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b-Functions and the representation theory of quivers

By

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§ 1. Introduction

We start with a classical formula

$$\det\left(\frac{\partial}{\partial v_{ij}}\right)\det(v)^{s+1} = (s+1)(s+2)\cdots(s+m)\cdot\det(v)^s \qquad (v \in M_m), \tag{1.1}$$

which is known since Cayley's time. From a modern view point, the identity (1.1) can be regarded as an example of b-functions of prehomogeneous vector spaces. In this note, we generalize (1.1) to the prehomogeneous vector spaces associated with quivers of type A. For this purpose, we use the result of [10], which asserts that under certain conditions, b-functions of reducible prehomogeneous vector spaces have decompositions correlated to the decomposition of representations. Moreover, in the latter half, we describe an algorithm to calculate the b-functions of several variables. This algorithm can be visualized by using the lace diagram, which now plays a remarkable role in singularity theory (cf. [3], [4]).

First we recall prehomogeneous vector spaces associated with quivers of type A. Let Q be a quiver of type A_r, i.e., a chain of r-vertices with arrows between them:

$$Q: \quad \overset{1}{\cdot} \longleftarrow \overset{2}{\cdot} \longleftarrow \overset{3}{\cdot} \longrightarrow \cdots \longleftarrow \overset{r}{\cdot}$$

For an r-tuple of positive integers $\underline{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{Z}^r_{>0}$, we define $GL(\underline{n})$ and $\mathrm{Rep}(Q, \underline{n})$ by

$$GL(\underline{n}) = GL(n_1) \times GL(n_2) \times \cdots \times GL(n_r),$$

$$\mathrm{Rep}(Q, \underline{n}) = \bigoplus_{i \to i+1 \text{ in } Q} M(n_{i+1}, n_i) \oplus \bigoplus_{j+1 \to j \text{ in } Q} M(n_j, n_{j+1}).$$

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(We call $\underline{n}$ the *dimension vector.*) Then $GL(\underline{n})$ acts on $\mathrm{Rep}(Q, \underline{n})$ by

$$g \cdot v = \big(\{g_{i+1}\, X_{i+1,i}\, g_i^{-1}\}_{i \to i+1 \text{ in } Q},\ \{g_j\, X_{j,j+1}\, g_{j+1}^{-1}\}_{j+1 \to j \text{ in } Q}\big)$$

for $g = (g_1, g_2, \ldots, g_r) \in GL(\underline{n})$ and $v = (\{X_{i+1,i}\}_{i \to i+1 \text{ in } Q}, \{X_{j,j+1}\}_{j+1 \to j \text{ in } Q}) \in \mathrm{Rep}(Q, \underline{n})$. It is well known that for any $\underline{n} \in \mathbb{Z}^r_{>0}$, $\mathrm{Rep}(Q, \underline{n})$ decomposes into a finite number of $GL(\underline{n})$-orbits. In particular, $(GL(\underline{n}), \mathrm{Rep}(Q, \underline{n}))$ is a prehomogeneous vector space.

Example 1.1. Let us consider the equioriented quiver of type A_5:

$$Q: \quad \overset{1}{\cdot} \longrightarrow \overset{2}{\cdot} \longrightarrow \overset{3}{\cdot} \longrightarrow \overset{4}{\cdot} \longrightarrow \overset{5}{\cdot}$$

For $\underline{n} = (n_1, \ldots, n_5) \in \mathbb{Z}^5_{>0}$, $GL(\underline{n})$ and $\mathrm{Rep}(Q, \underline{n})$ are given by

$$\begin{aligned} GL(\underline{n}) &= GL(n_1) \times GL(n_2) \times GL(n_3) \times GL(n_4) \times GL(n_5), \\ \mathrm{Rep}(Q, \underline{n}) &= M(n_2, n_1) \oplus M(n_3, n_2) \oplus M(n_4, n_3) \oplus M(n_5, n_4), \end{aligned}$$

and for $g = (g_1, \ldots, g_5) \in GL(\underline{n})$ and $v = (X_{2,1},\ X_{3,2},\ X_{4,3},\ X_{5,4}) \in \mathrm{Rep}(Q, \underline{n})$, we have

$$g \cdot v = (g_2 X_{2,1} g_1^{-1},\ g_3 X_{3,2} g_2^{-1},\ g_4 X_{4,3} g_3^{-1},\ g_5 X_{5,4} g_4^{-1}).$$

Remark. When Q is equioriented, $(GL(\underline{n}), \mathrm{Rep}(Q, \underline{n}))$ can be regarded as a prehomogeneous vector space of parabolic type arising from a special linear Lie algebra $\mathfrak{sl}(N)$ with $N = n_1 + \cdots + n_r$ ([8, 9]).

Example 1.2. Let us consider the following quiver of type A_5:

$$Q: \quad \overset{1}{\cdot} \longrightarrow \overset{2}{\cdot} \longleftarrow \overset{3}{\cdot} \longrightarrow \overset{4}{\cdot} \longleftarrow \overset{5}{\cdot}$$

For $\underline{n} = (n_1, \ldots, n_5) \in \mathbb{Z}^5_{>0}$, $GL(\underline{n})$ and $\mathrm{Rep}(Q, \underline{n})$ are given by

$$\begin{aligned} GL(\underline{n}) &= GL(n_1) \times GL(n_2) \times GL(n_3) \times GL(n_4) \times GL(n_5), \\ \mathrm{Rep}(Q, \underline{n}) &= M(n_2, n_1) \oplus M(n_2, n_3) \oplus M(n_4, n_3) \oplus M(n_4, n_5), \end{aligned}$$

and for $g = (g_1, \ldots, g_5) \in GL(\underline{n})$ and $v = (X_{2,1},\ X_{2,3},\ X_{4,3},\ X_{4,5}) \in \mathrm{Rep}(Q, \underline{n})$, we have

$$g \cdot v = (g_2 X_{2,1} g_1^{-1},\ g_2 X_{2,3} g_3^{-1},\ g_4 X_{4,3} g_3^{-1},\ g_4 X_{4,5} g_5^{-1}).$$

§ 2. Relative invariants

In this section, we describe a condition for $(GL(\underline{n}), \mathrm{Rep}(Q, \underline{n}))$ to have relative invariants and give their explicit construction. For the details, see Abeasis [1], Koike [7].

Let Q be a quiver of type A_r with arbitrary orientation. The orientation of Q is determined by the sequence

$$\{1 = \nu(0) < \nu(1) < \nu(2) < \cdots < \nu(h) < \nu(h+1) = r\}$$

which consists of the sinks and sources of Q. Note that if Q^* is the quiver obtained from Q by reversing all the arrows, then $(GL(\underline{n}), \mathrm{Rep}(Q^*, \underline{n}))$ is the dual prehomogeneous vector space $(GL(\underline{n}), \mathrm{Rep}(Q, \underline{n})^*)$.

Now we fix a dimension vector $\underline{n} = (n_1, \ldots, n_r)$ and consider the fundamental relative invariants of $(GL(\underline{n}), \mathrm{Rep}(Q, \underline{n}))$. First, for a given pair (p, q) with $1 \leq p < q \leq r$, we define indices $\alpha = \alpha(p, q)$ and $\beta = \beta(p, q)$ by the conditions

$$\nu(\alpha - 1) \leq p < \nu(\alpha), \qquad \nu(\beta) < q \leq \nu(\beta + 1).$$

When p, q are clear from the context, we just write α, β instead of $\alpha(p, q), \beta(p, q)$. Then $I_{\underline{n}}(Q)$ is defined to be the totality of pairs (p, q) with $1 \leq p < q \leq r$ which satisfy the following conditions (I1)$\sim$(I4):

(I1) For t with $p < t \leq \nu(\alpha)$, it follows that $n_t > n_p$,

(I2) For $\kappa = 0, 1, \ldots, \beta - \alpha - 1$ and t with $\nu(\alpha + \kappa) < t \leq \nu(\alpha + \kappa + 1)$, it follows that

$$n_t > n_{\nu(\alpha+\kappa)} - n_{\nu(\alpha+\kappa-1)} + \cdots + (-1)^{\kappa} n_{\nu(\alpha)} + (-1)^{\kappa+1} n_p,$$

(I3) For t with $\nu(\beta) < t < q$, it follows that

$$n_t > n_{\nu(\beta)} - n_{\nu(\beta-1)} + \cdots + (-1)^{\beta-\alpha} n_{\nu(\alpha)} + (-1)^{\beta-\alpha+1} n_p,$$

(I4) $n_q = n_{\nu(\beta)} - n_{\nu(\beta-1)} + \cdots + (-1)^{\beta-\alpha} n_{\nu(\alpha)} + (-1)^{\beta-\alpha+1} n_p.$

By Abeasis [1], we have the following lemma.

Lemma 2.1. *There exists a one-to-one correspondence between $I_{\underline{n}}(Q)$ and the set of $GL(\underline{n})$-orbits in $\mathrm{Rep}(Q, \underline{n})$ of codimension one. In particular, there exists a one-to-one correspondence between $I_{\underline{n}}(Q)$ and the fundamental relative invariants of $(GL(\underline{n}), \mathrm{Rep}(Q, \underline{n}))$.*

The explicit construction of an irreducible relative invariant corresponding to $(p, q) \in I_{\underline{n}}(Q)$ is given as follows. When there exist no sink and source between two vertices μ, ν ($\mu < \nu$) of Q, either the following (a) or (b) holds:

$$\text{(a)} \qquad \overset{\mu}{\cdot} \longrightarrow \overset{\mu+1}{\cdot} \longrightarrow \cdots \longrightarrow \overset{\nu-1}{\cdot} \longrightarrow \overset{\nu}{\cdot}$$

$$\text{(b)} \qquad \overset{\mu}{\cdot} \longleftarrow \overset{\mu+1}{\cdot} \longleftarrow \cdots \longleftarrow \overset{\nu-1}{\cdot} \longleftarrow \overset{\nu}{\cdot}$$

In the case of (a), we put

$$X_{\nu,\mu} = X_{\nu,\nu-1}X_{\nu-1,\nu-2}\cdots X_{\mu+1,\mu},$$

and in the case of (b), we put

$$X_{\mu,\nu} = X_{\mu,\mu+1}X_{\mu+1,\mu+2}\cdots X_{\nu-1,\nu}.$$

Now suppose that p is a source and q is a sink:

$$\overset{p}{\cdot} \longrightarrow \cdots \longrightarrow \overset{\nu(\alpha)}{\cdot} \longleftarrow \overset{\nu(\alpha)+1}{\cdot} \longleftarrow \cdots \longleftarrow \overset{\nu(\alpha+1)}{\cdot} \longrightarrow \cdots \longleftarrow \overset{\nu(\beta)}{\cdot} \longrightarrow \cdots \longrightarrow \overset{q}{\cdot}$$

In this case, for $v \in \mathrm{Rep}(Q, \underline{n})$, we define a matrix $Y_{(p,q)}$ by

$$Y_{(p,q)} = \begin{pmatrix} X_{\nu(\alpha),p} & X_{\nu(\alpha),\nu(\alpha+1)} & 0 & \cdots & 0 & 0 \\ 0 & X_{\nu(\alpha+2),\nu(\alpha+1)} & X_{\nu(\alpha+2),\nu(\alpha+3)} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & X_{\nu(\beta-1),\nu(\beta-2)} & X_{\nu(\beta-1),\nu(\beta)} \\ 0 & 0 & 0 & \cdots & 0 & X_{q,\nu(\beta)} \end{pmatrix},$$

and put $f_{(p,q)}(v) = \det Y_{(p,q)}$. Then it is easy to see that $f_{(p,q)}(v)$ is a relative invariant of $(GL(\underline{n}), \mathrm{Rep}(Q, \underline{n}))$.

Next we consider the case where both p and q are sources.

$$\overset{p}{\cdot} \longrightarrow \cdots \longrightarrow \overset{\nu(\alpha)}{\cdot} \longleftarrow \overset{\nu(\alpha)+1}{\cdot} \longleftarrow \cdots \longleftarrow \overset{\nu(\alpha+1)}{\cdot} \longrightarrow \cdots \longrightarrow \overset{\nu(\beta)}{\cdot} \longleftarrow \cdots \longleftarrow \overset{q}{\cdot}$$

Then we define a matrix $Y_{(p,q)}$ by

$$Y_{(p,q)} = \begin{pmatrix} X_{\nu(\alpha),p} & X_{\nu(\alpha),\nu(\alpha+1)} & 0 & \cdots & 0 & 0 \\ 0 & X_{\nu(\alpha+2),\nu(\alpha+1)} & X_{\nu(\alpha+2),\nu(\alpha+3)} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & X_{\nu(\beta-2),\nu(\beta-1)} & 0 \\ 0 & 0 & 0 & \cdots & X_{\nu(\beta),\nu(\beta-1)} & X_{\nu(\beta),q} \end{pmatrix}$$

and put $f_{(p,q)}(v) = \det Y_{(p,q)}$. Then it is easy to see that $f_{(p,q)}(v)$ is a relative invariant of $(GL(\underline{n}), \mathrm{Rep}(Q, \underline{n}))$. One can easily find similar expressions of $Y_{(p,q)}$ for the other cases, i.e., where "p is a sink and q is a source" or "both p and q are sinks".

Example 2.2. In Example 1.1, assume that $n_1 < n_2 < n_3 = n_4$, $n_5 = n_1$. Then we have $I_{\underline{n}}(Q) = \{(1,5), (3,4)\}$. The fundamental relative invariants are given explicitly by

$$f_{(3,4)}(v) = \det X_{4,3}, \qquad f_{(1,5)}(v) = \det X_{5,1} = \det(X_{5,4}X_{4,3}X_{3,2}X_{2,1}).$$

Example 2.3. In Example 1.2, assume that $n_1 + n_3 = n_2 + n_4$, $n_1 < n_2 < n_3$, $n_5 = n_1$. Then we have $I_{\underline{n}}(Q) = \{(1,4), (2,5)\}$. The fundamental relative invariants are given explicitly by

$$f_{(1,4)}(v) = \det \begin{pmatrix} X_{2,1} & X_{2,3} \\ O & X_{4,3} \end{pmatrix}, \qquad f_{(2,5)}(v) = \det \begin{pmatrix} X_{2,3} & O \\ X_{4,3} & X_{4,5} \end{pmatrix}.$$

Example 2.4. Let Q be the following quiver of type A_8:

$$Q : \overset{1}{\cdot} \longrightarrow \overset{2}{\cdot} \longrightarrow \overset{3}{\cdot} \longleftarrow \overset{4}{\cdot} \longrightarrow \overset{5}{\cdot} \longrightarrow \overset{6}{\cdot} \longleftarrow \overset{7}{\cdot} \longleftarrow \overset{8}{\cdot}$$

If the dimension vector $\underline{n}$ satisfies

$$\begin{aligned} &n_t > n_1 \ (t = 2,3), \qquad n_t > n_3 - n_1 \ (t = 4), \qquad n_t > n_4 - n_3 + n_1 \ (t = 5,6) \\ &n_7 > n_6 - n_4 + n_3 - n_1, \qquad n_8 = n_6 - n_4 + n_3 - n_1, \end{aligned}$$

then $(1,8) \in I_{\underline{n}}(Q)$ and the corresponding relative invariant is given by

$$f_{(1,8)}(v) = \det \left(\begin{array}{c|c|c} X_{3,1} & X_{3,4} & O \\ \hline O & X_{6,4} & X_{6,8} \end{array} \right) = \det \left(\begin{array}{c|c|c} X_{3,2}X_{2,1} & X_{3,4} & O \\ \hline O & X_{6,5}X_{5,4} & X_{6,7}X_{7,8} \end{array} \right).$$

§ 3. b-Functions of one variable

It follows from the theory of prehomogeneous vector spaces (cf. [6]) that there exists a polynomial $b_{(p,q)}(s) \in \mathbb{C}[s]$ satisfying

$$f_{(p,q)}\left(\frac{\partial}{\partial v}\right) f_{(p,q)}(v)^{s+1} = b_{(p,q)}(s) \cdot f_{(p,q)}(v)^s.$$

By using the decomposition formula of b-functions proved in F. Sato and Sugiyama [10], we can determine $b_{(p,q)}(s)$.

Example 3.1. Let us consider the equioriented quiver of type A_r:

$$Q : \overset{1}{\cdot} \longrightarrow \overset{2}{\cdot} \longrightarrow \overset{3}{\cdot} \longrightarrow \cdots \longrightarrow \overset{r}{\cdot}$$

Then, for $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r_{>0}$, we have

$$\begin{aligned} GL(\underline{n}) &= GL(n_1) \times GL(n_2) \times GL(n_3) \times \cdots \times GL(n_r), \\ \operatorname{Rep}(Q, \underline{n}) &= M(n_2, n_1) \oplus M(n_3, n_2) \oplus \cdots \oplus M(n_r, n_{r-1}), \end{aligned}$$

and the action is given by

$$g \cdot v = \left(g_2 X_{2,1} g_1^{-1},\ g_3 X_{3,2} g_2^{-1},\ \ldots, g_r X_{r,r-1} g_{r-1}^{-1}\right)$$

for $g = (g_1, g_2, g_3, \ldots, g_r) \in GL(\underline{n})$ and $v = (X_{2,1}, X_{3,2}, \ldots, X_{r,r-1}) \in \mathrm{Rep}(Q, \underline{n})$. Now we assume that $n_1 = n_r < n_2, n_3, \ldots, n_{r-1}$. Then

$$f(v) := \det (X_{r,r-1} \cdots X_{3,2} X_{2,1})$$

is an irreducible relative invariant corresponding to the character $\chi(g) = \det g_r \det g_1^{-1}$. Let us introduce the following graphical notation to indicate this polynomial.

$$f(v) = \det \left(\overset{n_1}{\circ} \longrightarrow \overset{n_2}{\circ} \longrightarrow \overset{n_3}{\circ} \longrightarrow \cdots \longrightarrow \overset{n_r}{\circ} \right).$$

We shall calculate the b-function $b_f(s)$ of f by using the result of [10]. We put

$$\begin{aligned} G' &= GL(n_3) \times \cdots \times GL(n_r), \\ E &= M(n_3, n_2) \oplus \cdots M(n_r, n_{r-1}), \\ F &= M(n_2, n_1), \\ GL(m) &= GL(n_2), \qquad GL(n) = GL(n_1) \end{aligned}$$

and regard $(GL(\underline{n}), \mathrm{Rep}(Q, \underline{n}))$ as a prehomogeneous vector space of the form [10, (2.2)]. Then we have $l = 0, d = 1$ in the notation of [10, Section 2] and thus we can apply [10, Theorem 2.6] in order to obtain the decomposition

$$b_f(s) = b_1(s) b_2(s).$$

Moreover, by [10, Theorem 3.3], we have

$$b_2(s) = \prod_{i=1}^{n_1} (s + n_2 - n_1 + i).$$

Note that $m = n_2, n = n_1, d = 1$ in the notation of [10, Theorem 3.3]. The last step is to calculate $b_1(s)$. Let $X^0_{2,1} = {}^t(I_{n_1} | 0_{n_1, n_2 - n_1}) \in F$ and put this into $f(v)$. Then we have

$$f(X^0_{2,1},\, X_{3,2},\, \ldots, X_{r,r-1}) = \det \left(X_{r,r-1} \cdots X'_{3,2} \right),$$

where $X'_{3,2}$ is given by

$$X_{3,2} = \left(X'_{3,2} \,\middle|\, X''_{3,2} \right) \in M(n_3, n_2), \qquad X'_{3,2} \in M(n_3, n_1), \qquad X''_{3,2} \in M(n_3, n_2 - n_1).$$

Hence $b_1(s)$ can be regarded as the b-function of the relative invariant

$$\det \left(\overset{n_1}{\circ} \longrightarrow \overset{n_3}{\circ} \longrightarrow \overset{n_4}{\circ} \longrightarrow \cdots \longrightarrow \overset{n_r}{\circ} \right)$$

of a prehomogeneous vector space arising from the equioriented quiver of type A_{r-1}. By repeating this cut-off operation, we get to the b-function of $\det \left(\overset{n_1}{\circ} \longrightarrow \overset{n_r}{\circ} \right)$, which is nothing but the formula (1.1). Summarizing the above argument, we obtain

$$b_f(s) = \prod_{t=2}^{r} \prod_{\lambda=1}^{n_1} (s + n_t - n_1 + \lambda).$$

The decomposition formula can be applied equally to $(GL(\underline{n}, \mathrm{Rep}(Q, \underline{n})))$ with Q being arbitrary orientation; for $\kappa = 0, 1, \ldots, \beta - \alpha$, we put

$$\begin{aligned}\overline{n}_{\nu(\alpha+\kappa)} &= \sum_{\tau=0}^{\kappa} (-1)^{\tau} n_{\nu(\alpha+\kappa-\tau)} + (-1)^{\kappa+1} n_p \\ &= n_{\nu(\alpha+\kappa)} - n_{\nu(\alpha+\kappa-1)} + \cdots + (-1)^{\kappa} n_{\nu(\alpha)} + (-1)^{\kappa+1} n_p.\end{aligned}$$

Then we have the following theorem.

Theorem 3.2.

$$\begin{aligned}b_{(p,q)}(s) = &\prod_{t=p+1}^{\nu(\alpha)} \prod_{\lambda=1}^{n_p} (s + n_t - n_p + \lambda) \\ &\times \prod_{\kappa=0}^{\beta-\alpha-1} \prod_{t=\nu(\alpha+\kappa)+1}^{\nu(\alpha+\kappa+1)} \prod_{\lambda=1}^{\overline{n}_{\nu(\alpha+\kappa)}} (s + n_t - \overline{n}_{\nu(\alpha+\kappa)} + \lambda) \\ &\times \prod_{t=\nu(\beta)+1}^{q} \prod_{\lambda=1}^{\overline{n}_{\nu(\beta)}} (s + n_t - \overline{n}_{\nu(\beta)} + \lambda).\end{aligned}$$

Example 3.3. In Example 2.2, we have

$$\begin{aligned}b_{(3,4)}(s) &= (s+1)(s+2)\cdots(s+n_3), \\ b_{(1,5)}(s) &= \prod_{t=2}^{5} \prod_{\lambda=1}^{n_1} (s + n_t - n_1 + \lambda) \\ &= (s+1)\cdots(s+n_1) \times (s + n_2 - n_1 + 1)\cdots(s+n_2) \\ &\quad \times (s + n_3 - n_1 + 1)^2 \cdots (s+n_3)^2.\end{aligned}$$

Example 3.4. In Example 2.3, we have

$$\begin{aligned}b_{(1,4)}(s) &= (s + n_2 - n_1 + 1)\cdots(s+n_2) \times (s + n_3 - n_2 + n_1 + 1)\cdots(s+n_3) \\ &\quad \times (s + n_4 - n_3 + n_2 - n_1 + 1)\cdots(s+n_4) \\ &= (s+1)\cdots(s+n_3) \times (s + n_2 - n_1 + 1)\cdots(s+n_2).\end{aligned}$$

Similarly, we have

$$b_{(2,5)}(s) = (s+1)\cdots(s+n_4) \times (s + n_3 - n_2 + 1)\cdots(s+n_3).$$

Example 3.5. The b-function $b_{(1,8)}(s)$ of the relative invariant $f_{(1,8)}(v)$ in Example 2.4 is given by

$$
\begin{aligned}
b_{(1,8)}(s) = & \prod_{t=2}^{3}(s+n_t-n_1+1)\cdots(s+n_t) \\
& \times \prod_{t=4}^{4}(s+n_t-n_3+n_1+1)\cdots(s+n_t) \\
& \times \prod_{t=5}^{6}(s+n_t-n_4+n_3-n_1+1)\cdots(s+n_t) \\
& \times (s+1)\cdots(s+n_7).
\end{aligned}
$$

§ 4. Some facts from the representation theory of quivers

To describe a combinatorial method to compute the b-functions of several variables, we quote some facts from the representation theory of quivers. For the details, we refer to Abeasis and Del Fra [2]. Assume that Q is a quiver of type A_r with arbitrary orientation.

$$
Q : \overset{1}{\cdot} \longrightarrow \overset{2}{\cdot} \longleftarrow \cdots \longleftarrow \overset{r-2}{\cdot} \longrightarrow \overset{r-1}{\cdot} \longrightarrow \overset{r}{\cdot}
$$

For the dimension vector $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{Z}_{>0}^r$, we take vector spaces L_i $(i = 1, \ldots, r)$ with $\dim L_i = n_i$. When $i \to i+1$ in Q, we take an arbitrary $A_{i+1,i} \in \mathrm{Hom}(L_i, L_{i+1})$, and when $j+1 \to j$ in Q, we take an arbitrary $A_{j,j+1} \in \mathrm{Hom}(L_{j+1}, L_j)$. If we fix a base of L_i and identify as $\mathrm{Hom}(L_i, L_{i+1}) \cong M(n_{i+1}, n_i)$ or $\mathrm{Hom}(L_{j+1}, L_j) \cong M(n_j, n_{j+1})$, then we can regard as

$$
A := (\{A_{i+1,i}\}_{i\to i+1 \text{ in } Q}, \{A_{j,j+1}\}_{j+1\to j \text{ in } Q}) \in \mathrm{Rep}(Q, \underline{n}).
$$

If $\nu(\kappa)$ is a sink and $\nu(\kappa-1), \nu(\kappa+1)$ are sources, then we can define the map ψ_κ^A by

$$
\psi_\kappa^A : L_{\nu(\kappa-1)} \oplus L_{\nu(\kappa+1)} \to L_{\nu(\kappa)} \quad ; \quad (z, w) \mapsto (A_{\nu(\kappa),\nu(\kappa-1)}z - A_{\nu(\kappa),\nu(\kappa+1)}w). \tag{4.1}
$$

For $1 \le i < j \le r$, we denote by $Q^{(i,j)}$ the subquiver of Q starting at i and terminating at j. Moreover, we define the map

$$
\varphi_{ij}^A : \bigoplus_t L_t \longrightarrow \bigoplus_{t'} L_{t'} \qquad \begin{pmatrix} t \text{ runs over all the sources of } Q^{(i,j)} \\ t' \text{ runs over all the sinks of } Q^{(i,j)} \end{pmatrix}
$$

to be the collection of the maps ψ_κ^A given as (4.1).

Example 4.1. Let us consider the following quiver of type A_5:

$$
\overset{1}{\cdot} \longrightarrow \overset{2}{\cdot} \longleftarrow \overset{3}{\cdot} \longrightarrow \overset{4}{\cdot} \longrightarrow \overset{5}{\cdot}
$$

Then we have

$$\begin{aligned}
\varphi_{14}^A : L_1 \oplus L_3 \to L_2 \oplus L_4 \quad &; \quad (z_1, z_3) \longmapsto (A_{2,1}z_1 - A_{2,3}z_3,\ A_{4,3}z_3) \\
\varphi_{25}^A : L_3 \to L_2 \oplus L_5 \quad &; \quad z_3 \longmapsto (-A_{2,3}z_3,\ A_{5,4}A_{4,3}z_3) \\
\varphi_{15}^A : L_1 \oplus L_3 \to L_2 \oplus L_5 \quad &; \quad (z_1, z_3) \longmapsto (A_{2,1}z_1 - A_{2,3}z_3,\ A_{5,4}A_{4,3}z_3).
\end{aligned}$$

For $A \in \mathrm{Rep}(Q, \underline{n})$, we define N_{ij}^A by

$$N_{ij}^A := \begin{cases} \operatorname{rank} \varphi_{ij}^A & (i < j) \\ \dim L_i = n_i & (i = j) \end{cases}.$$

Then the *rank parameter* $N_A := \{N_{ij}^A\}_{1 \le i \le j \le r}$ is an invariant which characterizes the $GL(\underline{n})$-orbit. That is, if we denote by $\mathcal{O}_A$ the $GL(\underline{n})$-orbit through $A \in \mathrm{Rep}(Q, \underline{n})$, then we have $\mathcal{O}_A = \mathcal{O}_B$ if and only if $N_{ij}^A = N_{ij}^B$ for $1 \le i \le j \le r$. Moreover, the partial ordering on the rank parameters coincides with the closure ordering on $GL(\underline{n})$-orbits. That is, we have $\mathcal{O}_A \subset \overline{\mathcal{O}}_B$ if and only if $N_{ij}^A \le N_{ij}^B$ for $1 \le i \le j \le r$.

Let us return to b-functions. Let $(p, q) \in I_{\underline{n}}(Q)$ (cf. Theorem 2.1). Note that $A \in \mathrm{Rep}(Q, \underline{n})$ satisfies $f_{(p,q)}(A) \ne 0$ if and only if φ_{pq}^A is an isomorphism. We denote by $N^{(p,q)}$ the rank parameter which is minimal among the rank parameters N_A such that φ_{pq}^A is an isomorphism, minimal with respect to the above-mentioned partial ordering. The orbit $\mathcal{O}^{(p,q)}$ corresponding to $N^{(p,q)}$ is the closed $GL(\underline{n})$-orbit in $\{A \in \mathrm{Rep}(Q, \underline{n})\,;\ f_{(p,q)}(A) \ne 0\}$, and thus it is unique (cf. Gyoja [5]).

For $N^{(p,q)} = \{N_{ij}^{(p,q)}\}_{1 \le i \le j \le r}$, we put

$$\begin{aligned}
\mathcal{F}^{(p,q)} := \Big\{ &\Big\{ N_{22}^{(p,q)} - N_{12}^{(p,q)} + 1, \dots, N_{22}^{(p,q)}\,(= n_2) \Big\}\ ; \\
&\Big\{ N_{33}^{(p,q)} - N_{23}^{(p,q)} + 1, \dots, N_{33}^{(p,q)}\,(= n_3) \Big\}\ ; \ \dots \\
&\dots\ ; \Big\{ N_{rr}^{(p,q)} - N_{r-1,r}^{(p,q)} + 1, \dots, N_{rr}^{(p,q)}\,(= n_r) \Big\} \Big\}.
\end{aligned} \tag{4.2}$$

Note that $\mathcal{F}^{(p,q)}$ is a set consisting of $r-1$ sets, and each set consists of consecutive natural numbers. However, if $N_{k-1,k}^{(p,q)} = 0$, then we regard $\{N_{kk}^{(p,q)} - N_{k-1,k}^{(p,q)} + 1, \dots, N_{kk}^{(p,q)}\,(= n_k)\}$ as the empty set $\emptyset$. Moreover, we define a "set" of linear forms $s + \mathcal{F}^{(p,q)}$ by

$$\begin{aligned}
s + \mathcal{F}^{(p,q)} := \Big\{ &\Big\{ s + N_{22}^{(p,q)} - N_{12}^{(p,q)} + 1, \dots, s + N_{22}^{(p,q)}\,(= s + n_2) \Big\}\ ; \\
&\Big\{ s + N_{33}^{(p,q)} - N_{23}^{(p,q)} + 1, \dots, s + N_{33}^{(p,q)}\,(= s + n_3) \Big\}\ ; \ \dots \\
&\dots\ ; \Big\{ s + N_{rr}^{(p,q)} - N_{r-1,r}^{(p,q)} + 1, \dots, s + N_{rr}^{(p,q)}\,(= s + n_r) \Big\} \Big\}.
\end{aligned}$$

Then we have the following theorem.

Theorem 4.2. *Let $(p,q) \in I_{\underline{n}}(Q)$. If we multiply all the linear forms contained in $s + \mathcal{F}^{(p,q)}$ together, then we obtain the b-function $b_{(p,q)}(s)$.*

Example 4.3. Let $\underline{n} = (2,5,6,6,2)$ in Example 3.3. Then we have

$$\begin{aligned} b_{(3,4)}(s) &= (s+1)(s+2)(s+3)(s+4)(s+5)(s+6), \\ b_{(1,5)}(s) &= (s+1)(s+2)(s+4)(s+5)^3(s+6)^2. \end{aligned}$$

To calculate the rank parameters $N^{(3,4)}$ and $N^{(1,5)}$ corresponding to the closed orbits, we construct the *lace diagrams* (cf. [3],[4]). Here, a lace diagram is a sequence of r-columns of dots, with n_i dots in the i-th column, together with line segments connecting dots of consecutive columns. Each dot may be connected to at most one dot in the column to the left of it, and to at most one dot in the column to the right of it. We identify the dots of the i-th column with chosen basis of L_i, and define each linear map $A_{i+1,i} \in \mathrm{Hom}(L_i, L_{i+1})$ or $A_{j,j+1} \in \mathrm{Hom}(L_{j+1}, L_j)$ according to the connections between the dots.

Now the lace diagrams corresponding to the closed orbits $\mathcal{O}^{(3,4)}$ and $\mathcal{O}^{(1,5)}$ are given as follows: Note that if any array in the diagram is erased, then the condition

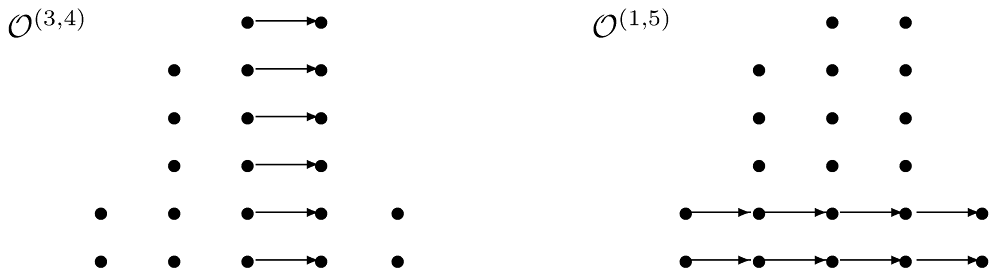

Figure 1. Lace diagrams corresponding to $\mathcal{O}^{(3,4)}$ and $\mathcal{O}^{(1,5)}$

$f_{(p,q)}(A) \neq 0$ is not satisfied, and conversely if some extra array is added, then the minimality condition is not satisfied. Thus we see that the rank parameters $N^{(3,4)}$ and $N^{(1,5)}$ are given as follows:

$$N^{(3,4)}: \quad \begin{array}{ccccc} 2 & 0 & 0 & 0 & 0 \\ & 5 & 0 & 0 & 0 \\ & & 6 & 6 & 0 \\ & & & 6 & 0 \\ & & & & 2 \end{array} \qquad N^{(1,5)}: \quad \begin{array}{ccccc} 2 & 2 & 2 & 2 & 2 \\ & 5 & 2 & 2 & 2 \\ & & 6 & 2 & 2 \\ & & & 6 & 2 \\ & & & & 2 \end{array}$$

By (4.2), the rank parameters read $\mathcal{F}^{(3,4)}$, $s+\mathcal{F}^{(3,4)}$ and $\mathcal{F}^{(1,5)}$, $s+\mathcal{F}^{(1,5)}$ as

$$\begin{aligned}
\mathcal{F}^{(3,4)} &= \{\emptyset\ ;\ \emptyset\ ;\ \{1,2,3,4,5,6\}\ ;\ \emptyset\}\,,\\
s+\mathcal{F}^{(3,4)} &= \{\emptyset\ ;\ \emptyset\ ;\ \{s+1,s+2,s+3,s+4,s+5,s+6\}\ ;\ \emptyset\}\,,\\
\mathcal{F}^{(1,5)} &= \{\{4,5\}\ ;\ \{5,6\}\ ;\ \{5,6\}\ ;\ \{1,2\}\}\,,\\
s+\mathcal{F}^{(1,5)} &= \{\{s+4,s+5\}\ ;\ \{s+5,s+6\}\ ;\ \{s+5,s+6\}\ ;\ \{s+1,s+2\}\}\,.
\end{aligned}$$

Note that if we multiply all the linear forms contained in $s+\mathcal{F}^{(3,4)}$ (resp. $s+\mathcal{F}^{(1,5)}$), then we obtain the b-function $b_{(3,4)}(s)$ (resp. $b_{(1,5)}(s)$).

Example 4.4. Let $\underline{n}=(2,5,7,4,2)$ in Example 3.4. Then we have

$$\begin{aligned}
b_{(1,4)}(s) &= (s+1)(s+2)(s+3)(s+4)^2(s+5)^2(s+6)(s+7),\\
b_{(2,5)}(s) &= (s+1)(s+2)(s+3)^2(s+4)^2(s+5)(s+6)(s+7).
\end{aligned}$$

The lace diagrams corresponding to $\mathcal{O}^{(1,4)}$ and $\mathcal{O}^{(2,5)}$ are given as follows: Here we

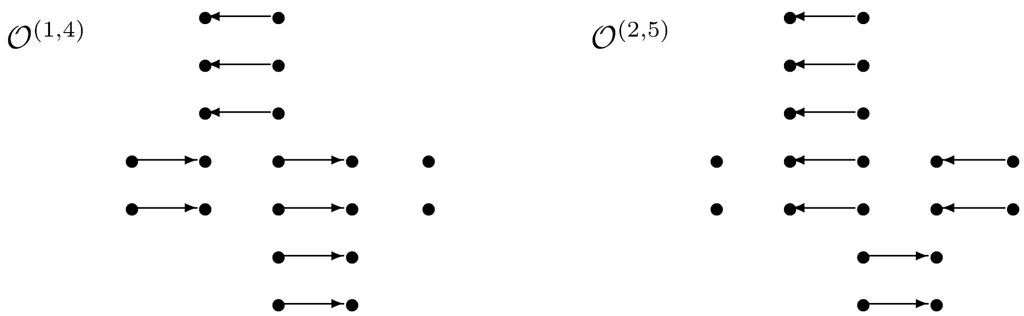

Figure 2. Lace diagrams corresponding to $\mathcal{O}^{(1,4)}$ and $\mathcal{O}^{(2,5)}$

follow the convention of [1, p. 467] or [3]. That is, two consecutive columns connected with a rightward arrow (resp. leftward arrow) are bottom-aligned (resp. top-aligned). Now we see that the rank parameters $N^{(1,4)}$ and $N^{(2,5)}$ are given by

$$N^{(1,4)}:\ \begin{array}{r} 2\,2\,5\,9\,9\\ 5\,3\,7\,7\\ 7\,4\,4\\ 4\,0\\ 2 \end{array} \qquad N^{(2,5)}:\ \begin{array}{r} 2\,0\,5\,7\,9\\ 5\,5\,7\,9\\ 7\,2\,4\\ 4\,2\\ 2 \end{array}$$

Hence we observe that $\mathcal{F}^{(1,4)}, s+\mathcal{F}^{(1,4)}$ and $\mathcal{F}^{(2,5)}, s+\mathcal{F}^{(2,5)}$ are

$$\begin{aligned}
\mathcal{F}^{(1,4)} &= \{\{4,5\}\ ;\ \{5,6,7\}\ ;\ \{1,2,3,4\}\ ;\ \emptyset\}\,,\\
s+\mathcal{F}^{(1,4)} &= \{\{s+4,s+5\}\ ;\ \{s+5,s+6,s+7\}\ ;\ \{s+1,s+2,s+3,s+4\}\ ;\ \emptyset\}\,,\\
\mathcal{F}^{(2,5)} &= \{\emptyset\ ;\ \{3,4,5,6,7\}\ ;\ \{3,4\}\ ;\ \{1,2\}\}\,,\\
s+\mathcal{F}^{(2,5)} &= \{\emptyset\ ;\ \{s+3,s+4,s+5,s+6,s+7\}\ ;\ \{s+3,s+4\}\ ;\ \{s+1,s+2\}\}\,,
\end{aligned}$$

and that if we multiply all the linear forms contained in $s+\mathcal{F}^{(1,4)}$ (resp. $s+\mathcal{F}^{(2,5)}$), then we obtain the b-function $b_{(1,4)}(s)$ (resp. $b_{(2,5)}(s)$).

§ 5. b-Functions of several variables

First we recall the definition of b-functions of several variables for general reductive prehomogeneous vector spaces (G,V) (cf. M. Sato [11]). Let $f_1,\dots,f_l$ be the fundamental relative invariants and χ_i the character of f_i. Let $f_1^*,\dots,f_l^*$ the fundamental relative invariants of the dual prehomogeneous vector space (G,V^*) such that the character of f_i^* is χ_i^{-1}. For a multi-variable $\underline{s}=(s_1,\dots,s_l)$, we put $\underline{f}^{\underline{s}}=\prod_{i=1}^l f_i^{s_i}$, $\underline{f}^{*\underline{s}}=\prod_{i=1}^l f_i^{*s_i}$. Then for any l-tuple $\underline{m}=(m_1,\dots,m_l)\in\mathbb{Z}_{\geq 0}^l$ of non-negative integers, we have

$$\underline{f}^{*\underline{m}}\left(\frac{\partial}{\partial v}\right)\underline{f}^{\underline{s}+\underline{m}}(v)=b_{\underline{m}}(\underline{s})\cdot\underline{f}^{\underline{s}}(v)$$

with some non-zero polynomial $b_{\underline{m}}(\underline{s})$. We call $b_{\underline{m}}(\underline{s})$ the b-function of $\underline{f}=(f_1,\dots,f_l)$.

Now we describe an algorithm to calculate the multi-variate b-function $b_{\underline{m}}(\underline{s})$ of $(GL(\underline{n}),\mathrm{Rep}(Q,\underline{n}))$. We take an arbitrary numbering on $I_{\underline{n}}(Q)$ and let

$$I_{\underline{n}}(Q)=\{(p_1,q_1),(p_2,q_2),\dots,(p_l,q_l)\}\,.$$

In the following, we write as $f_{(p_1,q_1)}=f_1$, $N^{(p_2,q_2)}=N^{(2)}$, $\mathcal{F}^{(p_3,q_3)}=\mathcal{F}^{(3)},\dots.$

1°. First, for given linear forms $s_{i_1}+\alpha$, $s_{i_2}+\alpha$, $\cdots$, $s_{i_t}+\alpha$ with the *same constant term* α, we define the *superposition operation* as follows:

$$s_{i_1}+\alpha,\ s_{i_2}+\alpha,\cdots,s_{i_t}+\alpha\quad\longmapsto\quad s_{i_1}+s_{i_2}+\cdots+s_{i_t}+\alpha \tag{5.1}$$

Carry out this operation on the $(k-1)$-th components

$$\begin{gathered}
\left\{s_1+N_{kk}^{(1)}-N_{k-1,k}^{(1)}+1,\dots,s_1+N_{kk}^{(1)}\,(=s_1+n_k)\right\},\\
\left\{s_2+N_{kk}^{(2)}-N_{k-1,k}^{(2)}+1,\dots,s_2+N_{kk}^{(2)}\,(=s_2+n_k)\right\},\\
\cdots\cdots\cdots\cdots\cdots\cdots\\
\left\{s_l+N_{kk}^{(l)}-N_{k-1,k}^{(l)}+1,\dots,s_l+N_{kk}^{(l)}\,(=s_l+n_k)\right\}
\end{gathered}$$

of $s_1+\mathcal{F}^{(1)}, s_2+\mathcal{F}^{(2)}, \ldots, s_l+\mathcal{F}^{(l)}$. Here we ignore the empty set $\emptyset$.

Example 5.1. For

$$\begin{aligned}\{s_1+3,\ s_1+4,\ s_1+5\},\\ \{s_2+4,\ s_2+5\},\\ \emptyset,\\ \{s_4+1,\ s_4+2,\ s_4+3,\ s_4+4,\ s_4+5\},\end{aligned}$$

we obtain the following new linear forms

$$s_4+1,\ s_4+2,\ s_1+s_4+3,\ s_1+s_2+s_4+4,\ s_1+s_2+s_4+5.$$

2°. Carry out the operation 1° for all $k=2,\ldots,r$.

3°. Substitute the linear forms obtained in 2° by the rule

$$s_{i_1}+s_{i_2}+\cdots+s_{i_t}+\alpha \quad\longmapsto\quad [s_{i_1}+s_{i_2}+\cdots+s_{i_t}+\alpha]_{m_{i_1}+m_{i_2}+\cdots+m_{i_t}},$$

and then multiply all of them together. Here the square parentheses symbol stands for

$$[A]_m := A(A+1)(A+2)\cdots(A+m-1).$$

The output of the operation 3° is the b-function $b_{\underline{m}}(\underline{s})$ of $\underline{f}=(f_1,\ldots,f_l)$.

Example 5.2. In Example 4.3, put $f_1 := f^{(3,4)}$, $f_2 := f^{(1,5)}$. For

$$\begin{aligned} s_1+\mathcal{F}^{(1)} &= s_1+\mathcal{F}^{(3,4)} = \{\emptyset\ ;\ \emptyset\ ;\ \{s_1+1,s_1+2,s_1+3,s_1+4,s_1+5,s_1+6\}\ ;\ \emptyset\},\\ s_2+\mathcal{F}^{(2)} &= s_2+\mathcal{F}^{(1,5)}\\ &= \{\{s_2+4,s_2+5\}\ ;\ \{s_2+5,s_2+6\}\ ;\ \{s_2+5,s_2+6\}\ ;\ \{s_2+1,s_2+2\}\},\end{aligned}$$

we carry out the operation 1°. Since the 1-st, 2-nd, 4-th components of $\mathcal{F}^{(3,4)}$ are the empty set, we obtain $\{s_2+4,s_2+5\}$ $\{s_2+5,s_2+6\}$, $\{s_2+1,s_2+2\}$ from the 1-st, 2-nd, 4-th components, and at the 3-rd component, we superpose the linear forms as follows:

$$\begin{matrix}\{s_1+1,s_1+2,s_1+3,s_1+4,s_1+5,s_1+6\}\\ \{s_2+5,s_2+6\}\end{matrix} \quad\longmapsto\quad \begin{matrix}s_1+1,\ s_1+2,\ s_1+3,\ s_1+4,\\ s_1+s_2+5,\ s_1+s_2+6\end{matrix}.$$

All the linear forms are aligned as

$$\begin{aligned}&s_1+1,\ s_1+2,\ s_1+3,\ s_1+4,\\ &s_2+1,\ s_2+2,\ s_2+4,\ (s_2+5)^{\times 2},\ s_2+6,\\ &s_1+s_2+5,\ s_1+s_2+6\end{aligned}$$

and by multiplying these factors according to 3°, we obtain the b-function $b_{\underline{m}}(\underline{s})$. That is,

$$\begin{aligned} b_{\underline{m}}(\underline{s}) = & [s_1+1]_{m_1}[s_1+2]_{m_1}[s_1+3]_{m_1}[s_1+4]_{m_1} \\ & \times [s_2+1]_{m_2}[s_2+2]_{m_2}[s_2+4]_{m_2}[s_2+5]^2_{m_2}[s_2+6]_{m_2} \\ & \times [s_1+s_2+5]_{m_1+m_2}[s_1+s_2+6]_{m_1+m_2}. \end{aligned}$$

The aspect of the superposition can be visualized as follows: First, as in Figure 3, we attach the linear forms in $s_1+\mathcal{F}^{(3,4)}$ (resp. $s_2+\mathcal{F}^{(1,5)}$) to the arrows in the lace diagram corresponding to $\mathcal{O}^{(3,4)}$ (resp. $\mathcal{O}^{(1,5)}$). Then we superpose these two diagrams. If two

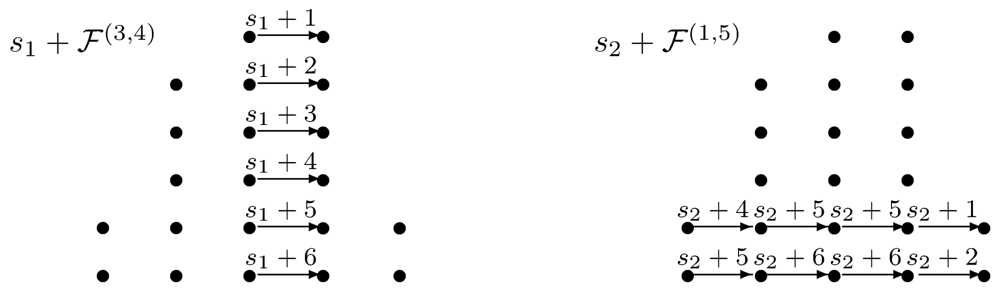

Figure 3. Lace diagrams corresponding to $s_1+\mathcal{F}^{(3,4)}$ and $s_2+\mathcal{F}^{(1,5)}$

linear forms are attached to the same arrow, those two linear forms are also superposed as in Figure 4.

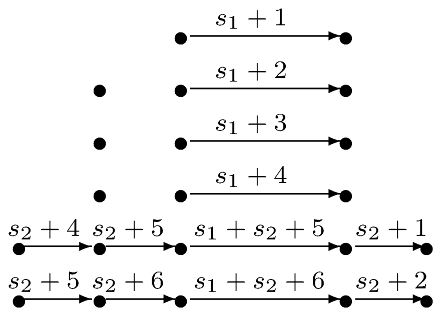

Figure 4. Superposition of the lace diagrams in Example 5.2

Example 5.3. In Example 4.4, we put $f_1 := f^{(1,4)}$, $f_2 := f^{(2,5)}$. For

$$s_1+\mathcal{F}^{(1,4)} = \{\{s_1+4, s_1+5\}\ ;\ \{s_1+5, s_1+6, s_1+7\}\ ;\ \{s_1+1, s_1+2, s_1+3, s_1+4\}\ ;\ \emptyset\},$$
$$s_2+\mathcal{F}^{(2,5)} = \{\emptyset\ ;\ \{s_2+3, s_2+4, s_2+5, s_2+6, s_2+7\}\ ;\ \{s_2+3, s_2+4\}\ ;\ \{s_2+1, s_2+2\}\},$$

we perform the operations $1^\circ, 2^\circ, 3^\circ$, and obtain

$$\begin{aligned} b_{\underline{m}}(\underline{s}) = {} & [s_1+1]_{m_1}[s_1+2]_{m_1}[s_1+4]_{m_1}[s_1+5]_{m_1} \\ & \times [s_2+1]_{m_2}[s_2+2]_{m_2}[s_2+3]_{m_2}[s_2+4]_{m_2} \\ & \times [s_1+s_2+3]_{m_1+m_2}[s_1+s_2+4]_{m_1+m_2}[s_1+s_2+5]_{m_1+m_2} \\ & \times [s_1+s_2+6]_{m_1+m_2}[s_1+s_2+7]_{m_1+m_2}. \end{aligned}$$

Also in this case, the aspect of the superposition can be interpreted visually. First, as

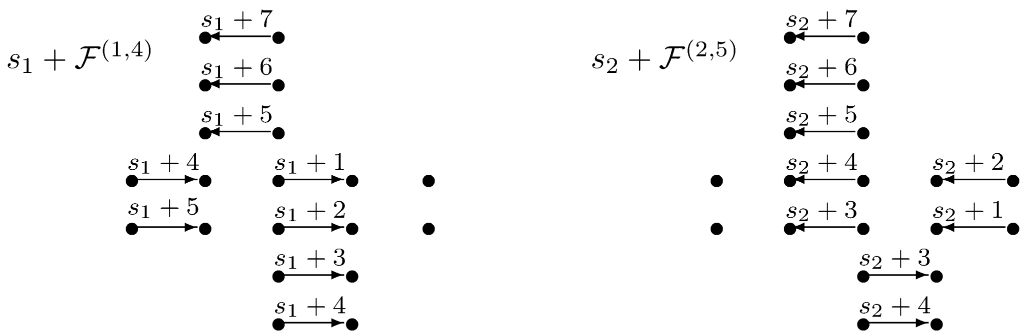

Figure 5. Lace diagrams corresponding to $s_1+\mathcal{F}^{(1,4)}$ and $s_2+\mathcal{F}^{(2,5)}$

in Figure 5, we attach the linear forms in $s_1+\mathcal{F}^{(1,4)}$ (resp. $s_2+\mathcal{F}^{(2,5)}$) to the arrows in the lace diagram corresponding to $\mathcal{O}^{(1,4)}$ (resp. $\mathcal{O}^{(2,5)}$). Here the linear forms in each column are attached upside down according as the arrow is leftward or rightward. As before, we superpose two diagrams and if two linear forms are attached to the same arrow, those two linear forms are also superposed as in Figure 6.

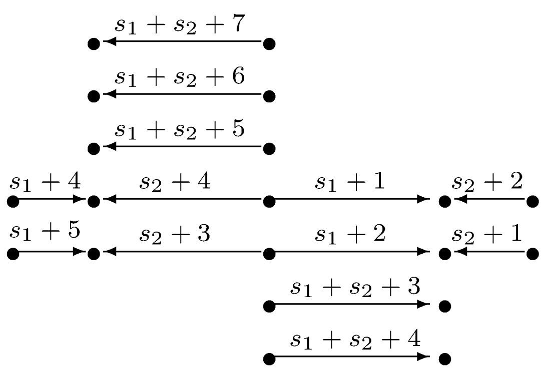

Figure 6. Superposition of the lace diagrams in Example 5.3

The validity of the algorithm for $b_{\underline{m}}(\underline{s})$ can be proved by using the structure theorem of b-functions (cf. [11]), and the localization of b-functions (cf. [13], [12]). The details will appear elsewhere.

Added in proof. The details of the algorithm have been published in the following paper; K. Sugiyama, b-Functions associated with quivers of type A, *Transformation Groups* **16**(2011), 1183–1222.

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