RIMS Kôkyûroku Bessatsu
B35 (2012), 31–40

Infinite dimensional relaxation oscillation in reaction-diffusion systems

By

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§ 1. Introduction

In systems of ordinary differential equations, oscillatory solutions or periodic solutions are often observed. Among such oscillatory solutions, there are relaxation oscillations as a specific behavior of oscillation. Most typical relaxation oscillation arises in the following van der Pol equation:

$$\frac{d^2u}{dt^2} - \frac{1}{\delta}(1-u^2)\frac{du}{dt} + u = 0, \tag{1.1}$$

where u is the dynamical variable and $\delta > 0$ is a parameter. Using the transformation $v = u - \frac{u^3}{3} - \delta\frac{du}{dt}$, the equation (1.1) can be written as

$$\begin{aligned} \frac{du}{dt} &= \frac{1}{\delta}(u - \frac{u^3}{3} - v), \\ \frac{dv}{dt} &= \delta u. \end{aligned} \tag{1.2}$$

In the case of sufficiently small $\delta \ll 1$, relaxation oscillation can be exhibited. The feature of relaxation oscillation is to consist of two dynamics, called fast and slow dynamics. When the value $(u(t), v(t))$ is away from the curve $v = u - u^3/3$, $(u(t), v(t))$ moves quickly in the horizontal direction due to $|\frac{dv}{dt}| = O(\delta)$. (See Figure 1.) This is called fast dynamics. When $(u(t), v(t))$ enter the region where $|u - \frac{u^3}{3} - v| = O(\delta^2)$, the both of $\frac{du}{dt}$ and $\frac{dv}{dt}$ are $O(\delta)$. Therefore, the solution $(u(t), v(t))$ moves slowly along the curve, which is called slow dynamics. For example, from the equation for v in

Received November 15, 2015.
2000 Mathematics Subject Classification(s): 35K57, 35B10, 35R15.
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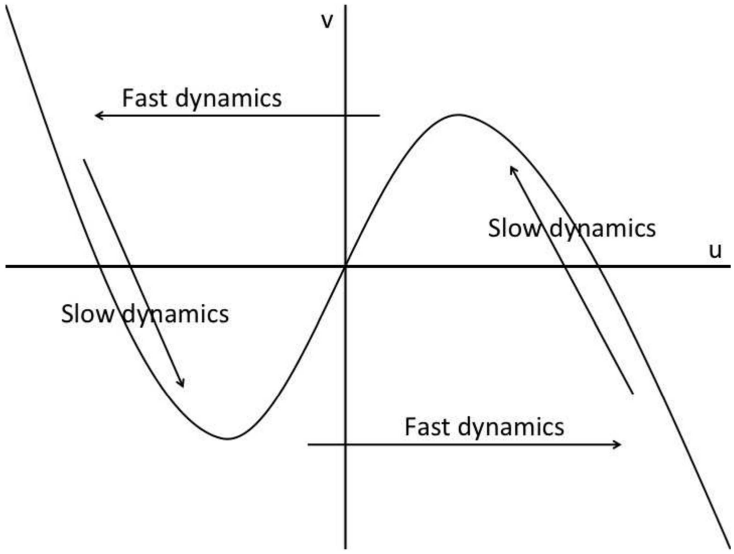

Figure 1. Flow for $\delta \ll 1$ of the van der Pol equation (1.2).

(1.2), if $u < 0$, v moves slowly in the negative direction. On the other hand, v moves slowly in the positive direction for $u > 0$. When the solution $(u(t), v(t))$ exits from this region, then the solution jumps to another branch, which is governed by fast dynamics. Consequently, as is shown in Figure 1, relaxation oscillation is observed. Another feature of relaxation oscillation is to possess a long period of oscillation due to slow dynamics. This oscillatory phenomenon was first found in a electric circuit by van der Pol([6]). Many results on relaxation oscillation in ODE systems are still reported (for instance [5]). Here we address a question: "Do partial differential equations give rise to relaxation oscillation?"

In this paper, we present relaxation oscillation arising in reaction-diffusion systems, which is called infinite dimensional relaxation oscillation in contrast with finite dimensional relaxation oscillation in systems of ordinary differential equations mentioned above. As long as we know, results on infinite dimensional relaxation oscillation is quite few([2],[3]). This implies that treatment of infinite dimensional relaxation oscillation even in one-spatial dimensional system is hard since a global bifurcation diagram of the system corresponding to Figure 1 is required. Though results on infinite dimensional relaxation oscillation is quite few, common mechanisms for the infinite dimensional relaxation oscillation appear. We explain the mechanism for infinite dimensional relaxation oscillation in an aggregation-growth system.

§ 2. Infinite dimensional relaxation oscillation

We consider the following one-spatial dimension reaction-diffusion system which describes aggregation phenomenon for social insects([2][4]):

$$
(2.1)\qquad \begin{cases}
u_{1t} = du_{1xx} + \dfrac{1}{\varepsilon}(k(v)u_2 - h(v)u_1) & \\
\qquad\qquad + \delta G_1(u_1, u_2), & \\
u_{2t} = (d+\alpha)u_{2xx} - \dfrac{1}{\varepsilon}(k(v)u_2 - h(v)u_1) & t > 0, \quad x \in (0,1), \\
\qquad\qquad + \delta G_2(u_1, u_2), & \\
v_t = Dv_{xx} + (u_1 + u_2) - v, & \\
u_{1x} = u_{2x} = v_x = 0, & t > 0, \quad x = 0, 1, \\
u_1(0,x) = u_{10}(x), & \\
u_2(0,x) = u_{20}(x), & x \in (0,1), \\
v(0,x) = v_0(x), &
\end{cases}
$$

where $G_i(u_1, u_2) = (1 - (u_1 + u_2))u_i$ $(i = 1, 2)$. u_1 and u_2 represent the population density of insect respectively. From the relation of diffusion coefficient, u_1 and u_2 are called less-active state and active state, respectively. v implies the concentration of aggregation pheromone which is secreted by u_1 and u_2 and is evaporated. These two states are convertible each other depending on the pheromone concentration v. Here we assume that the functions $k(v)$ and $h(v)$ are monotonically increasing and decreasing functions, respectively. If the pheromone concentration is high, u_2 converts into u_1 and stays there. On the other hand, if the pheromone concentration is low, u_1 switches into u_2 and diffuses faster. Further $G_i (i = 1, 2)$ means growth term. Positive constants $1/\varepsilon$ and δ imply the quickness of conversion and growth rate, respectively. On more detailed description of the model, refer to [2] and [4].

In this section, we sketch a mechanism of relaxation oscillation, which occurs in (2.1) with sufficiently small δ. Figure 2 shows that the periodic solution consists of two different dynamics, namely, fast and slow dynamics. In order to explain the fast and slow dynamics, define

$$\mu := \int_0^1 (u_1 + u_2)\, dx.$$

Then, from the equations for u_1 and u_2 in (2.1), after integration by parts, we obtain the following:

$$\frac{d}{dt}\mu = \delta \int_0^1 \{G_1(u_1, u_2) + G_2(u_1, u_2)\}\, dx.$$

In order to obtain the dynamics of time scale t, which corresponds to fast dynamics in

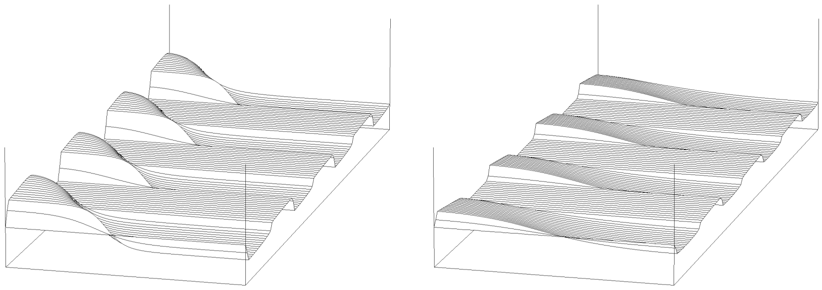

Figure 2. Spatio-temporal periodic solution of (2.1), where $\delta = 0.01$, $d = 0.05$, $\alpha = 0.1$, $D = 0.1$, $\varepsilon = 0.0001$, $k(v) = \frac{\tanh(10(v-1))+1}{2}$ and $h(v) = 1 - k(v)$. The left- and right-hand figures show the time evolutions of $u_1 + u_2$ and v, respectively, for $0 \le t \le 900$.

(2.1), formally setting $\delta = 0$ in (2.1), we obtain the following:

$$
\begin{cases}
u_{1t} = du_{1xx} + \dfrac{1}{\varepsilon}(k(v)u_2 - h(v)u_1), \\
u_{2t} = (d+\alpha)u_{2xx} - \dfrac{1}{\varepsilon}(k(v)u_2 - h(v)u_1), \\
v_t = Dv_{xx} + (u_1 + u_2) - v, \\
\dfrac{d}{dt}\mu = 0.
\end{cases} \tag{2.2}
$$

Since the fourth equation in (2.2) leads to $\mu(t) \equiv \mu$, which is independent of t, the fast dynamics in (2.1) are governed by the following system:

$$
\begin{cases}
u_{1t} = du_{1xx} + \dfrac{1}{\varepsilon}(k(v)u_2 - h(v)u_1), \\
u_{2t} = (d+\alpha)u_{2xx} - \dfrac{1}{\varepsilon}(k(v)u_2 - h(v)u_1), \\
v_t = Dv_{xx} + (u_1 + u_2) - v, \\
\displaystyle\int_0^1 (u_1 + u_2)dx = \mu.
\end{cases} \tag{2.3}
$$

When δ is sufficiently small, the solution (u_1, u_2, v) of (2.1) is expected to immediately converge to a stable equilibrium solution while the mass $\int_0^1 (u_1 + u_2)dx$ is preserved according to (2.3). The global structure of equilibrium solutions of (2.1) with $\delta = 0$ is shown in Figure 3 for $\alpha = 0.1$, which is obtained by AUTO([1]).

We denote the spatially constant equilibrium solution of (2.1) by

$$\Phi_0(\mu) := (\overline{u}_1, \overline{u}_2, \overline{v}) = (k(\frac{a}{b}\mu), h(\frac{a}{b}\mu), \frac{a}{b}\mu)$$

with parameter $\mu > 0$. The spatially constant equilibrium solution is destabilized at $\mu = \mu_c$ as μ increases, so that non-constant equilibrium solution branches bifurcate

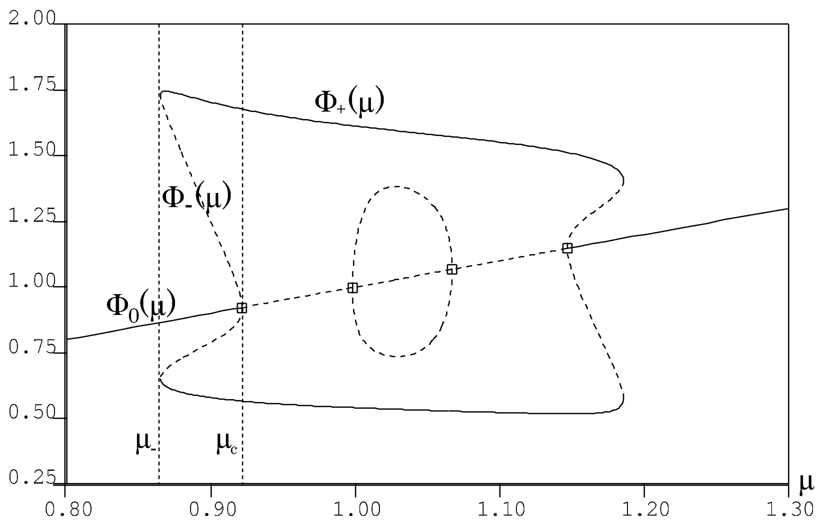

Figure 3. Global structure of equilibrium solutions of (2.1) with $\delta = 0$, where $d = 0.05$, $\alpha = 0.1$, $D = 0.1$, $\varepsilon = 0.0001$, $k(v) = \frac{\tanh(10(v-1))+1}{2}$ and $h(v) = 1-k(v)$. The horizontal and vertical axes denote $\mu = \int_0^1 (u_1+u_2)dx$ and u_1+u_2 at $x = 0$, respectively. The solid and dashed lines represent stable and unstable branches, respectively. The $\square$ symbol indicates stationary bifurcation points.

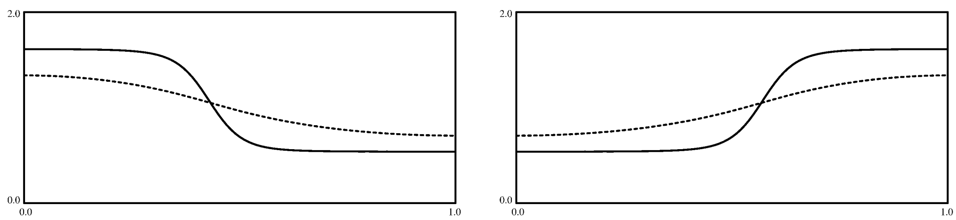

Figure 4. Stable equilibrium solutions at $\mu = 1$ in Figure 3. The solid and dashed curves represent $u_1 + u_2$ and v, respectively.

subcritically via pitchfork bifurcation. The non-trivial branches are unstable near $\mu = \mu_c$, but recover stability through saddle-node bifurcation at $\mu = \mu_-$, and the stable non-trivial branches become stable. As in Figure 3, we hereinafter denote the unstable and stable equilibrium solution branches by $\Phi_-(\mu)$ and $\Phi_+(\mu)$, respectively, which are parameterized with μ. For $\mu = 1$, the profiles of two stable equilibrium solutions are shown in Figure 4.

Next, we consider the slow dynamics of the solution to (2.1) for time scale $T = \delta t$. Using $T = \delta t$ in (2.1) and formally setting $\delta = 0$, we have

$$\text{(2.4)} \qquad \begin{cases} 0 = du_{1xx} + \dfrac{1}{\varepsilon}(k(v)u_2 - h(v)u_1), \\ 0 = (d+\alpha)u_{2xx} - \dfrac{1}{\varepsilon}(k(v)u_2 - h(v)u_1), \\ 0 = Dv_{xx} + (u_1 + u_2) - v, \\ \dfrac{d}{dT}\mu = \displaystyle\int_0^1 \{G_1(u_1, u_2) + G_2(u_1, u_2)\}\, dx. \end{cases}$$

Using the slow dynamics (2.4), we find that the solution (u_1, u_2, v) of (2.1) moves gradually along the equilibrium solution branches $\Phi_+(\mu)$ or $\Phi_0(\mu)$. We know that the fourth equation decides the moving direction. For the solution on the nontrivial branch $\Phi_+(\mu)$, we formally write the fourth equation in (2.4) as follows:

$$\frac{d}{dT}\mu = \int_0^1 G(\Phi_+(\mu))dx = F_1(\mu),$$

where $G(u_1, u_2, v) := G_1(u_1, u_2) + G_2(u_1, u_2) = (1 - (u_1 + u_2))(u_1 + u_2)$. On the other hand, for the solution on the trivial branch $\Phi_0(\mu)$, we have

$$\frac{d}{dT}\mu = \int_0^1 (1-\mu)\mu dx = (1-\mu)\mu = F_2(\mu).$$

We numerically obtain the sign of $F_1(\mu)$ on the stable branch $\Phi_+(\mu)$. Taking the same parameter values as described in Figure 3, Figure 5 shows that $F_1(\mu) < 0$ and $F_2(\mu) > 0$ for $\mu < 1$ and $F_2(\mu) < 0$ for $\mu > 1$. If $F_i(\mu) < 0$ $(i = 1, 2)$, the fourth equation in (2.4) says that μ decreases in T, whereas, if $F_i(\mu) > 0$, then μ increases in T.

Assume that a solution satisfying $\mu_- < \int_0^1 \{u_1(t_0) + u_2(t_0)\}\, dx < \mu_c$ at time t_0 is on the $\Phi_+(\mu)$ branch in Figure 3. Since $F_1(\mu) < 0$ for $\mu_- < \mu < \mu_c$, as in Figure 5, μ decreases. Therefore, the solution satisfying $\mu_- < \int_0^1 \{u_1(t) + u_2(t)\}dx < \mu_c$ moves to the left on the $\Phi_+(\mu)$ branch by the slow dynamics (2.4). When the solution reaches the point $\Phi_+(\mu_-)$, i.e., $\mu_- = \int_0^1 \{u_1(t_1) + u_2(t_1)\}dx$ at time t_1, the fast dynamics (2.3) becomes dominant, and the solution transfers to the point $\Phi_0(\mu_-)$. Then, the slow dynamics (2.4) become dominant again. Since $F_2(\mu) > 0$ on the $\Phi_0(\mu)$ branch, μ increases. Therefore, the solution moves to the right on the $\Phi_0(\mu)$ branch. When the

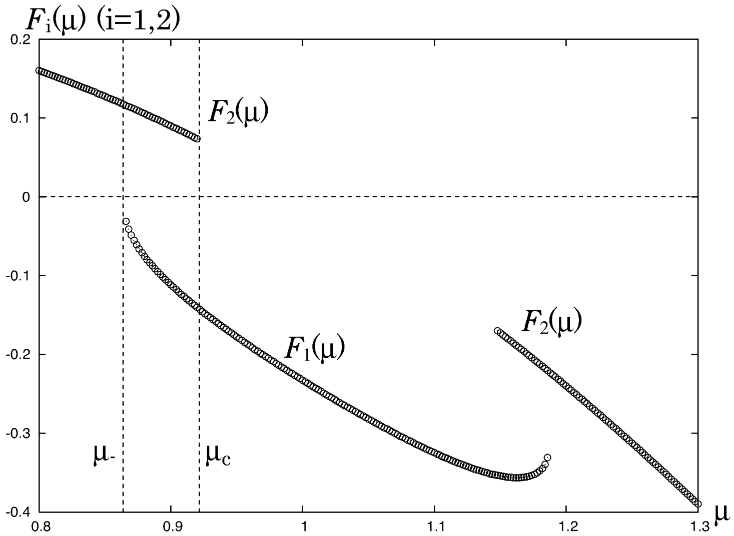

Figure 5. Graphs of $F_1(\mu)$ and $F_2(\mu)$ corresponding to stable branches $\Phi_+(\mu)$ and $\Phi_0(\mu)$ in Figure 3. The horizontal and vertical axes represent μ and $F_1(\mu)$ and $F_2(\mu)$, respectively.

solution crosses the point $\Phi_0(\mu_c)$, that is, $\mu_c = \int_0^1 \{u_1(t_2) + u_2(t_2)\}dx$ at some time t_2, the fast dynamics (2.3) becomes dominant and the solution is lifted up to the point $\Phi_+(\mu)$ for $\mu > \mu_c$. Due to $F_1(\mu) < 0$ on the $\Phi_+(\mu)$ branch, the solution travels to the left on the $\Phi_+(\mu)$ branch. The repetition of this cycle suggests the occurrence of infinite-dimensional relaxation oscillation as in Figure 6.

Under several assumptions, we can show the existence of relaxation oscillations for the system (2.1).

Let $\boldsymbol{u} := (u_1, u_2, v)$, and write (2.1) as $\boldsymbol{u}_t = \mathcal{L}(\boldsymbol{u}) + \delta \boldsymbol{G}$, where

$$\mathcal{L}(\boldsymbol{u}) := \begin{pmatrix} du_{1xx} + \frac{1}{\varepsilon}(k(v)u_2 - h(v)u_1) \\ (d+\alpha)u_{2xx} - \frac{1}{\varepsilon}(k(v)u_2 - h(v)u_1) \\ Dv_{xx} + (u_1 + u_2) - v \end{pmatrix}$$

and

$$\boldsymbol{G}(\boldsymbol{u}) := \begin{pmatrix} G_1(u_1, u_2) \\ G_2(u_1, u_2) \\ 0 \end{pmatrix} = \begin{pmatrix} u_1(1 - (u_1 + u_2)) \\ u_2(1 - (u_1 + u_2)) \\ 0 \end{pmatrix}.$$

Let us consider the following unperturbed equation:

$$\boldsymbol{u}_t = \mathcal{L}\boldsymbol{u}. \tag{2.5}$$

Let $L_0(\mu) := \mathcal{L}'(\Phi_0(\mu))$ be the linearized operator of $\mathcal{L}(\boldsymbol{u})$ with respect to $\Phi_0(\mu)$. Hereinafter, for constants $0 < \mu_- < \mu_c < 1$ we impose the following five assumptions (A1)

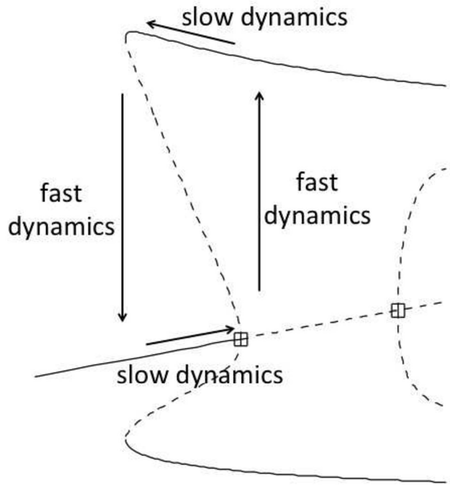

Figure 6. Flow for $\delta \ll 1$ of (2.1).

$\sim$ (A5):

(A1) The spectrum of $L_0(\mu)$ consists of $\Sigma_0(\mu) \cup \Sigma_1(\mu) \cup \Sigma_2(\mu)$, where $\Sigma_0(\mu) := \{0\}$ $\Sigma_1(\mu) := \{-\lambda_1(\mu)\}$ and $\Sigma_2(\mu) \subset \{\lambda \in \boldsymbol{C};\ Re(\lambda) < -2\gamma_0\}$ for a positive constant $\gamma_0 > 0$ and a continuous function $\lambda_1(\mu) \in \boldsymbol{R}$ satisfying $\lambda_1(\mu) < \gamma_0$ for $0 < \mu \leq 1$.

(A2) $\lambda_1(\mu) > 0$ for $\mu_- < \mu < \mu_c$, $\lambda_1(\mu_c) = 0$, and $\lambda_1(\mu) < 0$ for $\mu > \mu_c$. Moreover, for $\mu \neq \mu_c$, $\lambda_1(\mu)$ is a simple eigenvalue of $L_0(\mu)$.
Let $L_0^*(\mu)$ be the adjoint operator of $L_0(\mu)$, and let $\psi(\mu)$ be the eigenfunction associated with the eigenvalue $-\lambda_1(\mu)$ satisfying $L_0(\mu)\psi(\mu) = -\lambda_1(\mu)\psi(\mu)$. Since $L_0(\mu)\partial_\mu \Phi_0 = 0$ holds and $L_0(\mu)$ always has a zero eigenvalue, $L_0^*(\mu)$ has the same properties, i.e., there exist eigenfunctions $\phi^*(\mu)$ and $\psi^*(\mu)$ satisfying $L_0^*(\mu)\phi^*(\mu) = 0$ and $L_0^*(\mu)\psi^*(\mu) = -\lambda_1(\mu)\psi^*(\mu)$.

(A3) ψ and ψ^* are odd, whereas ϕ^* is even.

(A4) There exist two branches expressed as $\{\Phi_-(\mu);\ \mu_- \leq \mu \leq \mu_c\}$ and $\{\Phi_+(\mu);\ \mu \geq \mu_-\}$ for $0 < \mu_- < \mu_c$ satisfying $\Phi_-(\mu_c) = \Phi_0(\mu_c)$, $\Phi_-(\mu_-) = \Phi_+(\mu_-)$ and $\langle\ \Phi_-(\mu), \phi^*\ \rangle_{L^2} = \langle\ \Phi_+(\mu), \phi^*\ \rangle_{L^2} = \mu$. $\Phi_+(\mu)$ for $\mu > \mu_-$ is linearly stable, whereas $\Phi_-(\mu)$ for $\mu_- < \mu < \mu_c$ is linearly unstable. Furthermore, $F_1(\mu) < 0$ for $\mu_- \leq \mu \leq 1$ and $F_2(\mu) > 0$ for $0 < \mu \leq 1$ hold, where $F_1(\mu) := \langle\ \boldsymbol{G}(\Phi_+(\mu)), \phi^*\ \rangle_{L^2}$ and $F_2(\mu) := \langle\ \boldsymbol{G}(\Phi_0(\mu)), \phi^*\ \rangle_{L^2}$.

(A5) At $\mu = \mu_-$, there exists an orbitally stable connecting orbit, say $\xi_-(t)$ of (2.5) with $\langle \xi_-(t), \phi^* \rangle_{L^2} = \mu_-$ satisfying $\xi_-(t) \to \Phi_+(\mu_-)$ as $t \to -\infty$ and $\xi_-(t) \to \Phi_0(\mu_-)$ as $t \to +\infty$. In the right-hand neighborhood of μ_c, i.e., $\mu \in [\mu_c, 1)$, there exists a class of orbitally stable connecting orbits, say $\xi_c(t;\mu)$ with $\langle \xi_c(t;\mu), \phi^* \rangle_{L^2} = \mu$ satisfying $\xi_c(t;\mu) \to \Phi_0(\mu)$ as $t \to -\infty$ and $\xi_c(t;\mu) \to \Phi_+(\mu)$ as $t \to +\infty$. More precisely, $\xi_c(t;\mu) \to \Phi_0(\mu) + r(t)\psi(\mu)$ with $r(t) \downarrow 0$ as $t \to -\infty$ and $\xi_-(t) \to \Phi_0(\mu_-) + r(t)\psi(\mu_-)$ with $r(t) \downarrow 0$ as $t \to +\infty$ hold.

Let μ^* be the value satisfying

$$\int_{\mu_-}^{\mu^*} \frac{\lambda_1(\mu)}{F_2(\mu)} d\mu = 0$$

and $\Gamma_0 := \{\Phi_0(\mu);\ \mu_- \le \mu \le \mu^*\} \cup \{\xi_c(t;\mu^*);\ -\infty < t < +\infty\} \cup \{\Phi_+(\mu);\ \mu_- \le \mu \le \mu^*\} \cup \{\xi_-(t);\ -\infty < t < +\infty\}$ and $U(\Gamma_0;\eta) := \{\boldsymbol{u} \in L^\infty(I);\ dist\{\boldsymbol{u}, \Gamma_0\} < \eta\}$. Note that $0 < \mu_- < \mu_c < \mu^* < 1$ holds.

Under (A1) $\sim$ (A5), we can prove the following existence theorem for relaxation oscillations:

Theorem 2.1. *For sufficiently small $\eta > 0$, there exist $\delta > 0$ and a periodic orbit $\Gamma(t;\delta)$ of (2.1) such that $\Gamma(t;\delta) \in U(\Gamma_0;\eta)$.*

Proof. See [2] for the proof. □

§ 3. Concluding remarks

We have discussed the existence of infinite dimensional relaxation oscillation arising in a reaction-diffusion system. In general, it is hard to find infinite dimensional relaxation oscillation compared with ODE case since a global bifurcation diagram of the system is required such as Figure 3. Moreover, the sign of $F_1(\mu) = \int_0^1 G(\Phi_+(\mu))dx$ in (2.4) is also important to decide the moving direction of the solution on the branch $\Phi_+(\mu)$. In our case, the global bifurcation diagram and the sign of $F_1(\mu)$ are numerically obtained. These features are significant to find infinite dimensional relaxation oscillation. We emphasize that the key mechanism of infinite dimensional relaxation oscillation is transition between two different stable equilibrium solution branches. In [3], the existence of infinite dimensional relaxation oscillation is explained by a similar mechanism. Under several assumptions including these numerical evidences, the existence of infinite dimensional relaxation oscillation can be proved.

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