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# Nonplanar traveling waves of a bistable reaction-diffusion equation in the multi-dimensional space

By

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**Abstract**

We survey the existence result for nonplanar traveling waves of a bistable reaction-diffusion equation in the space $\mathbb{R}^{n+1}$, which gives a heteroclinic connection of the trivial solution and the unstable standing wave in $\mathbb{R}^n$. Since the problem can be formulated in a monostable case, the equation allows a continuous family of the traveling waves with the speed in a semi-infinite interval.

## § 1. Introduction

In this article we consider the following scalar reaction-diffusion equation in the multi-dimensional space $\mathbb{R}^{n+1}$:

$$u_t = \Delta u + u_{yy} + f(u), \tag{1.1}$$

where

$$x = (x_1, \cdots, x_n) \in \mathbb{R}^n,\ y \in \mathbb{R}, \quad u_t = \frac{\partial u}{\partial t}, \quad \Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}, \quad u_{yy} = \frac{\partial^2 u}{\partial y^2}.$$

We assume that $f(u)$ is $C^2$ on an open interval containing $[0,1]$ and that $f(u)$ satisfies

$$f(0) = f(a) = f(1) = 0, \quad f'(0) < 0, \quad f'(a) > 0, \quad f'(1) < 0, \tag{1.2}$$

$$f(u) \neq 0 \ \text{ for } \ u \in (0,a) \cup (a,1), \tag{1.3}$$

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where $0 < a < 1$, and

$$f''(u) \geq 0 \quad (0 \leq u \leq \beta), \quad f''(u) \leq 0 \quad (\beta \leq u \leq 1) \tag{1.4}$$

for a number $\beta \in (0, 1)$. The condition (1.2) implies that constant solutions $u = 0, 1$ are asymptotically stable equilibria of the diffusion free equation of (1.1) while $u = a$ is an unstable one. The equation (1.1) satisfying (1.2) and (1.3) is called a bistable reaction-diffusion equation. It is known that (1.1) with (1.2)-(1.3) allows a planar traveling wave, that is, there is a solution $u = \phi(y - ct)$ satisfying

$$\begin{cases} \phi_{zz} + c\phi_z + f(\phi) = 0, \quad \phi(z) > 0 \qquad (z \in \mathbb{R}), \\ \lim_{z \to -\infty} \phi(z) = 1, \quad \lim_{z \to \infty} \phi(z) = 0. \end{cases}$$

where $z = y - ct$ (for instance, see [7]). Since $\phi'(z) < 0$ holds for the solution $\phi(z)$, it is called a monotone planar traveling wave (solution). Henceforth we assume

$$\int_0^1 f(u)du > 0,$$

which implies that the speed $c$ is positive.

We note that the stationary equation

$$\Delta v + f(v) = 0 \quad (x \in \mathbb{R}^n), \qquad v(x) > 0, \qquad \lim_{|x| \to \infty} v(x) = 0$$

allows a one hump solution $v(x)$ for $n = 1$ or for $n \geq 2$ a radially symmetric solution $v(x) = \Phi(r)$ $(r = |x|)$

$$\begin{cases} \Phi_{rr} + \dfrac{n-1}{r}\Phi_r + f(\Phi) = 0, \quad \Phi(r) > 0 \quad (0 < r < \infty), \\ \Phi_r(0) = 0, \quad \lim_{r \to \infty} \Phi(r) = 0 \end{cases} \tag{1.5}$$

with the monotone profile $\Phi_r(r) < 0$ $(0 < r < \infty)$. These solutions $v(x)$ are called standing wave solutions and they are unique up to translation. Moreover, those are unstable as equilibrium solutions to

$$v_t = \Delta v + f(v) \qquad (x \in \mathbb{R}^n)$$

for $n = 1$ and $n \geq 2$ respectively.

We survey the results for the traveling wave connecting $u = 1$ (at $y = -\infty$) with $u = v(x)$ (at $y = \infty$). We look for the solution with the form $u = U(x, y - ct)$, namely, we consider the equation

$$\begin{cases} \Delta U + U_{zz} + cU_z + f(U) = 0 \quad ((x, z) \in \mathbb{R}^{n+1}), \\ \lim_{z \to -\infty} U(x, z) = 1, \quad \lim_{z \to \infty} U(x, z) = v(x). \end{cases} \tag{1.6}$$

Similarly, we can also consider the solution of

$$\begin{cases} \Delta U + U_{zz} + cU_z + f(U) = 0 \quad ((x,z) \in \mathbb{R}^{n+1}), \\ \lim\limits_{z\to-\infty} U(x,z) = 0, \quad \lim\limits_{z\to\infty} U(x,z) = v(x). \end{cases}$$

The same argument can apply to the latter case, so we omit the statement for this case, though the profile of $U(x,z)$ turns to be monotone increasing in $z$ while decreasing in (1.6).

The problem of (1.6) reminds us of the monostable case as in [15] and [1, 2]. Therefore, we can expect that there is a family of traveling waves with the speed $c$ in a semi-infinite interval. We show that this is certainly true in §3.

## § 2. Stationary problem in $\mathbb{R}^n$

Assume (1.2) and (1.3) and consider

$$\Delta v + f(v) = 0 \quad (x \in \mathbb{R}^n), \qquad v(x) > 0, \quad \lim_{|x|\to\infty} v(x) = 0. \tag{2.1}$$

For $n = 1$, the equation

$$v_{xx} + f(v) = 0 \tag{2.2}$$

is converted to a Hamilton system and we can easily obtain a unique solution of (2.1) up to translation by a homoclinic orbit of the Hamiltonian system. Moreover, the linearized eigenvalue problem for such a solution $v(x)$

$$\varphi_{xx} + f'(v(x))\varphi = \mu\varphi, \qquad \lim_{|x|\to\infty} \varphi(x) = 0$$

tells that the first eigenvalue is simple and positive while the second one is zero with the corresponding eigenfunction $\varphi = v_x(x)$.

For $n \geq 2$ we further assume (1.4). Given a positive $R$, we first consider the following problem in the ball with the radius $R$:

$$\Delta v + f(v) = 0 \quad (|x| < R), \qquad v = 0 \quad (|x| = R) \tag{2.3}$$

By the famous result [8] all positive solutions to (2.3) is radially symmetric, so put $v = v(r), r = |x|$, and they satisfy

$$v'' + \frac{n-1}{r}v' + f(v) = 0 \quad (0 < r < R), \qquad v'(0) = v(R) = 0. \tag{2.4}$$

Then the solution satisfies $v'(r) < 0$ $(0 < r < R)$. In fact, using the shooting method, we can classify the positive solutions to (2.4). In addition, the local bifurcation theorem applied to this case tells the following result:

**Proposition 2.1.** ([3], [19]) *There is a number* $R^* > 0$ *such that* (2.4) *with* (1.2)$\sim$(1.4) *has no positive solution for* $R < R^*$, *exactly one positive solution for* $R = R^*$ *and for* $R > R^*$ *exactly two positive solutions, denoted by* $v = v_R^+, v_R^-$, *satisfying* $0 < v_R^-(r) < v_R^+(r) < 1$.

We note that for the linearized eigenvalue problem of a positive solution $v_R$ to (2.4), defined by

$$\mathcal{L}_R[w] := \Delta w + f'(v_R)w = \mu w \quad (|x| < R), \qquad w = 0 \quad (|x| = R), \tag{2.5}$$

the eigenfunction $w$ corresponding to a positive eigenvalue $\mu$ $(> 0)$ is radially symmetric. This implies that the Morse index for the solution $v_R$ in (2.3) coincides with that in (2.4). Consequently, we have the next result.

**Proposition 2.2.** ([3], [19]) *Assume* $R > R^*$. *Then the spectrum of the linearized operator* $\mathcal{L}_R$ *for* $v = v_R^+$ *consists of negative eingenvalues while for* $v = v_R^-$ *it consists of one positive eigenvalue and negative eigenvalues.*

Now we consider the problem in the whole space.

$$\Delta v + f(v) = 0 \quad (x \in \mathbb{R}^n), \qquad v(x) > 0, \quad \lim_{|x|\to\infty} v(x) = 0. \tag{2.6}$$

By the argument in [9] we see that a positive solution to (2.6) must be radially symmetric. In fact, we have more properties for the solution to (2.6).

**Theorem 2.3.** ([3], [19]) *Assume* (1.2)$\sim$(1.4). *Then there is a positive solution to* (2.6) *which is radially symmetric and unique up to translation. For the solution denoted by* $\Phi(|x|)$, *consider the spectrum* $\sigma(\mathcal{L})$ *of*

$$\mathcal{L} := \Delta + f'(\Phi) : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \tag{2.7}$$

*Then the following properties hold:*

(i) $\sigma(\mathcal{L}) = \sigma_p(\mathcal{L}) \cup \sigma_e(\mathcal{L})$, *where* $\sigma_p(\mathcal{L})$ *is the point spectrum and* $\sigma_e(\mathcal{L})$ *is the essential spectrum.*

(ii) $\sigma_e(\mathcal{L}) = (-\infty, f'(0)]$, $\sigma_p(\mathcal{L}) \subset (f'(0), \infty)$.

(iii) *For* $\mu \in \sigma_p(\mathcal{L})$ *the corresponding eigenfunction* $\psi$ *satisfies*

$$|\psi(x)| \le C_\varepsilon e^{-\sqrt{(-f'(0)+\mu+\varepsilon)/2}|x|}$$

*for any* $\varepsilon > 0$ *and a constant* $C_\varepsilon > 0$ *depending on* $\varepsilon$.

(iv) *For* $\mu \in (0, \infty) \cap \sigma_p(\mathcal{L})$ *the corresponding eigenfunction is radially symmetric.*

(v) *The principal eigenvalue* $\lambda_1 > 0$ *is simple and the corresponding eigenfunction can be chosen positive.*

(vi) *The second eigenvalue is* $\lambda_2 = 0$ *and the eigenspace for* $\lambda_2 = 0$ *is spanned by*

$$\frac{\partial \Phi}{\partial x_i} \quad (i = 1, 2, \cdots, n).$$

We remark that as $R \to \infty$, the solution $v_R^+$ to (2.4) converges to $v = 1$ uniformly in any compact set while $v_R^-$ converges to the solution $\Phi$ of (1.5).

## § 3. Nonplanar traveling waves

By [6], [10], [11, 12, 13] and [17, 18] the existence and the stability of nonplanar traveling waves, called $V$-shaped waves and conical waves, have been extensively studied. Moreover, the existence of pyramidal shaped fronts is also given in [20, 21]. Here we introduce another type of traveling waves studied in [14] and [16].

For $n = 1$, as mentioned in the previous section, the stationary problem (2.2) is converted to a Hamiltonian system, thus we see that it has a family of periodic solutions $\{v_p(x; T)\}_{\omega < T < \infty}$ ($\omega := 2\pi\sqrt{f'(a)}$), where $T$ is the period of the solution, and a homoclinic solution $v_h(x)$. Then $v_p(\cdot; T) \to a$ as $T \to \omega + 0$, while as $T \to \infty$, $v_p(\cdot; T) \to v_h(\cdot)$ in any compact set.

The authors of [14] prove that (1.1) with (1.2) and (1.3) allows a heteroclinic connection between $u = 1$ ($y \to -\infty$) and $u = v_p(\cdot; T)$ ($y \to \infty$). More precisely, for each $T \in (\omega, \infty)$, there is a positive number $c_{min}(T)$ such that the equation allows a traveling wave solution given by solving (1.6) with $v = v_p(\cdot; T)$ if and only if $c \in [c_{min}(T), \infty)$. Moreover, the solution is unique up to translation.

They also prove that $c_{min}(T) \to c^*$ ($T \to \omega + 0$) for a number $c_{min} > 0$ and that for each $c > c^*$, the solution denoted by $U(x, z; c, T)$, converges to the planer traveling wave of (1.1), say $U^{1a}(z; c)$, connecting $u = 1$ ($z \to -\infty$) with $u = a$ ($z \to \infty$).

On the other hand, there is $c_{min} > 0$ such that $c_{min}(T) \to c_{min}$ ($T \to \infty$) and for each $c > c_{min}$ the solution $U(x, z; c, T)$ converges to $U(\cdot, \cdot; c)$ of (1.6) with $v = v_h$ as $T \to \infty$ with an appropriate shift.

In sequel they succeed to prove the existence of the traveling wave together with the characterization of the minimum speed for the solution to (1.6).

**Theorem 3.1.** ([14]) *Assume* (1.2) *and* (1.3). *For* $n = 1$ *there is a solution* $U = U(x, z; c)$ *to* (1.6) *satisfying* $U(-x, z; c) = U(x, z; c), U_z(\cdot, z; c) < 0$ *if and only if* $c \in [c_{min}, \infty)$. *This solution is unique up to* $z$ *translation.*

As for the higher dimensional case we have the next result.

**Theorem 3.2.** ([16]) *Let* $\Phi$ *be the positive solution to* (1.5) *which is unique. Then there is a solution to* (1.6) *satisfying* $U_z(x,z) \le 0$ *if* $c \ge 2\sqrt{\kappa}$, *where*

$$\kappa := \max\left\{\max_{0\le u\le 1}|f'(u)|, \lambda_1\right\}$$

($\lambda_1$ *is the first eigenvalue of* $\mathcal{L}$ *defined in* (2.7)).

The proof is done by the comparison method, that is, constructing appropriate subsolution and supersolution[1]. Unfortunately, this result is not so strong as Theorem 3.1 since the existence of the minimum speed is not discussed. Namely, $c \ge 2\sqrt{\kappa}$ is a technical condition to ensure the existence of the traveling wave. The uniqueness of the solution to (1.6) up to translation is neither discussed.

To apply the similar argument in [14] for $n \ge 2$, it is natural to consider the equation in the cylindrical domain given by

$$\begin{cases} \Delta W + W_{zz} + cW_z + f(w) = 0 \quad ((x,z) \in \{|x| < R\} \times \mathbb{R}), \\ \lim_{z\to-\infty} W(x,z) = v_R^+(x), \quad \lim_{z\to\infty} W(x,z) = v_R^-(x), \end{cases} \tag{3.1}$$

where $v_R^+(r)$ and $v_R^-(r)$ are the stable solution and the unstable solution of (2.4) with (1.2)$\sim$(1.4) respectively. As mentioned in the previous section, those solutions are radially symmetric and ordered as $0 < v_R^-(r) < v_R^+(r)$. Moreover,

$$\lim_{R\to\infty} v_R^-(r) = \Phi(r), \quad \lim_{R\to\infty} v_r^+(r) = 1 \qquad \text{(in any bounded interval } [0,K])$$

hold. By the results of the previous section and applying the results found in [4] or [22, 23], we obtain the next result.

**Lemma 3.3.** *Assume* (1.2)$\sim$(1.4). *Then there is a number* $c_m^R > 0$ *such that for each* $c \in [c_m^R, \infty)$ *the equation* (3.1) *allows a unique solution* $W(r,z)$ $(r = |x|)$ *satisfying* $W_z < 0$ *up to translation in* $z$.

Thus, the solution $u_R(x,y,t) := W(|x|, y - ct)$ gives the traveling wave connecting $u_R = v_R^+$ $(y = -\infty)$ and $u_R = v_R^-$ $(y \to \infty)$ in the cylindrical domain.

In conclusion we need a further discussion to prove the convergence of the minimum speed $c_m^R$ as $R \to \infty$, which is a future work.

[1] The definition of $\kappa$ in [16] should be corrected as in the above theorem.

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