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Some mathematical aspects of spiral wave pattern

By

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Abstract

In this survey, we discuss some mathematical aspects for spiral waves. This includes the steadily rotating spiral curves in the plane, the propagating wave segments in the plane, and the rotating wave patterns in a disk.

§ 1. Introduction

Wave propagation in excitable media has been studied extensively both by theoreticians and experimentalists, due to its wide applications in physical model, chemical reaction, and biological system. Among them, spiral wave has been recognized as a fascinating and important spatio-temporal pattern, such as waves of oxidation in the BZ reaction [24], waves of cyclic-AMP signaling in the social amoeba colonies of *Dioctyostelium discoideum* [12], waves of neuromuscular in the heart muscle [23, 19], and so on. We refer to reader to [22, 13, 17, 3] for some nice survey papers on spirals.

In weakly excitable 2-D media, spiral wave patterns are usually modeled by a reaction-diffusion system, such as the FitzHugh-Nagumo system

$$\frac{\partial u}{\partial t} = D\nabla^2 u + 3u - u^3 - v,$$
$$\frac{\partial v}{\partial t} = \epsilon[u - \delta + I(t)],$$

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where $I(t)$ is the light intensity, constants $D > 0$, $0 < \epsilon, \delta + \sqrt{3} \ll 1$, u is the activator and v is the inhibitor in the BZ reaction, for example. A spiral wave pattern corresponds to the domain of excitation in the medium. As $\epsilon \to 0$, the sharp transition layer becomes a curve which enclose the excited region. In general, we can divide the boundary of the spiral wave pattern into two parts, namely, the *wave front* and the *wave back*, by the phase change point (or, tip).

In this paper, we shall discuss some mathematical aspects for spiral waves. This includes the steadily rotating spiral curve in the plane from the kinematic equation approach, the propagating wave segment from the wave front interaction model proposed by Zykov and Showalter [26], and the rotating wave patterns in a disk [27].

§ 2. Rotating spiral curves in the plane

In a spiral wave pattern, the sharply located spiral wave fronts are modeled as a family of planar curves with one free end. In general, a family of planar curves with one free end parametrized by the time variable t can be described by the following so-called *kinematic equation*:

$$\kappa_t + u_{ss} + \left(\kappa \int_0^s \kappa u d\xi\right)_s + G(t)\kappa_s = 0,\ s > 0,\ t > 0, \tag{2.1}$$

where s is the arc length measured from the free end, $\kappa = \kappa(s,t)$ is the (signed) curvature, $u = u(s,t)$ is the normal velocity, and $G(t)$ is the tangential velocity of the free end. Indeed, if we choose the normal vector to be the left-hand normal to the tangent vector and the curvature to be positive when the curve is winding in the clockwise direction, then we can derive (2.1) from the definitions of normal and tangent vectors, normal and tangential velocities, and the Frenet-Serret Theorem in the plane (cf. [5]).

In a steadily rotating spiral curve, the family of curves are keeping the same shape with a constant positive angular frequency ω, and their free ends rotate along a circle in the counterclockwise direction with zero tangential velocity. Hence, by imposing $G(t) \equiv 0$ and κ, u are independent of t, for a steadily rotating spiral curve, (2.1) is reduced to

$$u''(s) + \left(\kappa(s) \int_0^s \kappa(\xi)u(\xi)d\xi\right)' = 0,\ s \geq 0. \tag{2.2}$$

By integrating (2.2) once, we obtain that u and κ satisfy the equation

$$u'(s) + \kappa(s) \int_0^s \kappa(\xi)u(\xi)d\xi = \omega, \tag{2.3}$$

where ω is the positive constant angular frequency of the wave.

Usually, the evolution of these curves are modeled by the curvature driven flow: $u = U(\kappa)$, e.g., linear eikonal equation $u = c - D\kappa$, with c the propagation velocity of the planar front (or, driving force) and D the diffusion coefficient of the activator.

When $c \neq 0$, by a normalization we may assume that $c = D = 1$ so that we have $u = 1 - \kappa$. In [1], it was shown that there is a critical value $\omega_* > 0$ such that a spiral curve solution with positive curvature exists if and only if $\omega \in (0, \omega_*]$. Moreover, we are able to count the exact number of such spiral curve solutions for any given $\omega \in (0, \omega_*]$. See also [2] for a different approach. We note that the radius of the circle of the free ends is given by $\rho = |u(0)|/\omega$ with the tangent pointing inward to the center of the circle if $u(0) < 0$; outward to the center if $u(0) > 0$. Also, at the free end, the position vector is always perpendicular to the normal vector. When $c = 0$, in [5] we studied the backward and forward self-similar solutions of the equation (2.1).

For $u = 1 - \kappa$, we set

$$v(s) := \int_0^s \kappa(\xi) u(\xi) d\xi, \ s > 0.$$

Then (2.3) is reduced to the following system

$$\frac{dv}{ds} = \kappa(1 - \kappa), \tag{2.4}$$

$$\frac{d\kappa}{ds} = v\kappa - \omega \tag{2.5}$$

with the initial condition

$$v(0) = 0, \ \kappa(0) = b, \tag{2.6}$$

where $b \in \mathbf{R}$ is the curvature at the free end. In [6], by a phase plane analysis, we study the system (2.4)-(2.5) and obtain a complete classification of solutions of this system. Besides providing another approach to derive the results obtained by [1, 2] for spiral curves with positive curvature, we also obtain spiral curve solutions with sign-changing curvature and with negative curvature. Note that the curvature function can change sign at most once.

It should be remark that the above steadily rotating spiral curves obtained in [1, 2, 6] are not exactly the front parts of spiral wave patterns. It could be a part of the front, or, it could be the whole part of the front plus a part of the back. It would be a very interesting question to characterize the exact front part and the back part of a spiral wave pattern.

§ 3. Propagating wave segments

A fundamental problem on spiral waves is to understand what is the region of the excitability of the medium for which spiral waves can exist. As shown in [25] that the existence of spiral waves is closely related to the existence of 1D pulse in 2D medium. In fact, spiral waves can exist in the medium with sufficiently high excitability [25, 22]. Moreover, there exists an excitability limit, which we denote by ζ_0, below which the propagation of 1D waves is not possible [18]. Therefore, below such a excitability limit ζ_0, the underlying medium cannot support the propagation of spiral waves. When the excitability of the medium decreases from a high value to a critical value, the corresponding wavelength tends to infinity and the associated wave pattern becomes an unbounded (nearly) planar wave with one free end which is known as a *critical finger* (cf. [18, 10]).

Recently, in [14, 15, 16], it is found that another wave pattern in the photosensitive BZ reaction, namely, a *wave segment* which has two free ends, and moves with a constant velocity and fixed shape. These wave segments are unstable, but can be stabilized by using a feedback control to continually adjust the excitability of the medium (by adjusting the incident light). Their experimental study and numerical simulations also showed that there is a unique stabilized wave segment for each given (admissible) excitability of the medium.

A propagating wave segment can be described by its two interfacial boundaries: the wave front and the wave back. These two boundaries separate the enclosed domain Ω of excitation from the refractory region. The stabilized propagating wave segment can be approximately described by the wave front interaction model proposed by Zykov and Showalter [26]. In this model, they use the free-boundary approach to reduce the reaction-diffusion system to two systems of ODEs.

The wave front can be described by the following system of equations for $(\tilde{x}, \tilde{y}, \tilde{\theta}) = (\tilde{x}, \tilde{y}, \tilde{\theta})(s)$:

$$\begin{cases} \tilde{x}' = \sin\tilde{\theta}, \\ \tilde{y}' = -\cos\tilde{\theta}, \\ \tilde{\theta}' = -1 + \sigma\cos\tilde{\theta}, \end{cases} \tag{3.1}$$

where $\sigma \in (0,1)$ is the normalized normal velocity at the midpoint of front, $(\tilde{x},\tilde{y})$ is the Euclidean coordinates of a point of the front, $\tilde{\theta}$ is the angle of the (left-hand) normal vector measured from the positive x-axis, and the arc length s is measured from the top point of the wave segment (the tip). By choosing the tip to be on the y-axis and assuming the wave segment is symmetric with respect to the x-axis, we have

$$(\tilde{x}(0),\tilde{y}(0),\tilde{\theta}(0)) = (0, W(\sigma), \pi/2),$$

where $W(\sigma)$ is the (unknown) half-width of the wave segment. Note that $\tilde{\theta} \in [0,\pi/2]$, since $\tilde{\theta}' < 0$. Moreover, we have $\tilde{y}(s) = 0$ when $\tilde{\theta}(s) = 0$.

Using $\tilde{\theta}' < 0$, we can solve system (3.1) to obtain

$$\begin{aligned} x_+ &= \frac{1}{\sigma}\log\frac{1}{1-\sigma\cos\tilde{\theta}}, \\ y_+ &= -\frac{\tilde{\theta}}{\sigma} + \frac{2}{\sigma\sqrt{1-\sigma^2}}\tan^{-1}\Big(\frac{(1+\sigma)\tan(\tilde{\theta}/2)}{\sqrt{1-\sigma^2}}\Big) \end{aligned}$$

for $\tilde{\theta} \in [0,\pi/2]$. In particular, the half-width of the wave segment $W(\sigma)$ is the evaluation of y_+ at $\tilde{\theta} = \pi/2$, i.e.,

$$W = W(\sigma) := -\frac{\pi}{2\sigma} + \frac{2}{\sigma\sqrt{1-\sigma^2}}\tan^{-1}\Big(\frac{1+\sigma}{\sqrt{1-\sigma^2}}\Big).$$

Note that $W(0^+) = 1$, $W' > 0$ and $W(\sigma) \to \infty$ as $\sigma \to 1^-$. This orbit (x_+, y_+) gives the relation $x_+ = f_\sigma(y_+)$. Note that f_σ (depending on σ) is a decreasing function defined on $[0, W(\sigma)]$ such that

$$\begin{aligned} &f_\sigma(0) = -\frac{\log(1-\sigma)}{\sigma},\ f_\sigma(W(\sigma)) = 0, \\ &f'_\sigma(0) = 0,\ f'_\sigma(W(\sigma)^-) = -\infty,\ f''_\sigma(0) = \sigma - 1 < 0, \\ &f_\sigma(0) \to \infty \quad \text{as } \sigma \to 1^-. \end{aligned}$$

By measuring the arclength from the tip and taking the left-hand normal vector, the wave back can be described by the initial value problem $(P_{\sigma,b})$ for (x,y,θ), namely, the equations

$$x' = \sin\theta, \tag{3.2}$$

$$y' = -\cos\theta, \tag{3.3}$$

$$\theta' = 1 + \sigma\cos\theta - b[f_\sigma(y) - x], \tag{3.4}$$

with the initial conditions

$$x(0) = 0, \ \ y(0) = W(\sigma), \ \ \theta(0) = -\pi/2, \tag{3.5}$$

where parameters $\sigma \in (0,1)$ and b (related to the medium excitability) is a positive constant. The constant b in the problem $(P_{\sigma,b})$ is to be determined so that

$$y(s^*) = 0, \ \theta(s^*) = 0 \ \text{ and } \ y' < 0 \text{ in } (0, s^*) \tag{3.6}$$

for some $s^* > 0$. The terminal condition is due to the symmetric assumption of the wave back with respect to the x-axis.

We can roughly classify the wave segments into:

(I) convex type : $\theta' > 0$ on $[0, s^*)$.

(II) non-convex type : θ' can change its sign in $(0, s^*)$.

Then we have the following theorem (cf. [7]).

Theorem 3.1. *For each $\sigma \in (0,1)$, there exists a unique $b^* = b^*(\sigma) > 0$ and $s^* = s^*(\sigma) > 0$ such that the solution (x, y, θ) of (P_{σ,b^*}) defined on $[0, s^*]$ satisfies $y' < 0$ on $(0, s^*)$, $y(s^*) = 0$, $\theta(s^*) = 0$ and $x < f_\sigma(y)$ on $(0, s^*]$. Moreover, the wave segment is convex when σ is small, while it becomes non-convex when σ is close to* 1.

The proof of this theorem is based on the shooting argument with the help of two reduced first order systems of two differential equations from $(P_{\sigma,b})$.

§ 4. Rotating wave patterns in a disk

The study of spiral patterns is always in the unbounded media and it cannot be applied to describe spiral waves rotating within a disk. In [27], two types of rigidly rotating patterns within a disk are studied by using the free-boundary approach. They are: *spots* moving along the disk boundary and *spiral waves* rotating around the disk center. The study of Zykov [27] indicates that a selection mechanism that uniquely determines the shape and angular velocity of these two patterns as a function of the medium excitability and the disk radius. Moreover, rotating spots are intrinsically unstable and can be observed in excitable media only under a stabilizing feedback as in the wave segments.

Let (x, y) be the Euclidean coordinates and (r, γ) be the polar coordinates in the plane. Let s be the arc length measured from the touching point of disk boundary for the front and from the tip (phase change point) for the back, let θ be the angle of the normal vector (right-hand to the tangent) measuring from the positive x-axis, and κ be the (signed) curvature. We choose the sign of curvature to be positive if the curve is winding in the counter-clock direction. We note that the difference between rotating spot and spiral wave is the curvature of touching point of the front on the domain disk boundary, namely, the curvature of this touching point is positive for a spot and is negative for a spiral wave.

A propagating wave pattern (the excited region) which lies inside a disk centered at the origin and is rotating counter-clockwise along the disk boundary with a positive angular speed ω can be described by

$$r(s,t) = r(s), \quad \gamma(s,t) = \gamma(s) + \omega t, \quad \theta(s,t) = \theta(s) + \omega t. \tag{4.1}$$

From

$$\frac{dx}{ds} = -\sin\theta, \quad \frac{dy}{ds} = \cos\theta, \tag{4.2}$$

$$\frac{dr}{ds} = \sin(\gamma - \theta), \quad \frac{d\gamma}{ds} = \frac{1}{r}\cos(\gamma - \theta), \quad \frac{d\theta}{ds} = \kappa, \tag{4.3}$$

we can compute the normal velocity by

$$V = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta = \frac{dr}{dt}\cos(\gamma - \theta) - r\frac{d\gamma}{dt}\sin(\gamma - \theta).$$

Let the functions describing the front/back be denoted by $(x_\pm, y_\pm, r_\pm, \gamma_\pm, \theta_\pm, V_\pm)$. Then from (4.1) it follows that

$$V_\pm = -\omega r_\pm \sin\varphi_\pm, \tag{4.4}$$

where $\varphi_\pm := \gamma_\pm - \theta_\pm$. We remark that the normal velocity vanishes at the tip which separates the front and the back of a rotating wave.

§ 4.1. Wave front

For the front, we take the normalized interface equation

$$V_+ = 1 - \kappa_+.$$

Using (4.3) and (4.4), we obtain

$$\left\{\begin{aligned} \frac{dr_+}{ds} &= \sin\varphi_+, \\ \frac{d\varphi_+}{ds} &= \frac{\cos\varphi_+}{r_+} - 1 - \omega r_+ \sin\varphi_+ \end{aligned}\right. \tag{4.5}$$

with the initial condition:

$$r_+|_{s=0} = R_D, \quad \varphi_+|_{s=0} = \frac{3}{2}\pi, \tag{4.6}$$

if we put the touching point of disk boundary for the front on the negative y-axis. We look for solution such that r is monotone. Hence $\varphi_+ \in [\pi, 2\pi]$, since $\varphi_+(0) = 3\pi/2$. Note that the tip is the point when $\varphi_+ = \pi$. This gives us the terminal condition for (4.5).

Note that when $\omega = 0$ we can easily see that the front is just a part of the unit circle centered at $(-1, -R_D)$.

To solve this boundary value problem for $\omega > 0$, we introduce the following useful transformation:

$$\begin{aligned} X(\tau) &:= \omega r_+(s)\cos\varphi_+(s), \ \tau = -s, \\ Y(\tau) &:= 1 + \omega r_+(s)\sin\varphi_+(s), \ \tau = -s. \end{aligned}$$

This gives the system

$$\left\{\begin{aligned} \frac{dX}{d\tau} &= Y(1-Y), \\ \frac{dY}{d\tau} &= -\omega + XY \end{aligned}\right.$$

with the terminal condition

$$X(0) = 0, \quad Y(0) = 1 - \omega R_D$$

and the initial condition

$$X(-s_1) = -\omega r_+(s_1), \quad Y(-s_1) = 1$$

for some $s_1 > 0$. Note that Y is the curvature function.

Then we have the following theorem.

Theorem 4.1 ([8]). *There exists a positive constant ω_* such that*

1. *Let $\omega \in (0, \omega_*]$. For each $R_D > 0$, there exists a unique solution (r_+, φ_+) of* (4.5)-(4.6) *defined on $[0, s_1]$ for some $s_1 = s_1(R_D, \omega) > 0$ such that $\varphi_+(s) \in (\pi, 2\pi)$ for all $s \in [0, s_1)$ and $\varphi_+(s_1) = \pi$.*

2. *Let $\omega > \omega_*$. Then there exists a positive constant $a^* = a^*(\omega)$ such that a unique solution (r_+, φ_+) of* (4.5)-(4.6) *defined on $[0, s_1]$ for some $s_1 = s_1(R_D, \omega) > 0$ with $\varphi_+(s) \in (\pi, 2\pi)$ for all $s \in [0, s_1)$ and $\varphi_+(s_1) = \pi$ exists if and only if $R_D \leq [1 + a^*(\omega)]/\omega$. Moreover, $a^*(\omega) \to \infty$ as $\omega \downarrow \omega_*$.*

3. *$\varphi_+(s; \omega) \in (\pi, 3\pi/2)$ for all $s \in (0, s_1(R_D, \omega))$ if and only if $\omega R_D \leq 1$.*

It follows from this theorem that, for each $R_D > 0$, there is a $\omega^*(R_D) > \omega_*$ such that a front exists if and only if $0 \leq \omega \leq \omega^*(R_D)$. Moreover, the front is *convex* if $\omega R_D \leq 1$; and is *nonconvex* if $\omega R_D > 1$. Define $(r_*, \theta_*, \gamma_*) = (r_+(s_1; \omega), \theta_+(s_1; \omega), \gamma_+(s_1; \omega))$. Note that $r_* > 0$ for $\omega < \omega^*(R_D)$ and $r_* = 0$ when $\omega = \omega^*(R_D)$. Also, we can define the inverse function $s = s(r)$ on (r_*, R_D) of $r = r_+(s)$ and obtain

$$\Gamma_+(r) := \gamma_+(s(r); \omega), \quad \Phi_+(r) := \varphi_+(s(r); \omega),$$

which shall be used in the description of the back as follows.

§ 4.2. Wave back

Fix a $R_D > 0$ and $\omega \in (0, \omega^*(R_D))$, the normalized interface equation for the back is given by

$$V_- = 1 - \kappa_- - b(\Gamma_+(r_-) - \gamma_-),$$

where b is a nonnegative constant to be determined. Indeed, the parameter b is related to the excitability of the medium

Dropping the subscript "minus sign", then the back of a rotating wave is governed by

$$\begin{cases} \dfrac{dr}{ds} = \sin\varphi, \\ \dfrac{d\gamma}{ds} = \dfrac{\cos\varphi}{r}, \\ \dfrac{d\varphi}{ds} = \dfrac{\cos\varphi}{r} - 1 - \omega r \sin\varphi + b(\Gamma_+(r) - \gamma). \end{cases} \tag{4.7}$$

with the initial condition

$$r|_{s=0} = r_*, \quad \gamma|_{s=0} = \gamma_*, \quad \varphi|_{s=0} = \pi. \tag{4.8}$$

We are looking for a solution (r,γ,φ) of (4.7)-(4.8) such that

$$r(s_2;\omega,b)=R_D,\ \varphi(s;\omega,b)\in(0,\pi)\ \text{ for } s\in(0,s_2),\ \varphi(s_2;\omega,b)=\frac{\pi}{2},$$

for some positive arc length s_2 and a certain constant $b\geq 0$.

For this, we consider the following open strip domain

$$\mathcal{Q}:=(r_*,R_D)\times\mathbb{R}\times(0,\pi)$$

and define the *exit-length* $S=S(b)$ and the *exit-point* $(r_e,\gamma_e,\varphi_e)(b)$ as follows:

1. if there is a positive number $\hat{s}$ such that the orbit stays in $\mathcal{Q}$ for $0<s<\hat{s}$ and $r(\hat{s})=R_D$, then $S=S(b)=\hat{s}$ and $(r_e,\gamma_e,\varphi_e)(b)=(R_D,\gamma(S),\varphi(S))$;
2. if there is a positive number $\overline{s}$ such that the orbit stays in $\mathcal{Q}$ for $0<s<\overline{s}$, $r(\overline{s})<R_D$ and $\varphi(\tau)>\pi$ for some $\tau>\overline{s}$ and close to $\overline{s}$, then $S=S(b)=\overline{s}$ and $(r_e,\gamma_e,\varphi_e)(b)=(r(S),\gamma(S),\pi)$;
3. if there is a positive number $\underline{s}$ such that the orbit stays in $\mathcal{Q}$ for $0<s<\underline{s}$, $r(\underline{s})<R_D$ and $\varphi(\tau)<0$ for some $\tau>\underline{s}$ and close to $\underline{s}$, then $S=S(b)=\underline{s}$ and $(r_e,\gamma_e,\varphi_e)(b)=(r(S),\gamma(S),0)$.

Note that r is increasing in s while the orbit stays in $\mathcal{Q}$, the orbit never touches the plane $r=r_*$. Hence we look for $b\geq 0$ such that $S(b)=\hat{s}$ and $\varphi(S)=\pi/2$.

Since $(dr/ds)(s)>0$ for all $s\in(0,S)$, the functions $\Gamma:=\Gamma(r)$ and $\Phi:=\Phi(r)$ are well-defined for $r\in(r_*,r(S))$. Moreover, (Γ,Φ) satisfies the system

$$\frac{d\Gamma}{dr}=f(r,\Gamma,\Phi),\quad \frac{d\Phi}{dr}=g(r,\Gamma,\Phi), \tag{4.9}$$

where $\Phi\in(0,\pi)$ and

$$f(r,\Gamma,\Phi):=\frac{\cos\Phi}{r\sin\Phi}$$
$$g(r,\Gamma,\Phi):=\frac{(\cos\Phi/r)-1-\omega r\sin\Phi+b(\Gamma_+(r)-\Gamma)}{\sin\Phi}.$$

Here the function $\Gamma_+(r)$ is well-defined for $r\in(r_*,r(S))$.

For the case of convex front, i.e., $\omega R_D\leq 1$, we have

Theorem 4.2 ([8]). *For a given $R_D>0$ and $\omega\in(0,\omega^*(R_D))$ with $\omega R_D\leq 1$, there is a unique constant $b^\sharp:=b^\sharp(\omega)$ such that a solution (Γ,Φ) of (4.9) with*

$$(\Gamma(r_*),\Phi(r_*))=(\gamma_*,\pi)$$

exists for $r \in (r_*, R_D]$ *and satisfies*

$$\Phi(R_D; b^\sharp) = \frac{\pi}{2}, \quad 0 < \Phi(r; b^\sharp) < \pi \ \ \textit{for } r_* < r < R_D.$$

This gives a rigorous proof of the existence and uniqueness of rotating spots. The case of rotating spirals is left open.

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