and $\theta_p = 1$. Note that in this situation we can remove the assumption with connected complement for the compact set K (see [30, p. 107]). Hence we obtain the main theorem.

§ 3.2. For zeta-functions

It is quite natural to generalize these results for several types of zeta- or L -functions. We quote the notation and theorems for the Steuding class $\tilde{\mathcal{S}}$ and the Selberg class \mathcal{S} from [30, Sections 2 and 6 or Notations]. In the sequel, we assume that a function $\mathcal{L}(s)$ has a representation as a Dirichlet series $\mathcal{L}(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ in some half-plane. We denote by $\tilde{\mathcal{S}}$ the class of Dirichlet series satisfying the axioms (i)-(v), and by \mathcal{S} the class of Dirichlet series satisfying the axioms (1)-(4).

- (i), (1) *Ramanujan hypothesis.* $a(n) = O(n^\varepsilon)$ for any $\varepsilon > 0$, where the implicit constant may depend on ε .
- (ii) *Analytic continuation.* There exists a real number $\sigma_{\mathcal{L}}$ such that $\mathcal{L}(s)$ has an analytic continuation to the half plane $\sigma > \sigma_{\mathcal{L}}$ with $\sigma_{\mathcal{L}} < 1$ except for at most a pole at $s = 1$.
- (2) *Analytic continuation.* There exists a non-negative integer k such that $(s-1)^k \mathcal{L}(s)$ is an entire function of finite order.
- (iii) *Finite order.* There exists a constant $\mu_{\mathcal{L}} \geq 0$ such that, for any fixed $\sigma > \sigma_{\mathcal{L}}$ and any $\varepsilon > 0$, $\mathcal{L}(\sigma + it) = O(|t|^{\mu_{\mathcal{L}} + \varepsilon})$ as $|t| \rightarrow \infty$; again the implicit constant may depend on ε .

- (3) *Functional equation.* The function $\mathcal{L}(s)$ satisfies a functional equation of type $\lambda_{\mathcal{L}}(s) = \omega \overline{\lambda_{\mathcal{L}}(1 - \bar{s})}$, where $\lambda_{\mathcal{L}}(s) := \mathcal{L}(s)Q^s \prod_{j=1}^f \Gamma(\lambda_j s + \mu_j)$ with positive real numbers Q, λ_j and complex numbers μ_j, ω with $\Re(\mu_j) \geq 0$ and $|\omega| = 1$.
- (iv) *Polynomial Euler product.* There exist $m, k \in \mathbb{N}$, and for every prime p , there are complex numbers $\alpha_j(p), 1 \leq j \leq k$, such that $\mathcal{L}(s) = \prod_p \prod_{j=1}^k (1 - \alpha_j(p)p^{-s})^{-1}$.
- (4) *Euler product.* $\mathcal{L}(s)$ satisfies $\mathcal{L}(s) = \prod_p \mathcal{L}_p(s)$ where $\mathcal{L}_p(s) = \exp(\sum_{j=1}^{\infty} b(p^j)p^{-js})$ with suitable coefficients $b(p^j)$ satisfying $b(p^j) = O(p^{j\theta})$ for some $\theta < 1/2$.
- (v) *Prime mean-square.* Let $\pi(x)$ denote the number of primes satisfying $p \leq x$. There exists a positive constant κ such that $\lim_{x \rightarrow \infty} \pi(x)^{-1} \sum_{p \leq x} |a(p)|^2 = \kappa$.

Define a so-called mean square half-plane by defining its abscissa σ_* as the infimum over all σ_1 such that $\int_{-T}^T |\mathcal{L}(\sigma + it)|^2 dt \sim T \sum_{n=1}^{\infty} |a(n)|^2 n^{-2\sigma}$ holds for all fixed $\sigma > \sigma_1$. Put $\mathcal{D} := \{s \in \mathbb{C} : \sigma_* < \Re(s) < 1\}$. Now we are in the position to state the universality theorem for $\mathcal{L} \in \tilde{\mathcal{S}}$ (see [30, Theorem 5.14]).

Theorem E. *Let $\mathcal{L} \in \tilde{\mathcal{S}}, \mathcal{K}$ be a compact subset of the strip \mathcal{D} with connected complement, and let $f(s)$ be a non-vanishing continuous function on \mathcal{K} which is analytic in the interior of \mathcal{K} . Then for any $\varepsilon > 0$, it holds that*

$$(3.1) \quad \liminf_{T \rightarrow \infty} \nu_T \left\{ \max_{s \in \mathcal{K}} |\mathcal{L}(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

Next, we generalize Theorem C. The degree of $\mathcal{L} \in \mathcal{S}$ is defined by $d_{\mathcal{L}} := 2 \sum_{j=1}^f \lambda_j$. Although the data of the functional equation are not unique, the degree is well-defined. Let $N_{\mathcal{L}}(\sigma, T)$ count the number of zeros $\rho = \beta + i\gamma$ of $\mathcal{L}(s)$ with $\beta > \sigma$ and $|\gamma| < T$ (counting multiplicities). As a generalization of the formula (1.3), Kaczorowski and Perelli [17, Lemma 3] proved that for every $\mathcal{L} \in \mathcal{S}$,

$$(3.2) \quad N_{\mathcal{L}}(\sigma, T) = O(T^{4(d_{\mathcal{L}}+3)(1-\sigma)+\varepsilon}),$$

uniformly for $1/2 \leq \sigma \leq 1$. Using the above formula, we have $N_{\mathcal{L}}(\sigma, T) = o(T)$ if $\sigma > \sigma_* = \sigma_*(\mathcal{L}) := 1 - 1/4(d_{\mathcal{L}} + 3)$. Therefore we have the following theorem which is a generalization of Theorem D (see [30, Theorem 8.4]).

Theorem F. *Let $\mathcal{L} \in \mathcal{S} \cap \tilde{\mathcal{S}}$. Then the function $\mathcal{L}(s)$ is non-vanishing in the half-plane $\sigma_* < \Re(s) < 1$ if and only if for any $\varepsilon > 0$, any z with $\sigma_* < \Re(z) < 1$ and any $0 < r < \min\{\Re(z) - \sigma_*, 1 - \Re(z)\}$,*

$$\liminf_{T \rightarrow \infty} \nu_T \left\{ \max_{|s-z| \leq r} |\mathcal{L}(s + i\tau) - \mathcal{L}(s)| < \varepsilon \right\} > 0.$$

At the end of this article, we state the following which is a generalization of Theorem 2.1. In [25, Theorem 2.5], the generalized strong recurrence for almost all real numbers for $\tilde{\mathcal{S}}$ are proved. The following theorem is the generalized strong recurrence for all non-zero real numbers for $\tilde{\mathcal{S}}$. We omit the proof since it is proved by a reformulation of the proof of Theorem 2.1 and [30, Theorem 4.3].

Theorem 3.3. *Let $\mathcal{L} \in \tilde{\mathcal{S}}$. For any $0 \neq d \in \mathbb{R}$, $\varepsilon > 0$ and compact set $\mathcal{K} \subset \mathcal{D}$,*

$$\liminf_{T \rightarrow \infty} \nu_T \left\{ \max_{s \in \mathcal{K}} |\mathcal{L}(s + i\tau) - \mathcal{L}(s + id\tau)| < \varepsilon \right\} > 0.$$

Note

Unfortunately, the papers [9] and [24] contain a gap in the proof of the main theorem, so actually their methods work only for the logarithm of the Riemann zeta function. Therefore ζ in Theorem 2.1 and \mathcal{L} in Theorem 3.3 should be replaced by $\log \zeta$ and $\log \mathcal{L}$, respectively. It should be noted that the author and Pańkowski [26] showed that Theorem D also holds for $\log \zeta$ instead of ζ .

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