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Remarks on a dynamical aspect of shortening-straightening flow for non-closed planar curves with fixed boundary

By

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Abstract

In this paper, we consider a geometric evolution equation defined on non-closed planar curves with finite length. The equation is given as a steepest descent flow for the geometric functional called the modified total squared curvature. We call the flow the shortening-straightening flow. The purpose of this paper is to prove a certain dynamical aspect of planar curve governed by shortening-straightening flow.

§ 1. Introduction

The steepest descent flow for various geometric functionals defined on curves have been studied by many people, for example, the shortening flow ([1], [4], [5]), the straightening flow for curve with fixed total length ([7], [14], [15]), and the straightening flow for curve with fixed local length ([6], [10]). In this paper, we focus on a geometric functional called "modified total squared curvature" or "modified one-dimensional Willmore functional" (see (1.1)). To begin with, we introduce the geometric functional.

Let γ be a planar curve and κ be the curvature. The length functional of γ is given by

$$\mathcal{L}(\gamma) = \int_\gamma ds,$$

where s denotes the arc length parameter of γ. The steepest descent flow for $\mathcal{L}(\gamma)$ is called curve shortening flow. For a given constant λ, the modified total squared

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curvature is defined as

$$E(\gamma) = \int_\gamma \kappa^2\, ds + \lambda^2 \mathcal{L}(\gamma). \tag{1.1}$$

The first term of the right hand side in (1.1) is well known as the total squared curvature or one-dimensional Willmore functional. The steepest descent flow for the total squared curvature is called curve straightening flow.

In this paper, we call the steepest descent flow for E "the shortening-straightening flow". The shortening-straightening flow is written as follows:

$$\partial_t \gamma = (-2\partial_s^2 \kappa - \kappa^3 + \lambda^2 \kappa)\boldsymbol{\nu}, \tag{1.2}$$

where $\boldsymbol{\nu}$ denotes the unit normal vector of γ pointing in the direction of the curvature.

We mention the known results of shortening-straightening flow. In 1996, first it has been proved by A. Polden ([13]) that the equation (1.2) admits smooth solutions globally defined in time, when the initial curve is smooth, closed, and has finite length (i.e., compact without boundary). Furthermore, he also proved that the solution converges to a stationary solution as time tends to infinity. In 2002, G. Dziuk, E. Kuwert, and R. Schätzle ([3]) extended the Polden's result of [13] to closed curves with finite length in $\mathbb{R}^n$.

Regarding the flow (1.2), we are interested in the following problem:

Problem 1.1. What is a dynamics of *non-closed* planar curves with finite length governed by shortening-straightening flow?

Concerning Problem 1.1, in particular we consider planar curves with fixed boundary. Indeed, let $\Gamma_0(x) : [0, L] \to \mathbb{R}^2$ be a smooth planar curve and $k_0(x)$ denote the curvature. Let $\Gamma_0(x)$ satisfy the following conditions:

$$\left|{\Gamma_0}'(x)\right| \equiv 1, \quad \Gamma_0(0) = (0,0), \quad \Gamma_0(L) = (R,0), \quad k_0(0) = k_0(L) = 0, \tag{C}$$

where $L > 0$ and $R > 0$ are given constants. For such curve Γ_0, let us consider the following initial boundary value problem:

$$\begin{cases} \partial_t \gamma = (-2\partial_s^2 \kappa - \kappa^3 + \lambda^2 \kappa)\boldsymbol{\nu}, \\ \gamma(0,t) = (0,0), \quad \gamma(L,t) = (R,0), \quad \kappa(0,t) = \kappa(L,t) = 0, \\ \gamma(x,0) = \Gamma_0(x). \end{cases} \tag{SSC}$$

Concerning the problem (SSC), we have proved the following result in [11]:

Theorem 1.1. *Let $\Gamma_0(x)$ be a smooth non-closed planar curve and satisfy the conditions* (C)*. Then there exists a unique classical solution $\gamma(x,t)$ of* (SSC) *for any $t > 0$. Moreover there exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $\gamma(x,t_i)$ converges to a stationary solution of* (SSC) *as $t_i \to \infty$.*

The stationary solution is obtained as a critical point of energy functional (1.1). The representation formula of all of the critical points is given by A. Linnér ([8]). However, to best of our knowledge, its stability is an outstanding question. In order to comprehend a dynamical aspect of solution of (SSC), we have to analyze the stability of the critical points. The purpose of this paper is to determine a certain dynamical aspect of solution of (SSC).

Before stating the main result, we define a certain set of non-closed planar curves. Let S_R denote a set of all non-closed smooth planar curves satisfying the following:

(i) One end point is fixed at $(0,0)$. And the another end point is fixed at $(R,0)$;

(ii) The curvature vanishes at the end points.

We state the main result of this paper in a concise form.

Theorem 1.2. *Let $0 < \theta < 1/4$. Let $\gamma_* \in S_R$ be the line segment. Then there exists a positive constant ε_* such that, for any smooth graph curve $\Gamma_0 \in S_R$ satisfying*

$$\|\Gamma_0 - \gamma_*\|_{h^{4+4\theta}([0,R])} < \varepsilon_*,$$

the solution $\gamma(x,t)$ of the problem

$$\begin{cases} \partial_t \gamma = (-2\partial_s^2 \kappa - \kappa^3 + \lambda^2 \kappa)\boldsymbol{\nu}, \\ \gamma(0,t) = (0,0),\ \gamma(R,t) = (R,0),\ \kappa(0,t) = \kappa(R,t) = 0, \\ \gamma(x,0) = \Gamma_0(x) \end{cases}$$

converges to $\gamma_*(x)$ in the C^∞ topology as $t \to \infty$.

Here $h^{4+4\theta}([0,R])$ denotes a little Hölder space.

This paper is organized as follows: Theorem 1.1 is shown in Section 2. In Section 3, we prove Theorem 1.2. Finally we shall announce a result concerning an application to non-compact case in Section 4.

§ 2. Compact case with fixed boundary

In this section, we shall prove a long time existence and a convergence of solution of (SSC). To begin with, we prove a short time existence of solution of (SSC).

§ 2.1. Short time existence

First we show a short time existence of solution to (SSC). For this purpose, let

$$\gamma(x,t) = \Gamma_0(x) + d(x,t)\boldsymbol{\nu}_0(x), \tag{2.1}$$

where $d(x,t) : [0,L] \times [0,\infty) \to \mathbb{R}$ is an unknown scalar function and $\boldsymbol{\nu}_0(x)$ denotes the unit normal vector of $\Gamma_0(x)$, i.e., $\boldsymbol{\nu}_0(x) = \mathfrak{R}{\Gamma_0}'(x) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right){\Gamma_0}'(x)$. The argument in [3] implies that (2.1) is a valid formulation for (SSC). Under the formulation (2.1), the boundary conditions $\gamma(0,t) = (0,0)$ and $\gamma(L,t) = (R,0)$ are reduced to

$$d(0,t) = d(L,t) = 0. \tag{2.2}$$

With the aid of Frenet-Serret's formula ${\Gamma_0}'' = k_0\boldsymbol{\nu}_0$ and ${\boldsymbol{\nu}_0}' = -k_0{\Gamma_0}'$, we have

$$\begin{aligned}
\partial_x\gamma &= (1-k_0d){\Gamma_0}' + \partial_x d\boldsymbol{\nu}_0,\\
\mathfrak{R}\partial_x\gamma &= -\partial_x d{\Gamma_0}' + (1-k_0d)\boldsymbol{\nu}_0,\\
\partial_x^2\gamma &= (-{k_0}'d - 2k_0\partial_x d){\Gamma_0}' + (\partial_x^2 d + k_0 - {k_0}^2 d)\boldsymbol{\nu}_0,\\
\kappa &= \frac{\partial_x^2\gamma \cdot \mathfrak{R}\partial_x\gamma}{|\partial_x\gamma|^3} = \frac{\partial_x d({k_0}'d + 2k_0\partial_x d) + (1-k_0d)(\partial_x^2 d + k_0 - {k_0}^2 d)}{\{(1-k_0d)^2 + (\partial_x d)^2\}^{3/2}}.
\end{aligned}$$

Thus the condition $\kappa(0,t) = \kappa(L,t) = 0$ is equivalent to

$$\partial_x^2 d(0,t) = \partial_x^2 d(L,t) = 0. \tag{2.3}$$

Since

$$s(x,t) = \int_0^x |\partial_x\gamma(x,t)|\, dx = \int_0^x \left\{(1-k_0(x)d(x,t))^2 + (\partial_x d(x,t))^2\right\}^{1/2} dx,$$

we have

$$\frac{\partial s}{\partial x} = \left\{(1-k_0(x)d(x,t))^2 + (\partial_x d(x,t))^2\right\}^{1/2} := |\gamma_d|\,. \tag{2.4}$$

Then we see that

$$\begin{aligned}
\partial_s^2\kappa &= \frac{\partial_x}{|\gamma_d|}\left(\frac{\partial_x}{|\gamma_d|}\left(\frac{\partial_x d(\partial_x k_0 d + 2k_0\partial_x d) + (1-k_0d)(\partial_x^2 d + k_0 - {k_0}^2 d)}{|\gamma_d|^3}\right)\right)\\
&= \frac{1}{|\gamma_d|^5}\partial_x^2\alpha_3 - \frac{7}{|\gamma_d|^6}\partial_x|\gamma_d|\,\partial_x\alpha_3 + \left\{-\frac{3}{|\gamma_d|^6}\partial_x^2|\gamma_d| + \frac{15}{|\gamma_d|^7}(\partial_x|\gamma_d|)^2\right\}\alpha_3,
\end{aligned}$$

where

$$\alpha_3 = \partial_x d(\partial_x k_0 d + 2k_0\partial_x d) + (1-k_0d)(\partial_x^2 d + k_0 - {k_0}^2 d).$$

Setting

$$\begin{aligned}
\alpha_1 &= \partial_x k_0 d + k_0\partial_x d,\\
\alpha_2 &= \partial_x d\partial_x^2 d + \alpha_1(k_0d - 1),\\
\alpha_4 &= \partial_x d\partial_x^3 d + (\partial_x^2 d)^2 + {\alpha_1}^2 + \partial_x\alpha_1(k_0d - 1),
\end{aligned}$$

we have

$$\partial_x |\gamma_d| = \frac{\alpha_2}{|\gamma_d|}, \qquad \partial_x^2 |\gamma_d| = -\frac{{\alpha_2}^2}{|\gamma_d|^3} + \frac{\alpha_4}{|\gamma_d|}.$$

Thus $\partial_s^2 \kappa$ is written as

$$\partial_s^2 \kappa = \frac{1}{|\gamma_d|^5} \partial_x^2 \alpha_3 - \frac{1}{|\gamma_d|^7} (7\alpha_2 \partial_x \alpha_3 + 3\alpha_3\alpha_4) + \frac{18}{|\gamma_d|^9} {\alpha_2}^2 \alpha_3.$$

Since $\kappa = \alpha_3 / |\gamma_d|^3$ and $\partial_t \gamma = \partial_t d \boldsymbol{\nu}_0$, we have

$$\begin{aligned} \partial_t d &= \left\{ -\frac{2}{|\gamma_d|^4} \partial_x^2 \alpha_3 + \frac{14}{|\gamma_d|^6} \alpha_2 \partial_x \alpha_3 + \frac{6}{|\gamma_d|^6} \alpha_3 \alpha_4 - \frac{36}{|\gamma_d|^8} {\alpha_2}^2 \alpha_3 - \frac{{\alpha_3}^3}{|\gamma_d|^8} + \frac{\lambda^2 \alpha_3}{|\gamma_d|^2} \right\} \frac{1}{1 - k_0 d} \\ &= -\frac{2}{|\gamma_d|^4} \partial_x^4 d + \Phi(d). \end{aligned}$$

Setting $A(d) = (-2/|\gamma_d|^4) \partial_x^4$, the problem (SSC) is written in terms of d as follows:

$$\begin{cases} \partial_t d = A(d) d + \Phi(d), \\ d(0,t) = d(L,t) = d''(0,t) = d''(L,t) = 0, \\ d(x,0) = d_0(x) = 0. \end{cases} \tag{2.5}$$

We shall find a smooth solution of (2.5) for a short time. To do so, we need to show the operator $A(d_0)$ is sectorial. Since $A(d_0) = -2\partial_x^4$, first we consider the boundary value problem

$$\begin{cases} \partial_x^4 \varphi + \mu \varphi = f, \\ \varphi(0) = \varphi(L) = \varphi''(0) = \varphi''(L) = 0, \end{cases} \tag{2.6}$$

where μ is a constant. The solution of (2.6) is written as

$$\varphi(x) = \int_0^L G(x,\xi) f(\xi)\, d\xi, \tag{2.7}$$

where $G(x,\xi)$ is a Green function given by

$$G(x,\xi) = \begin{cases} \dfrac{1}{(2\mu_*)^3} (g_1(\xi) g_2(x) + g_3(\xi) g_4(x)) & \text{for} \quad 0 \le x \le \xi, \\ \dfrac{1}{(2\mu_*)^3} (g_1(x) g_2(\xi) + g_3(x) g_4(\xi)) & \text{for} \quad \xi < x \le L. \end{cases} \tag{2.8}$$

Here the functions g_1, g_2, g_3, g_4, and constants K_0, K_1, K_2, μ_* are given by

$$\begin{aligned}
g_1(\zeta) &= \cos\mu_*\zeta\sinh\mu_*\zeta - \sin\mu_*\zeta\cosh\mu_*\zeta,\\
g_2(\zeta) &= e^{\mu_*\zeta}\cos\mu_*\zeta - \frac{K_1}{K_0}\cos\mu_*\zeta\sinh\mu_*\zeta + \frac{K_2}{K_0}\sin\mu_*\zeta\cosh\mu_*\zeta,\\
g_3(\zeta) &= \cos\mu_*\zeta\sinh\mu_*\zeta + \sin\mu_*\zeta\cosh\mu_*\zeta,\\
g_4(\zeta) &= -e^{\mu_*\zeta}\sin\mu_*\zeta + \frac{K_1}{K_0}\sin\mu_*\zeta\cosh\mu_*\zeta + \frac{K_2}{K_0}\cos\mu_*\zeta\sinh\mu_*\zeta,\\
K_0 &= 2\cos^2\mu_*L\sinh^2\mu_*L + 2\sin^2\mu_*L\cosh^2\mu_*L,\\
K_1 &= \frac{e^{2\mu_*L} - \cos 2\mu_*L}{2}, \quad K_2 = -\frac{\sin 2\mu_*L}{2}, \quad \mu_* = \frac{\mu^{1/4}}{\sqrt{2}}.
\end{aligned}$$

By virtue of (2.7) and (2.8), we see that the solution of (2.6) satisfies

$$\|\varphi\|_{W_p^4(0,L)} \le C\,\|f\|_{L^p(0,L)}\,. \tag{2.9}$$

Using the a priori estimate (2.9), we show that the operator $A(d_0)$ generates an analytic semigroup on $L^p(0,L)$. Moreover we can verify that $A_0 : h_B^{4+4\theta}([0,L]) \to h_B^{4\theta}([0,L])$ is an infinitesimal generator of an analytic semigroup on $h_B^{4\theta}([0,L])$, where $0 < \theta < 1/4$ (for example, see [9]). Here $h_B^{\alpha}([0,L])$ is a little Hölder space with boundary condition:

(2.10)

$$h_B^{\alpha}([0,L]) = \begin{cases} \{u \in h^{\alpha}([0,L]) \mid u(0) = u(L) = u''(0) = u''(L) = 0\} & \text{if} \quad \alpha > 2,\\ \{u \in h^{\alpha}([0,L]) \mid u(0) = u(L) = 0\} & \text{if} \quad 0 < \alpha < 2. \end{cases}$$

Since the equation in (2.5) is a fourth order quasilinear parabolic equation, we shall prove a short time existence of (2.5) as follows. Letting $B(d) := A(d) - A_0$, the system (2.5) is written as

$$\begin{cases} \partial_t d = A_0 d + B(d)d + \Phi(d),\\ d(0,t) = d(L,t) = d''(0,t) = d''(L,t) = 0,\\ d(x,0) = d_0(x) = 0. \end{cases} \tag{2.11}$$

And then, we find a solution of (2.11) for a short time by using contraction mapping principle. Indeed, making use of the maximal regularity property and continuous interpolation spaces, we see that there exists a unique classical solution of (2.11), i.e., (2.5), in the class $C([0,T]; h_B^{4+4\theta}([0,L])) \cap C^1([0,T]; h_B^{4\theta}([0,L]))$, where $T > 0$ is sufficiently small. And then we obtain the regularity by a standard bootstrap argument (see [9]). Then we obtain the following:

Lemma 2.1. *Let Γ_0 be a smooth curve satisfying* (C). *Then there exists a constant $T > 0$ such that the problem* (2.5) *has a unique smooth solution for $0 \le t < T$.*

Lemma 2.1 implies the existence of unique solution of (SSC) for a short time:

Theorem 2.1. *Let $\Gamma_0(x)$ be a smooth curve satisfying* (C). *Then there exist a constant $T > 0$ and a smooth curve $\gamma(x,t)$ such that $\gamma(x,t)$ is a unique classical solution of the problem* (SSC) *for $0 \le t < T$.*

§ 2.2. Long time existence

Next we shall prove a long time existence of solution to (SSC). Let us set

$$F^\lambda = 2\partial_s^2\kappa + \kappa^3 - \lambda^2\kappa. \tag{2.12}$$

Then the gradient flow (1.2) is written as

$$\partial_t\gamma = -F^\lambda \boldsymbol{\nu}. \tag{2.13}$$

The fact that the arc length parameter s depends on time t yields the following:

Lemma 2.2. *Under* (1.2), *the following formula holds*:

$$\partial_t\partial_s = \partial_s\partial_t - \kappa F^\lambda \partial_s. \tag{2.14}$$

Proof. Since $\partial_s = \partial_x / |\partial_x\gamma(x,t)|$, we obtain

$$\partial_t\partial_s = \partial_t\left(\frac{\partial_x}{|\partial_x\gamma|}\right) = \left(\frac{\partial_x}{|\partial_x\gamma|}\right)\partial_t - \frac{\partial_x\partial_t\gamma\cdot\partial_x\gamma}{|\partial_x\gamma|^3}\partial_x = \partial_s\partial_t - (\partial_s\partial_t\gamma\cdot\partial_s\gamma)\,\partial_s.$$

From the equation (2.13), we find

$$\partial_s\partial_t\gamma\cdot\partial_s\gamma = (-\partial_s F^\lambda\boldsymbol{\nu} - F^\lambda\partial_s\boldsymbol{\nu})\cdot\partial_s\gamma = \kappa F^\lambda.$$

Hence we obtain (2.14). □

Lemma 2.2 gives us the following:

Lemma 2.3. *Let $\gamma(x,t)$ satisfy* (1.2). *Then the curvature $\kappa(x,t)$ of $\gamma(x,t)$ satisfies*

$$\begin{aligned}\partial_t\kappa &= -\partial_s^2 F^\lambda - \kappa^2 F^\lambda \\ &= -2\partial_s^4\kappa - 5\kappa^2\partial_s^2\kappa + \lambda^2\partial_s^2\kappa - 6\kappa(\partial_s\kappa)^2 - \kappa^5 + \lambda^2\kappa^3.\end{aligned} \tag{2.15}$$

Furthermore, the line element ds of $\gamma(x,t)$ satisfies

$$\partial_t ds = \kappa F^\lambda ds = (2\kappa\partial_s^2\kappa + \kappa^4 - \lambda^2\kappa^2)ds. \tag{2.16}$$

Proof. Since $\kappa = \partial_s^2\gamma \cdot \boldsymbol{\nu}$, first we have

$$\partial_t \kappa = \partial_t \partial_s^2 \gamma \cdot \boldsymbol{\nu} + \partial_s^2 \gamma \cdot \partial_t \boldsymbol{\nu}.$$

By virtue of Lemma 2.2, the terms in the right-hand side are written as follows:

$$\begin{aligned}
\partial_t \partial_s^2 \gamma &= \partial_s \partial_t \partial_s \gamma - \kappa F^\lambda \partial_s^2 \gamma \\
&= \partial_s(\partial_s \partial_t \gamma - \kappa F^\lambda \partial_s \gamma) - \kappa^2 F^\lambda \boldsymbol{\nu} \\
&= \partial_s^2 \partial_t \gamma - \partial_s(\kappa F^\lambda)\partial_s \gamma - 2\kappa^2 F^\lambda \boldsymbol{\nu}, \\
\partial_t \boldsymbol{\nu} &= \mathfrak{R}(\partial_s \partial_t \gamma - \kappa F^\lambda \partial_s \gamma).
\end{aligned}$$

Using Frenet-Serret's formula $\partial_s^2\gamma = \kappa\boldsymbol{\nu}$ and $\partial_s \boldsymbol{\nu} = -\kappa \partial_s \gamma$, we obtain (2.15). Moreover the relation (2.16) is followed from the following calculation:

$$\partial_t ds = \partial_t \left| \partial_x \gamma \right| dx = \frac{\partial_x \gamma}{|\partial_x \gamma|} \cdot \partial_x \partial_t \gamma \, dx = \partial_s \gamma \cdot \partial_s \partial_t \gamma \, ds = \kappa F^\lambda ds.$$

□

Here we introduce the following notation:

Definition 2.1. ([2]) *We use the symbol* $\mathfrak{q}^r(\partial_s^l \kappa)$ *for a polynomial with constant coefficients such that each of its monomials is of the form*

$$\prod_{i=1}^{N} \partial_s^{j_i} \kappa \quad \text{with} \quad 0 \le j_i \le l \quad \text{and} \quad N \ge 1$$

with

$$r = \sum_{i=1}^{N} (j_i + 1).$$

Lemmas 2.2 and 2.3 give us a representation of $\partial_t \partial_s^j \kappa$:

Lemma 2.4. *For any* $j \in \mathbb{N}$, *the following formula holds*:

$$\partial_t \partial_s^j \kappa = -2\partial_s^{j+4}\kappa - 5\kappa^2 \partial_s^{j+2}\kappa + \lambda^2 \partial_s^{j+2}\kappa + \lambda^2 \mathfrak{q}^{j+3}(\partial_s^j \kappa) + \mathfrak{q}^{j+5}(\partial_s^{j+1}\kappa). \tag{2.17}$$

Proof. The case $j = 0$ in (2.17) has been already proved in Lemma 2.3, where $\mathfrak{q}^5(\partial_s \kappa) = -6\kappa(\partial_s \kappa)^2 - \kappa^5$ and $\mathfrak{q}^3(\kappa) = \kappa^3$. Next suppose that the formula (2.17) holds for $j - 1$. Then we have

$$\begin{aligned}
\partial_t \partial_s^j \kappa &= \partial_s \partial_t \partial_s^{j-1} \kappa - \kappa F^\lambda \partial_s^j \kappa \\
&= \partial_s \left\{ -2\partial_s^{j+3}\kappa - 5\kappa^2 \partial_s^{j+1}\kappa + \lambda^2 \partial_s^{j+1}\kappa + \lambda^2 \mathfrak{q}^{j+2}(\partial_s^{j-1}\kappa) + \mathfrak{q}^{j+4}(\partial_s^j \kappa) \right\} \\
&\qquad - \kappa(2\partial_s^2\kappa + \kappa^3 - \lambda\kappa^2)\partial_s^j \kappa \\
&= -2\partial_s^{j+4}\kappa - 5\kappa^2 \partial_s^{j+2}\kappa + \lambda^2 \partial_s^{j+2}\kappa + \lambda^2 \mathfrak{q}^{j+3}(\partial_s^j \kappa) + \mathfrak{q}^{j+5}(\partial_s^{j+1}\kappa).
\end{aligned}$$

We complete the proof. □

From the boundary condition of (SSC), we observe a certain behavior of κ at the boundary.

Lemma 2.5. *Let $\kappa(x,t)$ be the curvature of $\gamma(x,t)$ satisfying* (SSC)*. Then, for any $m \in \mathbb{N}$, it holds that*

$$\partial_s^{2m}\kappa(0,t) = \partial_s^{2m}\kappa(L,t) = 0. \tag{2.18}$$

Proof. First we show the case where $m = 1$, 2. Differentiating the boundary condition $\gamma(0,t) = (0,0)$ and $\gamma(L,t) = (R,0)$ with respect to t, we have $\partial_t\gamma(0,t) = \partial_t\gamma(L,t) = 0$. From $\kappa(0,t) = \kappa(L,t) = 0$ and the equation (1.2), we see that $\partial_s^2\kappa(0,t) = \partial_s^2\kappa(L,t) = 0$. Since $\partial_t\kappa(0,t) = \partial_t\kappa(L,t) = 0$, the equation (2.15) yields $\partial_s^4\kappa(0,t) = \partial_s^4\kappa(L,t) = 0$.

Next, suppose that $\partial_s^{2n}\kappa(0,t) = \partial_s^{2n}\kappa(L,t) = 0$ holds for any natural number $n \leq m$. Lemma 2.4 gives us

$$\partial_t\partial_s^{2m-2}\kappa = -2\partial_s^{2m+2}\kappa - 5\kappa^2\partial_s^{2m}\kappa + \lambda^2\partial_s^{2m}\kappa + \lambda^2\mathfrak{q}^{2m+1}(\partial_s^{2m-2}\kappa) + \mathfrak{q}^{2m+3}(\partial_s^{2m-1}\kappa).$$

Since any monomials of $\mathfrak{q}^{2m+1}(\partial_s^{2m-2}\kappa)$ and $\mathfrak{q}^{2m+3}(\partial_s^{2m-1}\kappa)$ contain at least one of the terms $\partial_s^{2l}\kappa$ $(l = 0, 1, 2, \cdots, m-1)$, we obtain $\partial_s^{2m+2}\kappa(0,t) = \partial_s^{2m+2}\kappa(L,t) = 0$. □

Let us define L^p norm with respect to the arc length parameter of γ. For a function $f(s)$ defined on γ, we write

$$\|f\|_{L_s^p} = \left\{\int_\gamma |f(s)|^p \, ds\right\}^{\frac{1}{p}}.$$

Similarly we define

$$\|f\|_{L_s^\infty} = \sup_{s\in[0,\mathcal{L}(\gamma)]} |f(s)|,$$

where $\mathcal{L}(\gamma)$ denotes the length of γ. Here we show the following interpolation inequalities:

Lemma 2.6. *Let $\gamma(x,t)$ be a solution of* (SSC)*. Let $u(x,t)$ be a function defined on γ and satisfy*

$$\partial_s^{2m}u(0,t) = \partial_s^{2m}u(L,t) = 0$$

for any $m \in \mathbb{N}$. Then, for integers $0 \leq p < q < r$, it holds that

$$\|\partial_s^q u\|_{L_s^2} \leq \|\partial_s^p u\|_{L_s^2}^{\frac{r-q}{r-p}} \|\partial_s^r u\|_{L_s^2}^{\frac{q-p}{r-p}}. \tag{2.19}$$

Moreover, for integers $0 \le p \le q < r$, it holds that

$$\|\partial_s^q u\|_{L_s^\infty} \le \sqrt{2}\,\|\partial_s^p u\|_{L_s^2}^{\frac{2(r-q)-1}{2(r-p)}} \|\partial_s^r u\|_{L_s^2}^{\frac{2(q-p)+1}{2(r-p)}}. \tag{2.20}$$

Proof. By the boundary condition of u, for any positive integer n, we have

$$\|\partial_s^n u\|_{L_s^2}^2 = \int_\gamma (\partial_s^n u)^2\,ds = -\int_\gamma \partial_s^{n-1}u \cdot \partial_s^{n+1}u\,ds \le \left\|\partial_s^{n-1}u\right\|_{L_s^2}\left\|\partial_s^{n+1}u\right\|_{L_s^2}.$$

This implies that $\log\|\partial_s^n u\|_{L_s^2}$ is convex with respect to $n > 0$. Thus we obtain the inequality (2.19).

Next we turn to (2.20). Since it holds that $\partial_s^{2m}u(0) = \partial_s^{2m}u(L) = 0$ for any $m \in \mathbb{N}$, the intermediate theorem implies that there exists at least one point $0 < \xi < L$ such that $\partial_s^{2m+1}u = 0$ at $x = \xi$. Hence, for each non-negative integer n, there exists a point $0 \le \xi_* \le L$ such that $\partial_s^n u = 0$ at $x = \xi_*$. Then we deduce that

$$\|\partial_s^n u\|_{L_s^\infty} \le \sqrt{2}\,\|\partial_s^n u\|_{L_s^2}^{\frac{1}{2}}\left\|\partial_s^{n+1}u\right\|_{L_s^2}^{\frac{1}{2}}. \tag{2.21}$$

Combining (2.19) with (2.21), we obtain (2.20). □

By virtue of Lemma 2.5, we are able to apply Lemma 2.6 to $\partial_s^n\kappa$ for any non-negative integer n. Making use of boundedness of energy functional at $\gamma = \Gamma_0$, we derive an estimate for $\|\kappa\|_{L_s^2}$.

Lemma 2.7. *Let γ be a solution of* (SSC)*. Then the curvature κ satisfies*

$$\|\kappa\|_{L_s^2}^2 \le \|k_0\|_{L^2(0,L)}^2 + \lambda^2\left(\mathcal{L}(\Gamma_0) - R\right). \tag{2.22}$$

Proof. Since the equation in (SSC) is the steepest descent flow for $E(\gamma) = \|\kappa\|_{L_s^2}^2 + \lambda^2\mathcal{L}(\gamma)$, we have

$$\|\kappa\|_{L_s^2}^2 + \lambda^2\mathcal{L}(\gamma) \le \|k_0\|_{L^2(0,L)}^2 + \lambda^2\mathcal{L}(\Gamma_0).$$

Since it is clear that $\mathcal{L}(\gamma) \ge R$, we obtain (2.22). □

We shall prove a long time existence of solution to (SSC) by using the energy method. To do so, we start with the following lemma:

Lemma 2.8. *For any non-negative integer j, it holds that*

$$\begin{aligned}\frac{d}{dt}\left\|\partial_s^j\kappa\right\|_{L_s^2}^2 &= -2\left\|\partial_s^{j+2}\kappa\right\|_{L_s^2}^2 - 2\lambda^2\left\|\partial_s^{j+1}\kappa\right\|_{L_s^2}^2 \\ &\quad + \lambda^2\int_\gamma \mathfrak{q}^{2j+4}(\partial_s^j\kappa)\,ds + \int_\gamma \mathfrak{q}^{2j+6}(\partial_s^{j+1}\kappa)\,ds.\end{aligned} \tag{2.23}$$

Proof. By virtue of Lemma 2.4, we have

$$\begin{aligned}\frac{d}{dt}\left\|\partial_s^j\kappa\right\|_{L_s^2}^2 &= \int_\gamma 2\partial_s^j\kappa\partial_t\partial_s^j\kappa\,ds + \int_\gamma (\partial_s^j\kappa)^2\kappa F^\lambda\,ds\\ &= \int_\gamma 2\partial_s^j\kappa\left\{-2\partial_s^{j+4}\kappa - 5\kappa^2\partial_s^{j+2}\kappa + \lambda^2\partial_s^{j+2}\kappa + \lambda^2\mathfrak{q}^{j+3}(\partial_s^j\kappa) + \mathfrak{q}^{j+5}(\partial_s^{j+1}\kappa)\right\}ds\\ &\quad + \int_\gamma \kappa\partial_s^j\kappa(2\partial_s^2\kappa + \kappa^3 - \lambda\kappa^2)\,ds.\end{aligned}$$

By integrating by parts, we get

$$\int_\gamma \kappa^2\partial_s^j\kappa\partial_s^{j+2}\kappa\,ds = -\int_\gamma \left\{2\kappa\partial_s\kappa\partial_s^j\kappa\partial_s^{j+1}\kappa + \kappa^2(\partial_s^{j+1}\kappa)^2\right\}ds.$$

Consequently we obtain (2.23). □

Using Lemmas 2.7 and 2.8, we derive the estimate for the derivative of $\|\partial_s^j\kappa\|_{L_s^2}^2$ with respect to t.

Lemma 2.9. *For any non-negative integer j, it holds that*

$$\frac{d}{dt}\left\|\partial_s^j\kappa\right\|_{L_s^2}^2 \le C\left\|\kappa\right\|_{L_s^2}^{4j+6} + C\left\|\kappa\right\|_{L_s^2}^{4j+10}.$$

Proof. We shall prove this lemma by estimating the right hand side of (2.23). In the process, Lemma 2.6 plays an important role. For the precise calculations, see [11]. □

Since the arc length parameter s depends on t, we need to estimate the local length of $\gamma(x,t)$.

Lemma 2.10. *Let $\gamma(x,t)$ be a solution of* (SSC) *for $0 \le t < T$. Then there exist positive constants C_1 and C_2 such that the inequalities*

$$\frac{1}{C_1(\Gamma_0,T)} \le |\partial_x\gamma(x,t)| \le C_1(\Gamma_0,T), \tag{2.24}$$

$$|\partial_x^m|\partial_x\gamma(x,t)|| \le C_2(\Gamma_0,T) \tag{2.25}$$

hold for any $(x,t)\in[0,L]\times[0,T]$ and integer $m \ge 1$.

We omit the proof. For the proof of Lemma 2.10, see [11]. By virtue of Lemma 2.10, we prove that the system (SSC) has a unique global solution in time.

Theorem 2.2. *Let Γ_0 be a smooth planar curve satisfying the conditions* (C). *Then there exists a unique classical solution of* (SSC) *for any time $t > 0$.*

Proof. Suppose not, then there exists a positive constant $\tilde{T}$ such that $\gamma(x,t)$ does not extend smoothly beyond $\tilde{T}$. It follows from Lemmas 2.7 and 2.9 that

$$\|\partial_s^m \kappa\|_{L_s^2}^2 \leq \|\partial_x^m k_0\|_{L^2(0,L)}^2 + C\tilde{T}$$

holds for any $0 \leq t \leq \tilde{T}$ and non-negative integer m. This yields that there exists a constant C such that

$$\|\partial_s^m \gamma\|_{L_s^2} \leq C \tag{2.26}$$

for $t \in [0, \tilde{T}]$. Here we have

$$\partial_x^m \gamma - |\partial_x \gamma|^m \, \partial_s^m \gamma = P(|\partial_x \gamma|, \cdots, \partial_x^{m-1} |\partial_x \gamma|, \gamma, \cdots, \partial_s^{m-1}\gamma), \tag{2.27}$$

where P is a certain polynomial. By virtue of (2.26), (2.27), and Lemma 2.10, we see that there exists a constant C such that

$$\|\partial_x^m \gamma\|_{L^2(0,L)} \leq C$$

for any $t \in [0, \tilde{T}]$ and $m \in \mathbb{N}$. Then $\gamma(x,t)$ extends smoothly beyond $\tilde{T}$ by Theorem 2.1. This is a contradiction. We complete the proof. $\square$

§ 2.3. Convergence

We prove that the solution $\gamma(x,t)$ of (SSC) converges to a stationary solution along a sequence of time in the C^∞ topology.

To begin with, we rewrite the equation (1.2) in terms of γ as follows:

$$\partial_t \gamma = -2\partial_s^4 \gamma + \left(\lambda^2 - 3\left|\partial_s^2 \gamma\right|^2\right) \partial_s^2 \gamma - 3\partial_s \left(\left|\partial_s^2 \gamma\right|^2\right) \partial_s \gamma. \tag{2.28}$$

By an argument similar to that in Lemma 2.3, we observe that the following rules hold:

$$\partial_t \partial_s = \partial_s \partial_t - G^\lambda \partial_s, \tag{2.29}$$

$$\partial_t ds = G^\lambda ds, \tag{2.30}$$

where $G^\lambda = \partial_s \partial_t \gamma \cdot \partial_s \gamma$. We start with the following lemma:

Lemma 2.11. *Let $\gamma(x,t)$ be the solution of* (SSC)*. Then, for any positive integer m, it holds that*

$$\partial_s^{2m} \gamma(0,t) = \partial_s^{2m} \gamma(L,t) = 0. \tag{2.31}$$

Proof. First we prove that the relation

$$\partial_s^n \gamma = \mathfrak{q}^{n-1}(\partial_s^{n-2}\kappa)\boldsymbol{\nu} + \mathfrak{q}^{n-1}(\partial_s^{n-2}\kappa)\partial_s\gamma \tag{2.32}$$

holds for any integers $n \geq 2$. Since $\partial_s^2\gamma = \kappa\boldsymbol{\nu}$, we see that (2.32) holds for $n = 2$. Suppose that (2.32) holds for any integers $2 \leq n \leq k$, where $k > 2$ is some integer. Then we have

$$\begin{aligned}
\partial_s^{k+1}\gamma &= \partial^s\{\mathfrak{q}^{k-1}(\partial_s^{k-2}\kappa)\}\boldsymbol{\nu} + \mathfrak{q}_s^{k-1}(\partial_s^{k-2}\kappa)\partial_s\boldsymbol{\nu} + \partial^s\{\mathfrak{q}^{k-1}(\partial_s^{k-2}\kappa)\}\partial_s\gamma + \mathfrak{q}_s^{k-1}(\partial_s^{k-2}\kappa)\partial_s^2\gamma \\
&= \left\{\partial^s\{\mathfrak{q}^{k-1}(\partial_s^{k-2}\kappa)\} + \kappa\mathfrak{q}_s^{k-1}(\partial_s^{k-2}\kappa)\right\}\boldsymbol{\nu} + \left\{\partial^s\{\mathfrak{q}^{k-1}(\partial_s^{k-2}\kappa)\} - \kappa\mathfrak{q}_s^{k-1}(\partial_s^{k-2}\kappa)\right\}\partial_s\gamma \\
&= \mathfrak{q}^k(\partial_s^{k-1}\kappa)\boldsymbol{\nu} + \mathfrak{q}^k(\partial_s^{k-1}\kappa)\partial_s\gamma.
\end{aligned}$$

This implies (2.32). Then, along the same line as in the proof of Lemma 2.5, we obtain the conclusion. $\square$

By virtue of Lemma 2.11, we can apply Lemma 2.6, i.e., interpolation inequalities, to $\partial_s^2\gamma$. Using the interpolation inequalities, we first prove the following estimate:

Lemma 2.12. *There exist positive constants C_1 and C_2 depending only on λ such that*

$$\left\|\partial_s^4\gamma\right\|_{L_s^2} \leq \|\partial_t\gamma\|_{L_s^2} + C_1\left\|\partial_s^2\gamma\right\|_{L_s^2}^5 + C_2\left\|\partial_s^2\gamma\right\|_{L_s^2}.$$

In order to derive the estimate of $\|\partial_s^n\gamma\|_{L_s^2}$ for $n \geq 5$, we are going to show the following:

Lemma 2.13. *For any $n \in \mathbb{N}$, it holds that*

$$\partial_t\partial_s^n\gamma = \partial_s^n\partial_t\gamma - \sum_{j=0}^{n-1}\partial_s^j(G^\lambda\partial_s^{n-j}\gamma).$$

Making use of Lemma 2.13, we prove the estimate of $\left\|\partial_s^{n+4}\gamma\right\|_{L_s^2}$ for any $n \in \mathbb{N}$:

Lemma 2.14. *For any $n \in \mathbb{N}$, the following estimate holds*:

$$\left\|\partial_s^{n+4}\gamma\right\|_{L_s^2} \leq \|\partial_s^n\partial_t\gamma\|_{L_s^2} + C\left\|\partial_s^2\gamma\right\|_{L_s^2}^{2n+5} + C\left\|\partial_s^2\gamma\right\|_{L_s^2}. \tag{2.33}$$

From Lemma 2.13, we observe a certain behavior of $\partial_t\gamma$ at the boundary.

Lemma 2.15. *Let $\gamma(x,t)$ be a solution of* (SSC). *Then it holds that*

$$\partial_s^{2m}\partial_t\gamma(0,t) = \partial_s^{2m}\partial_t\gamma(L,t) = 0 \tag{2.34}$$

for any non-negative integer m.

Proof. It is followed from Lemma 2.13 that

$$\partial_t \partial_s^n \gamma = \partial_s^n \partial_t \gamma - \sum_{j=0}^{n-1} \partial_s^j (G^\lambda \partial_s^{n-j} \gamma), \tag{2.35}$$

where

$$\begin{aligned} G^\lambda &= \partial_s^2 \left(\left| \partial_s^2 \gamma \right|^2 \right) - 2 \left| \partial_s^3 \gamma \right|^2 + \left(3 \left| \partial_s^2 \gamma \right|^2 - \lambda^2 \right) \left| \partial_s^2 \gamma \right|^2 \\ &= \partial_s^2 \gamma \cdot \partial_s^4 \gamma + \left(3 \left| \partial_s^2 \gamma \right|^2 - \lambda^2 \right) \left| \partial_s^2 \gamma \right|^2 . \end{aligned} \tag{2.36}$$

Moreover Lemma 2.5 gives us

$$\partial_t \partial_s^{2m} \gamma(0, t) = \partial_t \partial_s^{2m} \gamma(L, t) = 0. \tag{2.37}$$

Since

$$\partial_s^j (G^\lambda \partial_s^{n-j} \gamma) = \sum_{k=0}^{j} {}_j C_k \partial_s^k G^\lambda \partial_s^{2m-k} \gamma,$$

Lemma 2.13 and (2.36) yield that

$$\partial_s^j (G^\lambda \partial_s^{n-j} \gamma) = 0 \tag{2.38}$$

at $x = 0$, L for any $t > 0$ and non-negative integer $j \leq n$. By (2.35), (2.37), and (2.38), we complete the proof. □

By virtue of Lemma 2.15, we are able to apply Lemma 2.6 to $\partial_t \gamma$. By way of Lemma 2.15, we obtain the following:

Lemma 2.16. *For any $n \in \mathbb{N}$, it holds that*

$$\| \partial_s^n \partial_t \gamma \|_{L_s^2} \to 0 \quad as \quad t \to \infty. \tag{2.39}$$

Proof. Here we show an outline of proof. For a detail of the proof, see [12]. First we have

(2.40)
$$\int_0^\infty \| \partial_t \gamma \|_{L_s^2}^2 \, dt = - \int_0^\infty \partial_t \left(\int_\gamma \kappa^2 \, ds + \lambda^2 \mathcal{L}(\gamma) \right) dt = \left[\int_\gamma \kappa^2 \, ds + \lambda^2 \mathcal{L}(\gamma) \right]_{t=\infty}^{t=0} < \infty.$$

Next we show

$$\partial_t \| \partial_t \gamma \|_{L_s^2}^2 \leq - \left\| \partial_s^2 \partial_t \gamma \right\|_{L_s^2}^2 + C \| \partial_t \gamma \|_{L_s^2}^2 (1 + \| \partial_t \gamma \|_{L_s^2}^2). \tag{2.41}$$

This implies that $\|\partial_t\gamma\|_{L^2_s} \to 0$ as $t \to +\infty$. In particular, $\|\partial_t\gamma\|_{L^2_s}$ is bonded for any $t > 0$. Then (2.41) is reduced to

$$\partial_t \|\partial_t\gamma\|^2_{L^2_s} \le -\left\|\partial_s^2\partial_t\gamma\right\|^2_{L^2_s} + C\|\partial_t\gamma\|^2_{L^2_s}. \tag{2.42}$$

Integrating (2.42) on $[0,\infty)$, we obtain

$$\int_0^\infty \left\|\partial_s^2\partial_t\gamma\right\|^2_{L^2_s} dt \le -\int_0^\infty \partial_t \|\partial_t\gamma\|^2_{L^2_s} dt + C\int_0^\infty \|\partial_t\gamma\|^2_{L^2_s} dt < \infty. \tag{2.43}$$

Along the same line, we obtain the conclusion inductively. □

Making use of Lemmas 2.12, 2.14, and 2.16, we prove the following:

Theorem 2.3. *Let γ be a solution of* (SSC). *Then there exist a sequence $\{t_i\}_{i=0}^\infty$ with $t_i \to \infty$ and a planar curve $\hat\gamma$ such that $\gamma(\cdot,t_i)$ converges to $\hat\gamma(\cdot)$ up to a reparametrization in the C^∞ topology as $t_i \to \infty$. Moreover $\hat\gamma$ is a stationary solution of* (SSC).

Proof. Since it holds that

$$R \le \mathcal{L}(\gamma(\cdot,t)) \le \frac{1}{\lambda^2}\left\{\int_0^L k_0^2\,dx - \int_\gamma \kappa^2\,ds\right\} + \mathcal{L}(\gamma_0) < C,$$

we reparameterize γ by its arc length, i.e., $\gamma = \gamma(s,t)$. By virtue of Lemmas 2.12, 2.14, and 2.16, we see that

$$\|\partial_s^n\gamma(\cdot,t)\|_{L^2_s} < \infty \tag{2.44}$$

for any integers $n \ge 2$. From Lemma 2.6, the inequality (2.44) yields

$$\|\partial_s^n\gamma(\cdot,t)\|_{L^\infty_s} < \infty.$$

Thus $\partial_s^n\kappa$ is uniformly bounded with respect to t for any non-negative integers n. Furthermore it follows from (2.44) that

$$|\partial_s^n\kappa(s_1,t) - \partial_s^n\kappa(s_2,t)| \le \left|\int_{s_2}^{s_1} \partial_s^{n+1}\kappa(s,t)\,ds\right| \le C\,|s_1 - s_2|\,,$$

for each $n \in \mathbb{N}$, where the constant C is independent of t. Thus $\partial_s^n\kappa$ is equi-continuous with respect to t. Thus, there exist a sequence $\{t_{1,j}\}_{j=1}^\infty$ and $\hat\kappa(x)$ such that $\kappa(\cdot,t_{1,j})$ uniformly converges to $\hat\kappa(\cdot)$ as $t_{1,j} \to \infty$. Similarly, for each $n \in \mathbb{N}$, there exists a subsequence $\{t_{n,j}\}_{j=1}^\infty \subset \{t_{n-1,j}\}_{j=1}^\infty$ such that $\partial_s^n\kappa(\cdot,t)$ uniformly converges to $\partial_\cdot^n\hat\kappa(\cdot)$ as $t_{n,j} \to \infty$. By virtue of the diagonal method, we see that there exist a sequence

$\{t_i\}_{i=1}^{\infty}$ and a function $\hat{\kappa}(\cdot)$ such that $\kappa(\cdot, t_i)$ converges to $\hat{\kappa}(\cdot)$ in the C^∞ topology. Since $\gamma(\cdot, t)$ is fixed at the boundary, a curve $\hat{\gamma}$ with curvature $\hat{\kappa}$ is uniquely determined. Moreover, by Lemma 2.16, $\partial_t \gamma(\cdot, t)$ uniformly converges to 0 as $t \to \infty$. Therefore the curve $\hat{\gamma}$ is a stationary solution of (SSC). $\square$

§ 3. A dynamical aspect of shortening-straightening flow

In the previous section, we see that there exists a unique smooth solution of (SSC) for any $t > 0$ and the solution converges to a stationary solution along a sequence of time. In this section, we shall determine a certain dynamical aspect of shortening-straightening flow. For the purpose, we start with a linearized stability of a stationary solution of (SSC). The stationary solution of (SSC) is obtained as a critical point of the following variational problem:

Problem 3.1. *Let $R > 0$. Minimize the modified total squared curvature*

$$E(\gamma) = \int_\gamma \kappa^2 \, ds + \lambda^2 \mathcal{L}(\gamma)$$

over the set S_R.

The purpose of this section is to determine a dynamical aspect of shortening-straightening flow in a neighborhood of trivial critical point of Problem 3.1.

In particular, we focus on the case where non-closed planar curve γ is written as the graph of function. Namely, let define $\gamma(x) : [0, R] \to \mathbb{R}^2$ as

$$\gamma(x) = (x, u(x)), \tag{3.1}$$

where $u(x) : [0, R] \to \mathbb{R}$. Let us set

$$S_{RG} := \{\gamma \in S_R \mid \gamma(x) = (x, u(x)), \quad u : [0, R] \to \mathbb{R}\}.$$

We restrict Problem 3.1 as follows:

Problem 3.2. *Minimize the modified total squared curvature*

$$E(\gamma) = \int_\gamma \kappa^2 \, ds + \lambda^2 \mathcal{L}(\gamma)$$

over the set S_{RG}.

In this section, we use " $'$ " instead of d/dx for short. Under the formulation (3.1), the total squared curvature is expressed as

$$E(\gamma) = \int_0^R \left\{\kappa(x)^2 + \lambda^2\right\} \sqrt{1 + u'(x)^2} \, dx, \tag{3.2}$$

where the curvature $\kappa(x)$ of γ is written as follows:

$$\kappa(x) = \frac{d}{dx}\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right) = \frac{u''(x)}{(1+u'(x)^2)^{3/2}}. \tag{3.3}$$

By the representation (3.3), we see that Navier boundary condition

$$u(0) = u(R) = u''(0) = u''(R) = 0 \tag{3.4}$$

is equivalent to $\gamma(x) = (x, u(x)) \in S_{RG}$. Thus Problem 3.2 is formulated as follows:

Problem 3.3. *Minimize the functional* (3.2) *under the condition* (3.4).

In order to find a critical point of Problem 3.3, we consider a variation of γ as follows:

$$\Gamma(x,\varepsilon) = (x, u(x) + \varphi(x,\varepsilon)), \tag{3.5}$$

where $\varphi \in C^\infty((-\varepsilon_0, \varepsilon_0); C^2(0,R))$. Then it is clear that the the following holds:

Lemma 3.1. *Let $\gamma(x) \in S_{RG}$. The variation $\Gamma(x,\varepsilon) = (x, u(x)+\varphi(x,\varepsilon))$ belongs to the set S_{RG} if and only if*

$$\varphi(0,\varepsilon) = \varphi(R,\varepsilon) = \varphi''(0,\varepsilon) = \varphi''(R,\varepsilon) = 0, \tag{3.6}$$

for any $\varepsilon \in \mathbb{R}$.

In the following, let $\varphi(x,\varepsilon) \in C^\infty((-\varepsilon_0, \varepsilon_0); H^4(0,R))$ satisfy (3.6), where $\varepsilon_0 > 0$ is sufficiently small. Next we prove that a existence and stability of trivial critical point of Problem 3.3:

Lemma 3.2. *Problem 3.3 has a trivial critical point $u = 0$. Moreover, the trivial critical point is linearized stable.*

Proof. To begin with, we derive a first variational formula of E. Let $k(x,\varepsilon)$ denote the curvature of $\Gamma(x,\varepsilon)$. In order to obtain the formula, we calculate the following:

$$\begin{aligned}\frac{d}{d\varepsilon}E(\Gamma(\cdot,\varepsilon))\bigg|_{\varepsilon=0} = \int_0^R \bigg\{ & 2k(x,\varepsilon)\frac{d}{d\varepsilon}k(x,\varepsilon)\sqrt{1+(u'(x)+\varphi'(x,\varepsilon))^2} \\ & + \left(k(x,\varepsilon)^2+\lambda^2\right)\frac{d}{d\varepsilon}\sqrt{1+(u'(x)+\varphi'(x,\varepsilon))^2}\bigg\}\,dx\,\bigg|_{\varepsilon=0}.\end{aligned}$$

In the following, we write $\varphi_\varepsilon(x)$ instead of $(d/d\varepsilon)\varphi(x,\varepsilon)|_{\varepsilon=0}$, for short. Since

$$\frac{d}{d\varepsilon}\sqrt{1+(u'(x)+\varphi'(x,\varepsilon))^2}\bigg|_{\varepsilon=0} = \frac{u'(x)}{\sqrt{1+u'(x)^2}}\varphi'_\varepsilon(x), \tag{3.7}$$

$$\frac{d}{d\varepsilon}k(x,\varepsilon)\bigg|_{\varepsilon=0} = \frac{\varphi''_\varepsilon(x)}{(1+u'(x)^2)^{3/2}} - 3\kappa(x)\frac{u'(x)}{1+u'(x)^2}\varphi'_\varepsilon(x), \tag{3.8}$$

we obtain

$$\frac{d}{d\varepsilon}E(\Gamma(\cdot,\varepsilon))\Big|_{\varepsilon=0} = \int_0^R \left\{ 2\kappa(x)\frac{\varphi_\varepsilon''(x)}{1+u'(x)^2} + (-5\kappa(x)^2+\lambda^2)\frac{u'(x)}{\sqrt{1+u'(x)^2}}\varphi_\varepsilon'(x) \right\} dx.$$

Integrating by parts, we have

$$\begin{aligned}\int_0^R 2\kappa(x)\frac{\varphi_\varepsilon''(x)}{1+u'(x)^2}\,dx &= \int_0^R \left\{-2\kappa'(x)\frac{\varphi_\varepsilon'(x)}{1+u'(x)^2} + 4\kappa(x)^2\frac{u'(x)}{\sqrt{1+u'(x)^2}}\varphi_\varepsilon'(x)\right\} dx \\ &= \int_0^R \left\{ \frac{2}{\sqrt{1+u'(x)^2}}\left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right)' - 4\kappa(x)^3 \right. \\ &\qquad \left. - 10\frac{\kappa(x)\kappa'(x)u'(x)}{\sqrt{1+u'(x)^2}} \right\} \varphi_\varepsilon(x)\,dx\end{aligned}$$

and

$$\begin{aligned}\int_0^R (-5\kappa(x)^2+\lambda^2)&\frac{u'(x)}{\sqrt{1+u'(x)^2}}\varphi_\varepsilon'(x)\,dx \\ &= \int_0^R \left\{5\kappa(x)^3 - \lambda^2\kappa(x) + 10\frac{\kappa(x)\kappa'(x)u'(x)}{\sqrt{1+u'(x)^2}}\right\}\varphi_\varepsilon(x)\,dx.\end{aligned}$$

Thus we obtain the first variational formula:

(3.9)

$$\frac{d}{d\varepsilon}E(\Gamma(\cdot,\varepsilon))\Big|_{\varepsilon=0} = \int_0^R \left\{ \frac{2}{\sqrt{1+u'(x)^2}}\left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right)' + \kappa(x)^3 - \lambda^2\kappa(x)\right\}\varphi_\varepsilon(x)\,dx.$$

In the following, let $\gamma(x) = (x, u(x))$ be a critical point of E and $\kappa(x)$ denote the curvature, i.e., $u(x)$ be a solution of the following problem:

$$\begin{cases} \dfrac{2}{\sqrt{1+u'(x)^2}}\left(\dfrac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right)' + \kappa(x)^3 - \lambda^2\kappa(x) = 0 \quad \text{in} \quad [0,R] \\ u(0) = u(R) = u''(0) = u''(R) = 0. \end{cases} \tag{3.10}$$

For a critical point γ, we derive a second variational formula of E. Since $u(x)$ is a solution of (3.10), it is sufficient to calculate the following:

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}E(\Gamma(\cdot,\varepsilon))\Big|_{\varepsilon=0} = \int_0^R \frac{d}{d\varepsilon}\left\{ \frac{2}{\sqrt{1+(u'(x)+\varphi'(x,\varepsilon))^2}}\left(\frac{k'(x,\varepsilon)}{\sqrt{1+(u'(x)+\varphi'(x,\varepsilon))^2}}\right)' \right. \\ \left. + k(x,\varepsilon)^3 - \lambda^2 k(x,\varepsilon) \right\}\Big|_{\varepsilon=0} \varphi_\varepsilon(x)\,dx.\end{aligned}$$

First we have

$$\int_0^R \frac{d}{d\varepsilon}\left\{k(x,\varepsilon)^3 - \lambda^2 k(x,\varepsilon)\right\}\varphi_\varepsilon(x)\,dx\,\Big|_{\varepsilon=0} = \int_0^R \left(3\kappa(x)^2 - \lambda^2\right)\frac{d}{d\varepsilon}k(x,\varepsilon)\,\Big|_{\varepsilon=0}\varphi_\varepsilon(x)\,dx.$$

By the same calculations as above, we obtain

$$\begin{aligned}
&\int_0^R \frac{d}{d\varepsilon}\left\{\frac{2}{\sqrt{1+(u'(x)+\varphi'(x,\varepsilon))^2}}\left(\frac{k'(x,\varepsilon)}{\sqrt{1+(u'(x)+\varphi'(x,\varepsilon))^2}}\right)'\right\}\varphi_\varepsilon(x)\,dx\,\Big|_{\varepsilon=0}\\
&= \int_0^R \Bigg[-\frac{2}{1+u'(x)^2}\frac{u'(x)}{\sqrt{1+u'(x)^2}}\left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right)'\varphi'_\varepsilon(x)\\
&\qquad + \frac{2}{\sqrt{1+u'(x)^2}}\left\{\frac{1}{\sqrt{1+u'(x)^2}}\frac{d}{d\varepsilon}k'(x,\varepsilon)\,\Big|_{\varepsilon=0} - \frac{\kappa'(x)u'(x)}{(1+u'(x)^2)^{3/2}}\varphi'_\varepsilon(x)\right\}'\Bigg]\varphi_\varepsilon(x)\,dx\\
&= \int_0^R \Bigg[\frac{u'(x)}{1+u'(x)^2}\left(\kappa(x)^3 - \lambda^2\kappa(x)\right)\varphi'_\varepsilon(x)\\
&\qquad + \frac{2}{\sqrt{1+u'(x)^2}}\left\{\frac{1}{\sqrt{1+u'(x)^2}}\frac{d}{d\varepsilon}k'(x,\varepsilon)\,\Big|_{\varepsilon=0} - \frac{\kappa'(x)u'(x)}{(1+u'(x)^2)^{3/2}}\varphi'_\varepsilon(x)\right\}'\Bigg]\varphi_\varepsilon(x)\,dx.
\end{aligned}$$

Integrating by parts, we reduce the second term on the right hand side to

$$\begin{aligned}
&\int_0^R \frac{2}{\sqrt{1+u'(x)^2}}\left\{\frac{1}{\sqrt{1+u'(x)^2}}\frac{d}{d\varepsilon}k'(x,\varepsilon)\,\Big|_{\varepsilon=0} - \frac{\kappa'(x)u'(x)}{(1+u'(x)^2)^{3/2}}\varphi'_\varepsilon(x)\right\}'\varphi_\varepsilon(x)\,dx\\
&= \int_0^R -\frac{2}{\sqrt{1+u'(x)^2}}\left\{\frac{1}{\sqrt{1+u'(x)^2}}\frac{d}{d\varepsilon}k'(x,\varepsilon)\,\Big|_{\varepsilon=0} - \frac{\kappa'(x)u'(x)}{(1+u'(x)^2)^{3/2}}\varphi'_\varepsilon(x)\right\}\varphi'_\varepsilon(x)\,dx\\
&\quad \int_0^R +2\kappa(x)u'(x)\left\{\frac{1}{\sqrt{1+u'(x)^2}}\frac{d}{d\varepsilon}k'(x,\varepsilon)\,\Big|_{\varepsilon=0} - \frac{\kappa'(x)u'(x)}{(1+u'(x)^2)^{3/2}}\varphi'_\varepsilon(x)\right\}\varphi_\varepsilon(x)\,dx.
\end{aligned}$$

Here we focus on a trivial critical point. Clearly, Problem (3.10) has the trivial solution $u = 0$. For the trivial critical point, we have

$$\frac{d}{d\varepsilon}k(x,\varepsilon)\,\Big|_{\varepsilon=0} = \varphi''_\varepsilon(x).$$

Hence the second variational formula for the trivial critical point $u = 0$ is written as follows:

$$\frac{d^2}{d\varepsilon^2}E(\Gamma(\cdot,\varepsilon))\,\Big|_{\varepsilon=0} = \int_0^R \left\{-2\varphi'''_\varepsilon(x)\varphi'_\varepsilon(x) - \lambda^2\varphi''_\varepsilon(x)\varphi_\varepsilon(x)\right\}\,dx \tag{3.11}$$

Integrating by parts, we see that

$$\frac{d^2}{d\varepsilon^2}E(\Gamma(\cdot,\varepsilon))\,\Big|_{\varepsilon=0} = \int_0^R \left\{2\varphi''_\varepsilon(x)^2 + \lambda^2\varphi'_\varepsilon(x)^2\right\}\,dx > 0$$

for any non-trivial variation. Therefore we complete the proof. □

By virtue of Lemma 3.2, we can determine a dynamical aspect of shortening-straightening flow starting from a “neighborhood” of a line segment.

Theorem 3.1. *Let $0 < \theta < 1/4$. Let $\gamma_* \in S_R$ be the line segment. Then there exists a positive constant ε_* such that, for any smooth curve $\Gamma_0 \in S_{RG}$ with*

$$\|\Gamma_0 - \gamma_*\|_{h^{4+4\theta}([0,R])} < \varepsilon_*,$$

the solution $\gamma(x,t)$ of the initial boundary value problem

$$\begin{cases} \partial_t \gamma = (-2\partial_s^2 \kappa - \kappa^3 + \lambda^2 \kappa)\boldsymbol{\nu}, \\ \gamma(0,t) = (0,0),\ \gamma(R,t) = (R,0),\ \kappa(0,t) = \kappa(R,t) = 0, \\ \gamma(x,0) = \Gamma_0(x) \end{cases}$$

converges to $\gamma_*(x)$ in the C^∞ topology as $t \to \infty$.

Proof. The line segment $\gamma_*(x) \in S_R$ is expressed as

$$\gamma_*(x) = (x,0), \quad (0 \le x \le R).$$

Let $\mathcal{O}_\varepsilon$ be a neighborhood of γ_* as follows:

$$\mathcal{O}_\varepsilon = \{\gamma \in S_R \mid \|\gamma - \gamma_*\|_{h^{4+4\theta}([0,R])} < \varepsilon\}. \tag{3.12}$$

Let $0 < \varepsilon_1 < 1$ fix arbitrarily. It is easy to check that $\mathcal{O}_{\varepsilon_1} \subset S_{RG}$ holds. Set

$$\mathcal{A} := \left\{ \varphi \in h^{4+4\theta}([0,R]) \;\middle|\; \|\varphi\|_{h^{4+4\theta}([0,R])} = 1,\ \varphi(0) = \varphi(R) = \varphi''(0) = \varphi''(R) = 0 \right\}.$$

Since $\mathcal{O}_{\varepsilon_1} \subset S_{RG}$, we see that any $\Gamma_0 \in \mathcal{O}_{\varepsilon_1}$ is expressed as follows:

$$\Gamma_0(x;\varepsilon) = \gamma_*(x) + \varepsilon(0, \varphi(x)),$$

where $\varphi \in \mathcal{A}$ and $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$. On the other hand, by virtue of Lemma 3.2, there exists a positive constant ε_2 such that

$$\frac{d^2}{d\varepsilon^2} E(\Gamma_0(\cdot;\varepsilon)) > 0 \tag{3.13}$$

holds for any $\varepsilon \in (-\varepsilon_2, \varepsilon_2)$ and $\varphi \in \mathcal{A}$. Set $\varepsilon_* := \min\{\varepsilon_1, \varepsilon_2\}$. Then the fact (3.13) implies that any $\Gamma_0 \in \mathcal{O}_{\varepsilon_*}$ satisfies

$$\frac{d}{d\varepsilon} E(\Gamma_0(\cdot;\varepsilon)) \neq 0 \tag{3.14}$$

and

$$E(\Gamma_0(\cdot;\varepsilon)) > E(\gamma_*) \tag{3.15}$$

for any $\varepsilon \in (-\varepsilon_*, \varepsilon_*) \setminus \{0\}$ and $\varphi \in \mathcal{A}$. Let $\varepsilon \in (-\varepsilon_*, \varepsilon_*) \setminus \{0\}$ and $\varphi \in \mathcal{A}$ fix arbitrarily. Then, for $\Gamma(x, \varepsilon) = \gamma_*(x) + \varepsilon(0, \varphi(x))$, combining (3.14)-(3.15) with Theorems 2.2 and 2.3, we see that there exists a unique classical solution $\gamma(x, t)$ of

$$\begin{cases} \partial_t \gamma = (-2\partial_s^2 \kappa - \kappa^3 + \lambda^2 \kappa)\boldsymbol{\nu}, \\ \gamma(0,t) = (0,0), \ \gamma(R,t) = (R,0), \ \kappa(0,t) = \kappa(R,t) = 0, \\ \gamma(x,0) = \Gamma_0(x;\varepsilon) \end{cases} \tag{3.16}$$

for any finite time, and the solution $\gamma(x, t)$ converges to the line segment γ_* in the C^∞ topology along a sequence of time. In the following we shall prove that the solution $\gamma(x, t)$ of (3.16) converges to γ_* in the C^∞ topology as $t \to \infty$. Suppose that there exists a sequence $\{t_i\}_{i=0}^\infty$ such that $\gamma(x, t_i)$ does not converge to γ_* as $t_i \to \infty$. Then there exists $\rho > 0$, for any $N \in \mathbb{N}$, there exists $j > N$ such that

$$\|\gamma(\cdot, t_j) - \gamma_*(\cdot)\|_{C^\infty(0,R)} > \rho. \tag{3.17}$$

However, by virtue of (3.15) and Theorem 2.3, we observe that there exists a subsequence $\{t_{j_k}\} \subset \{t_j\}_{j=0}^\infty$ such that $\gamma(x, t_{j_k})$ converges to γ_* in the C^∞ topology as $t_{j_k} \to \infty$. This contradicts (3.17). Therefore we obtain the conclusion. $\square$

§ 4. Application to non-compact case

We close this paper with an announcement of result concerning an application to non-compact case. We are also interested in the following problem:

Problem 4.1. What is a dynamics of *non-closed* planar curves with *infinite* length governed by shortening-straightening flow?

Concerning Problem 4.1, first we consider an initial value problem for the flow (1.2) in [12]. To begin with, we shall state the initial condition. Let $\gamma_0(x) = (\phi_0(x), \psi_0(x)) : \mathbb{R} \to \mathbb{R}^2$ be a smooth non-closed curve and satisfy the following conditions:

$$|\partial_x \gamma_0(x)| \equiv 1, \tag{A1}$$

$$\kappa_0, \partial_x^m \kappa_0 \in L^2(\mathbb{R}) \quad \text{for all} \ \ m \in \mathbb{N}, \tag{A2}$$

$$\lim_{x\to\infty} \phi_0(x) = \infty, \qquad \lim_{x\to-\infty} \phi_0(x) = -\infty, \qquad \lim_{|x|\to\infty} \phi_0'(x) = 1, \tag{A3}$$

$$\psi_0(x) = O(|x|^{-\alpha}) \ \text{for some} \ \alpha > \frac{1}{2} \ \text{as} \ |x| \to \infty, \quad \psi_0' \in L^2(\mathbb{R}), \tag{A4}$$

where κ_0 denotes the curvature. The definition of γ_0 and the assumption (A1) imply that $\gamma_0(x)$ has infinite length. The assumptions (A3) and (A4) means that the initial curve $\gamma_0(x)$ is close to an axis in C^1 sense as $|x| \to \infty$.

For γ_0 satisfying (A1)–(A4), we consider the following initial value problem:

$$\text{(SS)} \qquad \begin{cases} \partial_t \gamma = (-2\partial_s^2 \kappa - \kappa^3 + \lambda^2 \kappa)\boldsymbol{\nu}, \\ \gamma(x, 0) = \gamma_0(x). \end{cases}$$

The main result of [12] is stated as follows:

Theorem 4.1. *Let $\gamma_0(x)$ be a smooth planar curve satisfying* (A1)–(A4). *Then there exists a smooth curve $\gamma(x, t) : \mathbb{R} \times [0, \infty) \to \mathbb{R}^2$ satisfying* (SS).

Generally, in order to prove that an initial value problem for a steepest descent flow of a geometric functional has a long time solution, the fact that the functional at initial state is bounded plays an important role. However, in our case, the functional at initial state $E(\gamma_0)$ is *not* finite. For, the initial curve γ_0 has infinite length. This is a difficulty of our problem (SS).

In order to overcome the difficulty, we construct an "approximate solution" of (SS) by using a solution of (1.2) for compact case with fixed boundary. For this purpose, we make use of Theorem 1.1. For the sequence defined by approximate solutions, we apply Arzelà-Ascoli's theorem and construct a solution of (SS). The point is to prove that the approximate solutions are uniformly bounded.

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