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An exposition of root systems and Lie algebras (affine and elliptic)

By

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Abstract

This is an exposition in order to give an explicit way to understand (1) a non-topological proof for an existence of a base of an affine root system, (2) a Serre-type definition of an elliptic Lie algebra with rank ≥ 2, and (3) the isotropic root multiplicities obtained from a viewpoint of the Saito-marking lines.

§ 1. Introduction

In 1985, K. Saito [16] introduced the notion of an *n-extended affine root system*. If $n = 0$ (respectively, $n = 1$), it is an irreducible finite root system (respectively, an affine root system). In [16], he also intensively studied 2-extended affine root systems, which are now called *elliptic root systems* (see [17]).

Recall that a root system R is called *reduced* if $2\alpha \notin R$ for any $\alpha \in R$. A reduced elliptic root system is called *reduced-marked* if it has a codimension-one quotient root system isomorphic to a reduced affine root system (see also [16, §5 A)]), that is, $g(\Pi) = \{\emptyset\}$ for some g defined in (4.7). Most of the reduced elliptic root systems are reduced-marked (see [1], [2]).

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Until now, various attempts have been made to construct Lie algebras whose non-isotropic roots form extended affine root systems. Among them are *toroidal Lie algebras* [15], *extended affine Lie algebras* [1], and *toral type extended affine Lie algebras* [4], [21]. See [18, Introduction] for the history.

In 2000, K. Saito and D. Yoshii [18] constructed certain Lie algebras by using the Borcherds lattice vertex algebras, called them *simply-laced elliptic Lie algebras* and showed that they are isomorphic to ADE-type (2-variable) toroidal Lie algebras of rank ≥ 2. They also gave two other definitions for their Lie algebras. One uses generators and relations. The other uses (affine-type) Heisenberg Lie algebras; this was generalized by D. Yoshii [20] in order to define Lie algebras associated with the reduced-marked elliptic root systems, which are now called *elliptic Lie algebras*, or, precisely, *reduced-marked elliptic Lie algebras*. In 2004, the second author [19] gave defining relations of the reduced-marked elliptic Lie algebras of rank ≥ 2. Theorem 5.3 in this paper accounts for why those should be called the elliptic Lie algebras.

The aim of this paper is to obtain the following, in a quite explicit way:

(1) A purely algebraic proof for the existence of a base of an affine root system (see Theorem 3.1), the result which is obtained in [13] using a topological argument.

(2) An extension of a result from [19] to that for any reduced elliptic root system R with rank ≥ 2; we define a Lie algebra $\mathfrak{g}$ with generators and finite relations (see Definition 5.1), and show that the non-isotropic roots of $\mathfrak{g}$ constitute R with multiplicity one (see Theorem 5.1). We also show that if a Lie algebra $\mathfrak{t}$ has R as its non-isotropic root system (and satisfies some extra conditions), there exists an epimorphism from $\mathfrak{g}$ to $\mathfrak{t}$ (see Theorem 5.3).

(3) A list of the multiplicities of the isotropic roots of $\mathfrak{g}$ (see Theorem 6.1; this is our own new result, and is obtained from Saito's view-point). To get the list, for a technical reason, the extension (2) is essential.

As for (2), we point out that our defining relations are closely related to defining relations, called *Drinfeld realization*, of the quantum affine algebras due to V.G. Drinfeld [7, Theorems 3 and 4]. Recently the same authors have written a paper [5], motivated by [22], giving a finite presentation of the universal coverings of some Lie tori.

We hope that the material presented here regarding affine root systems, in particular the existence of a base, would give another point of view to readers interested in the subject, specially to those reading the book [14] by I.G. MacDonald. (Incidentally, in order to read [14], we also hope that the paper [8] would also be helpful in being familiar with Coxeter groups, especially the Matsumoto theorem.)

§ 2. Preliminary

In this section, we mention elemental properties of (Saito's) extended affine root systems, especially (2.5).

§ 2.1. Basic notation and terminology

As usual, we let $\mathbb{Z}$ denote the ring of integers, $\mathbb{N}$ the set of positive integers, $\mathbb{R}$ the field of real numbers, and $\mathbb{C}$ the field of complex numbers. For a set S, let $|S|$ denote the cardinal number of S. If S is a subset of $\mathbb{R}$, we let $S^\times := \{s \in S | s \neq 0\}$, $S_+ := \{s \in S | s \geq 0\}$, and $S_- := \{s \in S | s \leq 0\}$.

For a unital subring X of $\mathbb{C}$, an X-module M, a subset Y of X, subsets S and S' of M, $x \in X$ and $m \in M$, we let $S + S' := \{m + m' \in M | m \in S, m' \in S'\}$, $m + S := \{m\} + S$, $YS := \{y_1 s_1 + \cdots + y_r s_r | r \in \mathbb{N}, y_i \in Y, s_i \in S\,(1 \leq i \leq r)\}$, $Ym := Y\{m\}$, $xS := \{x\}S$ and $-S := (-1)S$; we understand $S + \emptyset = \emptyset$, $\emptyset S = \emptyset$ and $Y\emptyset = \emptyset$.

Throughout this paper, for any $\mathbb{F}$-linear space $\mathcal{V}$ with a symmetric bilinear form $(\,,\,) : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, we set $\mathcal{V}^0 := \{v \in \mathcal{V} | (v, v) = 0\}$ and $\mathcal{V}^\times := \mathcal{V} \setminus \mathcal{V}^0$; for each $v \in \mathcal{V}^\times$, we set $v^\vee := \frac{2v}{(v,v)}$ and define $s_v \in \mathrm{GL}(\mathcal{V})$ by $s_v(z) = z - (v^\vee, z)v$ ($z \in \mathcal{V}$); for any non-empty subset S of $\mathcal{V}^\times$, we denote by W_S the subgroup of $\mathrm{GL}(\mathcal{V})$ generated by $\{s_v | v \in S\}$, i.e.,

$$W_S := \langle s_v | v \in S \rangle, \tag{2.1}$$

and moreover, let $W_S \cdot S' := \{w(z') \in \mathcal{V} | w \in W_S, z' \in S'\}$, $W_S \cdot z := W_S \cdot \{z\}$ for a subset S' of $\mathcal{V}$ and $z \in \mathcal{V}$, and say that a subset S of $\mathcal{V}^\times$ is *connected* if there exists no non-empty proper subset S' of S with $(S', S \setminus S') = \{0\}$. For a subset $\mathcal{V}'$ of $\mathcal{V}$, let $(\mathcal{V}')^0 := \mathcal{V}' \cap \mathcal{V}^0$, and $(\mathcal{V}')^\times := \mathcal{V}' \cap \mathcal{V}^\times$. We call an element of $\mathcal{V}^0$ *isotropic*.

In this paper, if $\mathcal{V}^0$ is a subspace of $\mathcal{V}$, we always let

$$\pi : \mathcal{V} \to \mathcal{V}/\mathcal{V}^0 \tag{2.2}$$

denote the canonical map.

§ 2.2. Extended affine root systems

Definition 2.1. Let $l \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. Let $\mathcal{V}$ be an $(l+n)$-dimensional $\mathbb{R}$-linear space. Recall $\mathcal{V}^0$ and $\mathcal{V}^\times$ from Subsection 2.1. Assume that there exists a positive semi-definite symmetric bilinear form $(\,,\,) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ such that $\dim_{\mathbb{R}} \mathcal{V}^0 = n$. Let R be a subset of $\mathcal{V}$. Then R (or more precisely, $(R, \mathcal{V})$) is an (n-)*extended affine root system* if R satisfies the following axioms:

(AX1) $R \subset \mathcal{V}^\times$, $\mathcal{V} = \mathbb{R}R$.

(AX2) $\mathbb{Z}R$ is free as a $\mathbb{Z}$-module, and $\mathrm{rank}_{\mathbb{Z}}\mathbb{Z}R = n + l(= \dim_{\mathbb{R}} \mathcal{V})$.
(AX3) $(\alpha^\vee, \beta) \in \mathbb{Z}$ for α, $\beta \in R$.
(AX4) $s_\alpha(R) = R$ for all $\alpha \in R$.
(AX5) R is connected.

(see [16, (1.2) Definition 1 and (1.3) Note 2 iii)] and see [2] for an equivalence to [1, Definition 2.1].) Let $W = W_R$ (see (2.1)).

Let R be as in Definition 2.1. It is well-known that for all $\alpha \in R$,

$$\begin{cases} R \cap \mathbb{R}\alpha = \{\alpha, -\alpha\},\ \{\alpha, 2\alpha, -\alpha, -2\alpha\} \text{ or } \{\alpha, \frac{1}{2}\alpha, -\alpha, -\frac{1}{2}\alpha\}, \\ (\text{so } -R = R). \end{cases} \tag{2.3}$$

We call R *reduced* (resp. *non-reduced*) if $R \cap 2R = \emptyset$ (resp. $R \cap 2R \neq \emptyset$).

We say that two extended affine root systems $(R, \mathcal{V})$ and $(R', \mathcal{V}')$ are *isomorphic* if there exist an $\mathbb{R}$-linear bijective map $f : \mathcal{V} \to \mathcal{V}'$ and $c \in \mathbb{R}$ with $c > 0$ such that $f(R) = R'$ and $(f(v), f(w)) = c(v, w)$ for v, $w \in \mathcal{V}$.

(2.4) We call this f a *root system isomomorphism.*

Let R, l and n be as above.

By [12, Theorem 5 of Chapter XV], since $\mathbb{Z}R/(\mathbb{Z}R)^0$ is torsion free, (AX1-5) imply that there exists an $\mathbb{R}$-basis $\{x_1, \dots, x_{l+n}\}$ of $\mathcal{V}$ such that $\{x_{l+1}, \dots, x_{l+n}\}$ is an $\mathbb{R}$-basis of $\mathcal{V}^0$, $\{x_1, \dots, x_{l+n}\}$ is a $\mathbb{Z}$-basis of the (torsion) free $\mathbb{Z}$-module $\mathbb{Z}R$ and $\{x_{l+1}, \dots, x_{l+n}\}$ is a $\mathbb{Z}$-basis of the (torsion) free $\mathbb{Z}$-module $(\mathbb{Z}R)^0$ (see Subsection 2.1 for notation), that is,

$$\begin{cases} \mathcal{V} = \mathbb{R}R = \bigoplus_{i=1}^{l+n} \mathbb{R}x_i,\ \mathcal{V}^0 = \bigoplus_{j=l+1}^{l+n} \mathbb{R}x_j, \\ \mathbb{Z}R = \bigoplus_{i=1}^{l+n} \mathbb{Z}x_i,\ (\mathbb{Z}R)^0 = \bigoplus_{j=l+1}^{l+n} \mathbb{Z}x_j, \\ \dim_{\mathbb{R}} \mathcal{V} = \mathrm{rank}_{\mathbb{Z}}\mathbb{Z}R = n + l,\ \dim_{\mathbb{R}} \mathcal{V}^0 = \mathrm{rank}_{\mathbb{Z}}(\mathbb{Z}R)^0 = n. \end{cases} \tag{2.5}$$

Let $\{a_1, \dots, a_n\}$ be a $\mathbb{Z}$-basis of $(\mathbb{Z}R)^0$. Then there exist $x_1, \dots, x_l \in \mathbb{Z}R$ such that $\{x_1, \dots, x_l, a_1, \dots, a_n\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}R$ as well as an $\mathbb{R}$-basis of $\mathcal{V} = \mathbb{R}R$ (see above). Let $1 \leq m \leq n$. Let $\pi' : \mathcal{V} \to \mathcal{V}/(\mathbb{R}a_m \oplus \cdots \oplus \mathbb{R}a_n)$ be the canonical map. Note that $\{\pi'(x_1), \dots, \pi'(x_l), \pi'(a_1), \dots, \pi'(a_{m-1})\}$ is an $\mathbb{X}$-basis of $\mathbb{X}\pi'(R)$ for $\mathbb{X} \in \{\mathbb{Z}, \mathbb{R}\}$. In particular, we see that

(2.6)
if $y_1, \dots, y_{l+m-1}$ are elements of $\mathbb{Z}R$ such that
$\{\pi'(y_1), \dots, \pi'(y_{l+m-1})\}$ is a $\mathbb{Z}$-base of the free $\mathbb{Z}$-module $\mathbb{Z}\pi'(R)$,
then $\{y_1, \dots, y_{l+m-1}, a_m, \dots, a_n\}$ is an $\mathbb{X}$-basis of $\mathbb{X}R$ for $\mathbb{X} \in \{\mathbb{Z}, \mathbb{R}\}$.

(2.7) We call l the *rank* of R. We call n the *nullity* of R.

If $n = 0$, then R is an *irreducible finite root system* (see [16, (1.3) Example 1 i)]). If $n = 1$, then R is an *affine root system* (see [16, (1.3) Example 1 ii)]), see also Remark 2.1 below. If $n = 2$, then R is an *elliptic root system* (see [16, (1.3) Example 1 iii)], [17] and [18]).

Remark 2.1. Assume $n = 1$. Here we give a sketch of a proof of an equivalence between affine root systems in the senses of [13], [14, §1.2] and [16] (i.e. our Definition 2.1). Let F and E be as in [14, §1.2]. Let S be a subset of F, and assume S is an irreducible affine root system in the sense of [14, §1.2]. Identify $\mathcal{V}$ with F, that is, we regard $\mathcal{V}$ as an $l+1$-dimensional $\mathbb{R}$-linear space of affine-linear functions $f : E \to \mathbb{R}$. Clearly S satisfies (AX1) and (AX3-5). Let $\lambda \in \mathcal{V}^\times$. Let $\mu \in \mathcal{V}^\times$ be such that $c\mu \in \lambda + \mathcal{V}^0$ for some $c \in \mathbb{R}^\times$. Then $\lambda - c\mu$ is a constant function on E, that is, $(\lambda - c\mu)(E) = \{d_{\lambda - c\mu}\}$ for some $d_{\lambda - c\mu} \in \mathbb{R}$. We have $s_\mu s_\lambda(x) = x - (\lambda^\vee, x)(\lambda - c\mu)$ for $x \in \mathcal{V}$. Further, for $e \in E$, we have $s_\mu s_\lambda \cdot e = e + \frac{2d_{\lambda - c\mu}}{(\lambda,\lambda)} D\lambda$, see [14, §1.1] for $D\lambda$. Then by using an argument similar to [16, (1.16) Assertion 1], we can see that S satisfies (AX2). Let R be as in Definition 2.1. Let T be the subgroup of W generated by $\{s_\alpha s_{\alpha'} \,|\, \alpha, \alpha' \in R, \mathbb{R}^\times \pi(\alpha) = \mathbb{R}^\times \pi(\alpha')\}$. Then T is a normal abelian subgroup, and W/T can be identified with the finite Weyl group $W_{\pi(R)}$ (cf. [16, (1.3) Note 2 ii)]). Then R satisfies (AR 4) of [14, §1.2].

§ 2.3. Base of an irreducible finite or affine root system

Assume that $n \in \{0, 1\}$ (cf. (2.7)). We call a subset Π of R formed by $(l+n)$-linearly independent elements a *base* if

$$R = (R \cap \mathbb{Z}_+\Pi) \cup (R \cap \mathbb{Z}_-\Pi). \tag{2.8}$$

(For $n = 0$, see [9, Theorem 10.1]. For $n = 1$, see Theorem 3.1 in this paper (cf. MacDonald [13, (4.6)] (see also [16, (3.3) i)-iii)]))). If Π is a base of R, then, for $\mathbb{X} \in \{\mathbb{Z}, \mathbb{R}\}$, we have

$$\Pi \text{ is an } \mathbb{X}\text{-basis of } \mathbb{X}R, \text{ that is, } \mathbb{X}R = \bigoplus_{\alpha \in \Pi} \mathbb{X}\alpha. \tag{2.9}$$

Assume that $n = 1$. Let $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_l\}$ be a base of R; we always assume α_0 is such that $\{\pi(\alpha_1), \dots, \pi(\alpha_l)\}$ is a base of $\pi(R)$ (see Theorem 3.1). Let $\delta(\Pi) \in \mathbb{Z}\Pi$ be such that

$$\delta(\Pi) \in \mathbb{N}\Pi \text{ and } \{\delta(\Pi)\} \text{ is a } \mathbb{Z}\text{-basis of } (\mathbb{Z}R)^0, \text{ that is, } \mathbb{Z}\delta(\Pi) = (\mathbb{Z}R)^0. \tag{2.10}$$

$\delta(\Pi)$ is unique by (2.5). By (2.6), for $\mathbb{X} \in \{\mathbb{Z}, \mathbb{R}\}$, we have

$$\{\alpha_1, \dots, \alpha_l, \delta(\Pi)\} \text{ is a } \mathbb{X}\text{-basis of } \mathbb{X}R, \text{ that is, } \mathbb{X}R = (\bigoplus_{i=1}^{n} \mathbb{X}\alpha_i) \oplus \mathbb{X}\delta(\Pi). \tag{2.11}$$

The following lemma is well-known, e.g., see [9, Theorem 10.3, Lemmas 10.4 C,D, §12 Excercises 3].

Lemma 2.1. *Assume that* $n = 0$ *(cf.* (2.7)*). Let* Π *be a base of* R *(cf.* (2.8)*). Then we have the following*:

(1) $W_\Pi = W$ *and* $W \cdot \Pi = R \setminus 2R$. *(see* (2.1) *for* W_Π *and see Definition* 2.1 *for* $W = W_R$*).*

(2) $W \cdot \alpha = \{\beta \in R | (\alpha, \alpha) = (\beta, \beta)\}$ *for each* $\alpha \in R$.

(3) *For each* $\alpha \in R$*, there exists a unique* $\alpha_+ \in W \cdot \alpha$ *such that* $W \cdot \alpha \subset \alpha_+ + \mathbb{Z}_- \Pi$.

(4) *Let* $r = |\{(\alpha, \alpha) | \alpha \in R\}|$. *Then* $1 \leq r \leq 3$. *Moreover, if* $r = 3$*, then* $R \cap 2R = \{\beta \in R \,|\, (\beta, \beta) \geq (\alpha, \alpha)$ *for all* $\alpha \in R\}$.

Proof of (3). Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$. Then α_+ is the element $\sum_{i=1}^{l} m_i \alpha_i \in W \cdot \alpha$ ($m_i \in \mathbb{Z}$) for which $\sum_{i=1}^{l} m_i$ is maximal. Let $w \in W_\Pi$ and let $w = s_{\alpha_1} \cdots s_{\alpha_r}$ be a reduced expression, that is, r is as small as possible. By [9, Corollary 10.2 C], we have $w.\alpha_+ = \alpha_+ - \sum_{j=1}^{r} (\alpha_j^\vee, \alpha_+) s_{\alpha_1} \cdots s_{\alpha_{j-1}}(\alpha_j) \in \alpha_+ + \mathbb{Z}_- \Pi$. □

For R and Π of Lemma 2.1, we let

$$\Theta(R, \Pi) := \{\alpha_+ \in R | \alpha \in R\}. \tag{2.12}$$

By checking directly (and using [9, §12 Table 2]), we have

$$(\mu, \nu) > 0 \text{ for } \mu,\, \nu \in \Theta(R, \Pi). \tag{2.13}$$

(The fact (2.13) can also be proved as follows. Let $\gamma_i \in \mathcal{V}$ ($1 \leq i \leq l$) be such that $(\gamma_i, \alpha_j) = \delta_{ij}$. Then $\mu = \sum_{i=1}^{l} x_i \gamma_i$ with $x_i \in \mathbb{R}_{\geq 0}$, and $x_j > 0$ for some j. Write $\nu = \sum_{i=1}^{l} y_i \alpha_i$ with $y_i \in \mathbb{Z}_+$ ($1 \leq i \leq l$). If $y_i = 0$ for some i, there exist i_1, $i_2 \in \{1, \ldots, l\}$ with $i_1 \neq i_2$, $y_{i_1} = 0$, $y_{i_2} > 0$ and $(\alpha_{i_1}, \alpha_{i_2}) < 0$, so $(\alpha_{i_1}, \nu) < 0$ which implies that $s_{\alpha_{i_1}}(\nu) = \nu - (\alpha_{i_1}^\vee, \nu)\alpha_{i_1} \notin \nu + \mathbb{Z}_- \Pi$, contradiction. Hence $y_i > 0$ for all $1 \leq i \leq l$. Hence $(\mu, \nu) \geq x_j y_j > 0$.)

§ 2.4. Notation $S_{\rm sh}$, $S_{\rm lg}$, $S_{\rm ex}$

Let R be an (n-)extended affine root system (see Definition 2.1). Define the subsets $R_{\rm sh}$, $R_{\rm lg}$ and $R_{\rm ex}$ of R by

$$R_{\rm sh} := \{\alpha \in R \,|\, (\alpha, \alpha) \leq (\beta, \beta) \text{ for all } \beta \in R\}, \tag{2.14}$$

$R_{\rm ex} := R \cap \pi^{-1}(2\pi(R_{\rm sh}))$ and $R_{\rm lg} := R \setminus (R_{\rm sh} \cup R_{\rm ex})$ (see (2.2) for π). Then we have

$$R = R_{\rm sh} \cup R_{\rm lg} \cup R_{\rm ex} \text{ (disjoint union)}. \tag{2.15}$$

For a subset S of R, let

$$S_{\mathrm{sh}} := S \cap R_{\mathrm{sh}},\ S_{\mathrm{lg}} := S \cap R_{\mathrm{lg}},\ S_{\mathrm{ex}} := S \cap R_{\mathrm{ex}}. \tag{2.16}$$

§ 3. A non-topological proof for the existence of a base of an affine root system

In this section we assume R is an affine root system, that is, we assume $n = 1$ (see (2.7)).

§ 3.1. The existence of a base of an affine root system

The following theorem seems to be well-known (see [13]), but we state and prove it for later use. The proof in [13] uses topological terminology. Our proof seems to be the first one without using topology. Besides we need a technically written statement of the following theorem for application.

Theorem 3.1. (*cf.* [13]) *Let $\delta' \in \mathcal{V}^0 \setminus \{0\}$ be such that $\mathbb{Z}\delta' = (\mathbb{Z}R)^0$ (cf. (2.5)). Let $\Pi' = \{\alpha_1, \dots, \alpha_l\}$ be a subset of R with $|\Pi'| = l$ such that $\pi(\Pi')$ is a base of the irreducible finite root system $(\pi(R), \mathcal{V}/\mathbb{R}\delta')$ (cf. (2.8) and (2.2)). (So $\mathbb{Z}R = \mathbb{Z}\delta' \oplus \mathbb{Z}\Pi'$ (cf. (2.6)).) Then there exists a unique*

$$\alpha_0 = \alpha_0(R, \Pi', \delta') \in R \tag{3.1}$$

such that $\{\alpha_0\} \cup \Pi'$ is a base of R and $\alpha_0 \in \mathbb{N}\delta' \oplus \mathbb{Z}\Pi'$. Moreover $\alpha_0 = \delta' - \theta$ for some $\theta \in \mathbb{N}\Pi'$ with $\pi(\theta) \in \Theta(\pi(R), \pi(\Pi'))$ (see (2.12)). In particular, $[(\alpha_i^\vee, \alpha_j)]_{0 \le i,j \le l}$ is a generalized Cartan matrix of affine-type in the sense of [10, §4.3 and Proposition 4.7]. Further, letting $\Pi_1 = \{\alpha_0\} \cup \Pi'$, for any base Π_2 of R we have $\Pi_2 = \epsilon w(\Pi_1)$ for some $\epsilon \in \{1, -1\}$ and $w \in W_{\Pi_1}$. In particular,

$$R \setminus 2R = W_{\Pi_1} \cdot \Pi_1 \text{ and } W = W_{\Pi_1}. \tag{3.2}$$

Proof. (*Strategy.* We use a linear map $f : \mathcal{V} \to \mathbb{R}$ (i.e., $f \in \mathcal{V}^*$) such that $f(\alpha_i) = 1$ $(1 \le i \le l)$ and $f(\delta')$ is sufficiently large (see (3.6)). Let Π^f be the subset of R formed by the elements $\beta \in R$ satisfying the condition that $f(\beta) > 0$ and β is not expressed as the summation of more than one elements β' of R with $f(\beta') > 0$ (see (3.8)). We show that Π^f is a base of R satisfying the properties of the statement. It is easy to see that $\Pi' \subset \Pi^f$ and $R = (R \cap \mathbb{Z}_+\Pi^f) \cup (R \cap \mathbb{Z}_-\Pi^f))$. We show $|\Pi^f| = l + 1$ by using (2.13).)

We proceed with the proof of the theorem in the following steps.

Step 1 (*Definition of f*). Notice that for $\mathbb{X} \in \{\mathbb{Z}, \mathbb{R}\}$,

$$\mathbb{X}R = \mathbb{X}\delta' \oplus (\oplus_{i=1}^{l} \mathbb{X}\alpha_i) \tag{3.3}$$

(see (2.6)). We may assume that $(\alpha_i, \alpha_i) \leq (\alpha_{i+1}, \alpha_{i+1})$ for $1 \leq i \leq l-1$. Also since $\pi(\Pi')$ is a base of $\pi(R)$, if $l \geq 2$, we may assume α_1 is such that there exists a unique $j \in \{2, \ldots, l\}$ such that $(\alpha_1, \alpha_j) \neq 0$. Let

$$R' := \begin{cases} W_{\Pi'} \cdot (\Pi' \cup \{2\alpha_1\}) & \text{if } l = 1, \\ W_{\Pi'} \cdot (\Pi' \cup \{2\alpha_1\}) & \text{if } l \geq 2 \text{ and } 2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2), \\ W_{\Pi'} \cdot \Pi' & \text{otherwise.} \end{cases} \tag{3.4}$$

Using [9, Theorem 10.3 (c) (and §12 Exercise 3)], we can see that $W_{\Pi'} \cdot \Pi'$ and R' are irreducible finite root systems with the base Π'. If $\pi(R)$ is reduced, then $\pi(R) = \pi(W_{\Pi'} \cdot \Pi')$. If $\pi(R)$ is not reduced, then $\pi(R) = \pi(R')$. In particular, we have

$$R \subset R' + \mathbb{Z}\delta'. \tag{3.5}$$

(see also (3.3)).

Define $f \in \mathcal{V}^*$ by

$$f(\alpha_i) = 1 \ (1 \leq i \leq l) \quad \text{and} \quad f(\delta') = 3M, \tag{3.6}$$

where $M := \max\{|f(\gamma)| | \gamma \in R'\}$ (notice $|R'| < \infty$). It follows from (3.5) that $f(\beta) \neq 0$ for $\beta \in R$.

Step 2 (*Definition of* Π^f). Let $R^{f,+} := \{\beta \in R | f(\beta) > 0\}$. By (3.6), we have

$$R^{f,+} = R \cap ((R' \cap \mathbb{Z}_+\Pi') \cup (\cup_{m=1}^{\infty}(m\delta' + R'))). \tag{3.7}$$

Let Π^f be a subset of R formed by the elements $\beta \in R^{f,+}$ satisfying the condition that there exist no $\beta_1, \ldots, \beta_r \in R^{f,+}$ with $r \geq 2$ such that $\beta = \beta_1 + \cdots + \beta_r$; namely,

$$\Pi^f := R^{f,+} \setminus (\bigcup_{r=2}^{\infty}\{\sum_{i=1}^{r}\beta_i | \beta_i \in R^{f,+}\}). \tag{3.8}$$

By (3.7), we have

$$\Pi' \subset \Pi^f. \tag{3.9}$$

Notice $\mathbb{Z}\Pi' \neq \mathbb{Z}R$ (by (3.3)). Then we have

$$\mathbb{Z}\Pi^f = \mathbb{Z}R, \ R = (R \cap \mathbb{Z}_+\Pi^f) \cup (R \cap \mathbb{Z}_-\Pi^f) \text{ and } |\Pi^f| \geq |\Pi'| + 1. \tag{3.10}$$

(As mentioned in our strategy, we show that Π^f is a base of R.)

Step 3 (*If* $\beta \in \Pi^f/\Pi'$, *then we have* $\pi(\beta) \in \Theta(\pi(R), \pi(\Pi'))$ (*for* $\Theta(\pi(R), \pi(\Pi'))$, *see* (2.12))). Let $\beta \in \Pi^f/\Pi'$ (see also (3.9)-(3.10)). We show that β is expressed as

$$\beta = m\delta' - \theta \tag{3.11}$$

for some $m \in \mathbb{N}$ and some θ with

$$\theta \in \Theta(R', \Pi') \tag{3.12}$$

(see (2.12) for $\Theta(R', \Pi')$). By (3.7), since $\Pi^f \subset R^{f,+}$, we have

$$\beta = m\delta' + \mu \tag{3.13}$$

for some $m \in \mathbb{N}$ and $\mu \in R'$. Let $\theta \in \Theta(R', \Pi') \cap W_{\Pi'} \cdot \mu$, where we recall from Lemma 2.1 (2)-(3) that $|\Theta(R', \Pi') \cap W_{\Pi'} \cdot \mu| = 1$. Notice $\{\mu, -\mu, \theta, -\theta\} \subset W_{\Pi'} \cdot \mu$ (cf. Lemma 2.1 (2)). Then $m\delta' - \theta \in R$ since $m\delta' - \theta \in m\delta' + W_{\Pi'} \cdot \mu = W_{\Pi'} \cdot (m\delta' + \mu) = W_{\Pi'} \cdot \beta \subset R$. By Lemma 2.1 (3), we have $\theta + \mu = \theta - (-\mu) \in \mathbb{Z}_+\Pi'$. Since $m\delta' - \theta \in R^{f,+}$ (cf. (3.7)), $\beta = (m\delta' - \theta) + (\theta + \mu)$ and $\beta \in \Pi^f$, we have $\theta + \mu = 0$ and (3.11), as desired.

Step 4 ($|\Pi^f| = l + 1$). We show

$$|\Pi^f \setminus \Pi'| = 1, \text{ i.e., } |\Pi^f| = l + 1 \tag{3.14}$$

(see also (3.9)-(3.10)).

Assume $|\Pi^f \setminus \Pi'| > 1$. Let β_1, $\beta_2 \in \Pi^f \setminus \Pi'$ and assume $\beta_1 \neq \beta_2$. Assume $(\beta_1, \beta_1) \leq (\beta_2, \beta_2)$. Then, by (2.13) and (3.11)-(3.12), we see that

$$(\beta_2^\vee, \beta_1) = \begin{cases} 1 \text{ if } \pi(\beta_1) \neq \pi(\beta_2), \\ 2 \text{ if } \pi(\beta_1) = \pi(\beta_2). \end{cases} \tag{3.15}$$

Assume $(\beta_2^\vee, \beta_1) = 1$. Then, since $\pm(\beta_1 - \beta_2) = s_{\beta_2}(\pm\beta_1) \in R$, we have $\beta_1 - \beta_2 \in R^{f,+}$ or $\beta_2 - \beta_1 \in R^{f,+}$. This contradicts the fact β_1, $\beta_2 \in \Pi^f$ since $\beta_1 = \beta_2 + (\beta_1 - \beta_2)$ and $\beta_2 = \beta_1 + (\beta_2 - \beta_1)$. Assume $(\beta_2^\vee, \beta_1) = 2$, so $\pi(\beta_1) = \pi(\beta_2)$. By (3.11), there exist n_1, $n_2 \in \mathbb{N}$ and $\theta \in \Theta(R', \Pi')$ such that

$$\beta_i = n_i\delta' - \theta \quad (i \in \{1, 2\}) \tag{3.16}$$

(so $\beta_2 - \beta_1 = (n_2 - n_1)\delta'$). Assume $n_1 < n_2$. Notice that for $i \in \{1, 2\}$ and $r \in \mathbb{Z}$,

$$\begin{aligned} R \ni\, & (s_{\beta_2} s_{\beta_1})^r(\beta_i) \quad \text{(by (AX4))} \\ & = (n_i + 2r(n_2 - n_1))\delta' - \theta \\ & = \begin{cases} (n_2 + (2r - 1)(n_2 - n_1))\delta' - \theta & \text{if } i = 1, \\ (n_2 + 2r(n_2 - n_1))\delta' - \theta & \text{if } i = 2. \end{cases} \end{aligned} \tag{3.17}$$

Hence

$$(n_2 + r(n_2 - n_1))\delta' - \theta \in R \quad \text{for all } r \in \mathbb{Z}. \tag{3.18}$$

Let $n_3 \in \mathbb{Z}_+$ and $t \in \mathbb{N}$ be such that $0 \leq n_3 < n_2 - n_1$ and $n_2 = t(n_2 - n_1) + n_3$. Assume $n_3 = 0$. By (3.18), $\{-\theta, (n_2 - n_1)\delta' - \theta\} \subset R$. Hence, by (3.7) (and (2.3)),

$\{\theta, (n_2 - n_1)\delta' - \theta\} \subset R^{f,+}$. Notice $t \geq 2$ (since $0 < n_1 < n_2$ and $n_3 = 0$). Since $\beta_2 = t((n_2 - n_1)\delta' - \theta) + (t-1)\theta$, we have $\beta_2 \notin \Pi^f$, contradiction. Assume $n_3 > 0$. Notice $2n_3 < n_2$ (since $2n_3 < (n_2 - n_1) + n_3 \leq t(n_2 - n_1) + n_3 = n_2$). Let $\beta_3 = n_3\delta' - \theta$. By (3.18), $\beta_3 \in R$. By (3.7), $\beta_3 \in R^{f,+}$. Notice $\beta_2 - 2\beta_3 = s_{\beta_3}(\beta_2) \in R$ (by (AX4)). Then by (3.7), we have

$$\beta_2 - 2\beta_3 = (n_2 - 2n_3)\delta' + \theta \in R^{f,+}. \tag{3.19}$$

Since $\beta_2 = (\beta_2 - 2\beta_3) + 2\beta_3$, we have $\beta_2 \notin \Pi^f$, contradiction. Hence $|\Pi^f| = l + 1$, as desired.

Step 5 (Π^f *is a base with* $\alpha_0 = \delta' - \theta$). Let α_0 be $\beta = m\delta' - \theta$ of (3.11). Then $\Pi^f = \Pi' \cup \{\alpha_0\}$, where we notice (3.9) and (3.14). It is clear that the elements of Π^f are linearly independent (cf. (3.3)). Hence, by (3.10), Π^f is a base of R (cf. (2.8)). Since $\mathbb{Z}\Pi' \oplus \mathbb{Z}\delta' = \mathbb{Z}\Pi' \oplus \mathbb{Z}\alpha_0$ (by (3.3) and (3.10)), we have $m = 1$.

Step 6 (*The last claim holds*). Let $\Pi_1 = \Pi' \cup \{\alpha_0\}$. Let Π_2 be a base of R. Define $h \in V^*$ by $h(\beta) := 1$ ($\beta \in \Pi_2$). Then $h(R) \subset \mathbb{Z} \setminus \{0\}$. By the same formula as in (3.17), we have $|\{(s_\theta s_{\alpha_0})^r(\alpha_0) \in R | r \in \mathbb{Z}\}| = \infty$ (notice that $(s_\theta s_{\alpha_0})^r(\alpha_0) \in R$ (by (AX4)) since $s_\theta = s_{\frac{1}{2}\theta}$ and $\theta \in R \cup 2R$ (see (3.12) and (3.4))). Hence $|R| = \infty$, which implies $|h(R)| = \infty$. Hence, by (3.5), since $|R'| < \infty$ (R' is an irreducible finite root system), we have $h(\delta') \neq 0$. We may assume

$$h(\delta') > 0 \tag{3.20}$$

(otherwise, we replace Π_2 with $-\Pi_2$). Let

$$\begin{aligned} m(\Pi_1, \Pi_2) &:= |(R \cap \mathbb{Z}_+\Pi_1 \cap \mathbb{Z}_-\Pi_2) \setminus 2R| \\ &= |\{\beta \in (R \cap \mathbb{Z}_+\Pi_1) \setminus 2R \,|\, h(\beta) < 0\}|. \end{aligned}$$

Since $\alpha_0 = \delta' - \theta$, we have $R \cap \mathbb{Z}_+\Pi_1 \subset R' + \mathbb{Z}_+\delta'$ (cf. (3.5)). Hence, since $|R'| < \infty$, by (3.20), we have $m(\Pi_1, \Pi_2) < \infty$.

We use induction on $m(\Pi_1, \Pi_2)$; if $m(\Pi_1, \Pi_2) = 0$, then, by (2.8), $R \cap \mathbb{Z}_+\Pi_1 = R \cap \mathbb{Z}_+\Pi_2$, so $\Pi_1 = \Pi_2$. Assume $m(\Pi_1, \Pi_2) > 0$. Then there exists $\alpha \in \Pi_1$ such that $\alpha \in \mathbb{Z}_-\Pi_2$ (notice that $R \subset \mathbb{Z}_-\Pi_2 \cup \mathbb{Z}_+\Pi_2$). By (2.8) (and (2.3)), we see

$$s_\alpha((R \cap \mathbb{Z}_+\Pi_1) \setminus 2R) = \{-\alpha\} \cup (((R \cap \mathbb{Z}_+\Pi_1) \setminus 2R) \setminus \{\alpha\}). \tag{3.21}$$

Then we have

$$\begin{aligned} &m(\Pi_1, s_\alpha(\Pi_2)) \\ &= |(R \cap \mathbb{Z}_+\Pi_1 \cap \mathbb{Z}_- s_\alpha(\Pi_2)) \setminus 2R| \\ &= |s_\alpha((R \cap \mathbb{Z}_+\Pi_1 \cap \mathbb{Z}_- s_\alpha(\Pi_2)) \setminus 2R)| \\ &= |(s_\alpha(R \cap \mathbb{Z}_+\Pi_1) \cap \mathbb{Z}_-\Pi_2) \setminus 2R| \\ &= m(\Pi_1, \Pi_2) - 1 \quad \text{(by (3.21) since } s_\alpha(\alpha) = -\alpha \notin \mathbb{Z}_-\Pi_2). \end{aligned}$$

Then, by the induction, we see that there exists $w \in W_{\Pi_1}$ such that $w(\Pi_2) = \Pi_1$, as desired.

Note that for any $\beta \in R \setminus 2R$, there exists a subset Π'' of R with $|\Pi''| = l$ such that $\beta \in \Pi''$ and $\pi(\Pi'')$ is a base of $\pi(R)$. Hence by the above argument, we have (3.2). This completes the proof. □

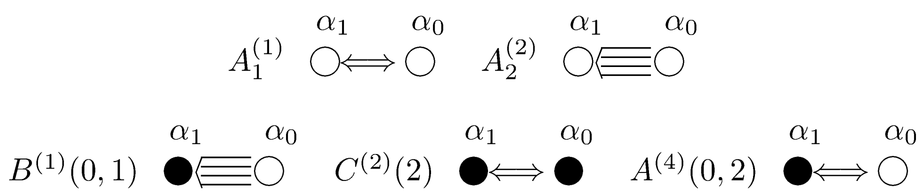
By (3.2), we have

$$
\begin{cases} R = W_\Pi \cdot (\Pi \cup (2\Pi \cap R)), \\ (\mathbb{Z}R)^\times \setminus R \\ \quad = W_\Pi \cdot \Big((2\Pi \setminus R) \cup (\bigcup_{r \in 3+\mathbb{Z}_+} r\Pi) \cup ((\mathbb{Z}R)^\times \setminus (\mathbb{Z}_+\Pi \cup \mathbb{Z}_-\Pi))\Big). \end{cases} \tag{3.22}
$$

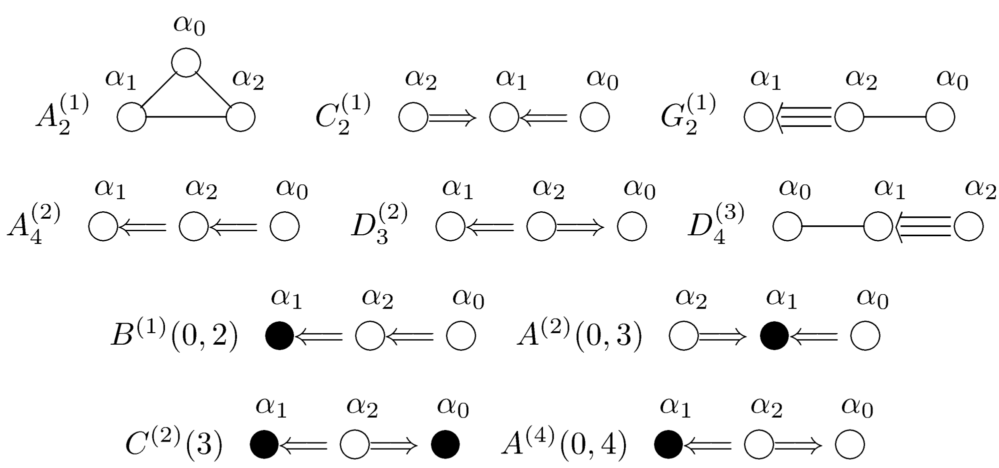
§ 3.2. Dynkin diagrams of affine root systems

Here we give the Dynkin diagrams for (R, Π) of Theorem 3.1. We assume that if $2\alpha_0 \in R$, then $2\alpha_i \in R$ for some $i \neq 0$, see $A^{(4)}(0, 2l)$ below. We describe them in the same manner as in [11, Table 1-4]; especially, if $2\alpha_i \notin R$ (resp. $2\alpha_i \in R$), then the i-th dot is white (resp. black). The names of them are also the same as in [11, Table 1-4].

(i) The case of $l = 1$:

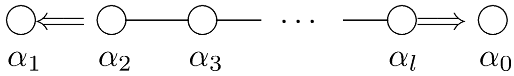


(ii) The case of $l = 2$:

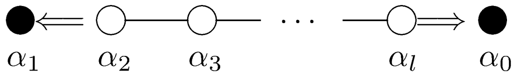


(iii) The case of $l \geq 3$:

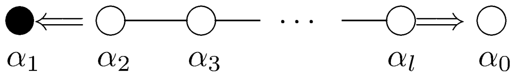
$D^{(2)}_{l+1}$



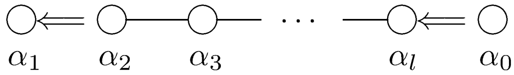
$C^{(2)}(l+1)$



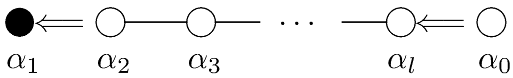
$A^{(4)}(0,2l)$



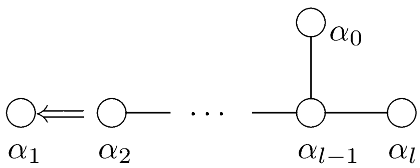
$A^{(2)}_{2l}$



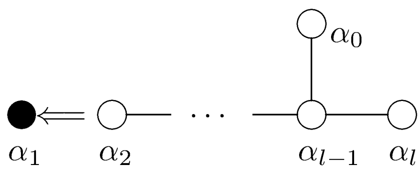
$B^{(1)}(0,l)$



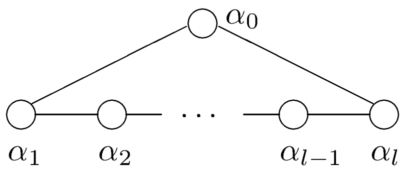
$B^{(1)}_l$



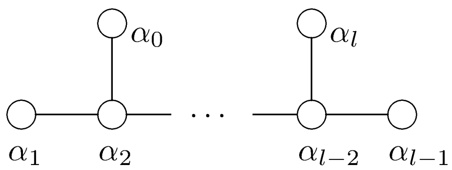
$A^{(2)}(0,2l-1)$



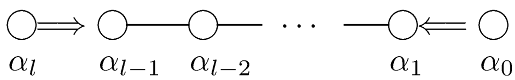
$A^{(1)}_l$



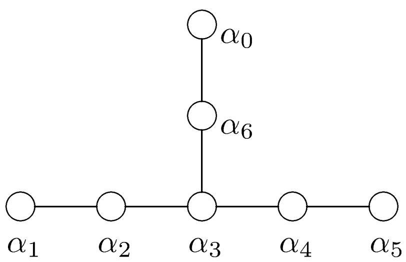
$D^{(1)}_l$



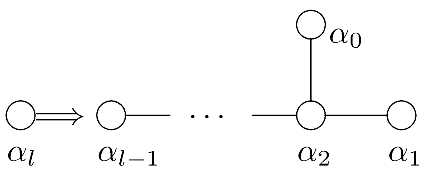
$C^{(1)}_l$



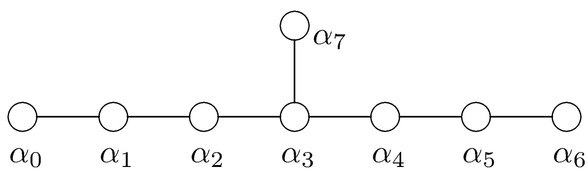
$E^{(1)}_6$

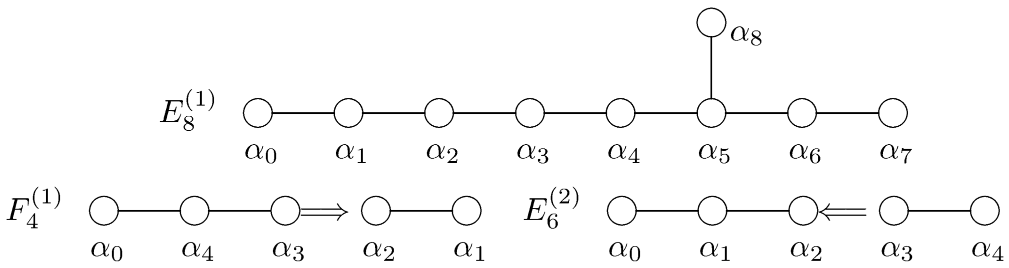


$A^{(2)}_{2l-1}$



$E^{(1)}_7$





§ 4. Elliptic root systems

In this section we assume R is a reduced elliptic root system, that is, $R \cap 2R = \emptyset$ and $n = 2$ (see (2.7)).

§ 4.1. Fundamental-set of an elliptic root system

Definition 4.1. (*Fundamental-set* $\Pi \cup \{a\}$) We say that a subset $\Pi \cup \{a\}$ of $\mathbb{Z}R$ is a *fundamental-set* of R if it satisfies the axioms (FS1)-(FS2) below; we always let

$$\pi_a : \mathcal{V} \to \mathcal{V}/\mathbb{R}a \tag{4.1}$$

denote the canonical map.

(FS1) $a \in (\mathbb{Z}R)^0$ and there exists $b \in (\mathbb{Z}R)^0$ such that $\{a, b\}$ is a basis of $(\mathbb{Z}R)^0$, i.e., $(\mathbb{Z}R)^0 = \mathbb{Z}a \oplus \mathbb{Z}b$.

(FS2) $|\Pi| = l + 1$, $\Pi \subset R$ and $\pi_a(\Pi)$ is a base of the affine root system $\pi_a(R)$.

Until end of this section, let $\Pi \cup \{a\} = \{\alpha_0, \ldots, \alpha_l\} \cup \{a\}$ denote a fundamental-set of R. We assume $\pi(\{\alpha_1, \ldots, \alpha_l\})$ is a base of $\pi(R)$.

Let $\delta(\Pi) \in \mathbb{Z}\Pi$ be such that

$$\delta(\Pi) \in \mathbb{N}\Pi \quad \text{and} \quad \mathbb{Z}\delta(\Pi) = (\mathbb{Z}\Pi)^0. \tag{4.2}$$

Then $\pi_a(\delta(\Pi)) = \delta(\pi_a(\Pi))$ (see (2.10) for $\delta(\pi_a(\Pi))$).

Let $\delta = \delta(\Pi)$ be as in (4.2). By (2.6), (2.11) and (2.8), for $\mathbb{X} \in \{\mathbb{Z}, \mathbb{R}\}$, we have

$$\begin{cases} \mathbb{X}R = \bigoplus_{\lambda \in \Pi \cup \{a\}} \mathbb{X}\lambda = (\bigoplus_{\alpha \in \Pi \setminus \{\alpha_0\}} \mathbb{X}\alpha) \bigoplus \mathbb{X}\delta \bigoplus \mathbb{X}a, \\ (\mathbb{X}R)^0 = \mathbb{X}\delta \oplus \mathbb{X}a, \\ R \subset (\mathbb{X}_+\Pi \oplus \mathbb{X}a) \cup (\mathbb{X}_-\Pi \oplus \mathbb{X}a). \end{cases} \tag{4.3}$$

§ 4.2. Maps k and g

Lemma 4.1. (1) *For any $\alpha \in R$, we have*

$$(\alpha + (\mathbb{Z} \setminus \{0\})a) \cap R \neq \emptyset. \tag{4.4}$$

(2) *Let S be a non-empty proper connected subset of* Π. *Let* $\mathcal{V}^S := \mathbb{R}S \oplus \mathbb{R}a$ *and* $R^S := R \cap \mathcal{V}^S$. *Then* $(R^S, \mathcal{V}^S)$ *is a reduced affine root system (we have assumed R is reduced), and* $(\pi_a(R^S), \mathcal{V}/\mathbb{R}a)$ *is an irreducible finite root system with the base* $\pi_a(S)$. *In particular,* $\mathbb{Z}R^S = \mathbb{Z}S \oplus \mathbb{Z}k_S a$ *for some* $k_S \in \mathbb{N}$.

Proof. (1) By (4.3), R cannot be included in $\mathbb{Z}\Pi$. Hence there exist $\mu \in R$ and $m \in \mathbb{Z} \setminus \{0\}$ such that $\mu \in ma + \mathbb{Z}\Pi$. Since $\pi_a(R)$ is an affine root system and $\pi_a(\Pi)$ is a base of $\pi_a(R)$, by the first equality of (3.22), there exist $\gamma \in \Pi$, $c \in \{1,2\}$ and $w \in W_\Pi$ such that $w(\mu) = c\gamma + ma$. Notice that

$$R \ni s_\gamma s_{c\gamma+ma}(\gamma) = s_\gamma(\gamma - (c^{-1}2)(c\gamma + ma)) = \gamma - 2c^{-1}ma. \tag{4.5}$$

(Hence (4.4) holds for this special γ.) Let $\lambda = \gamma - 2c^{-1}ma$. For $\beta \in R$, we have

$$R \ni s_\gamma s_\lambda(\beta) = s_\gamma(\beta - (\gamma^\vee, \beta)\lambda) = \beta + (\gamma^\vee, \beta) \cdot 2c^{-1}ma. \tag{4.6}$$

By (AX5) and (4.3), by repetition of equations similar to (4.6), we see that (4.4) holds for any $\alpha \in R$.

(2) This follows from (1) and (4.3). $\square$

By Lemma 4.1 (2), for each $\alpha \in \Pi$, $R^{\{\alpha\}}$ is a rank-one reduced affine root system and $\{\pi_a(\alpha)\}$ is a base of a rank-one irreducible finite root system $\pi_a(R^{\{\alpha\}})$. By Theorem 3.1, we can define maps

$$k : \Pi \to \mathbb{N} \text{ and } g : \Pi \to \{\emptyset, 2\mathbb{Z}+1\} \tag{4.7}$$

by

$$R \cap (\mathbb{R}\alpha \oplus \mathbb{R}a) = \bigcup_{\varepsilon \in \{1,-1\}} ((\varepsilon\alpha + \mathbb{Z}k(\alpha)a) \cup (2\varepsilon\alpha + g(\alpha)k(\alpha)a)) \tag{4.8}$$

$(\alpha \in \Pi)$ (see also (4.3)).

Since $\pi_a(R) \setminus 2\pi_a(R) = W_{\pi_a(\Pi)} \cdot \pi_a(\Pi)$ (see Theorem 3.1), we have

$$R = \bigcup_{w \in W_\Pi} (\bigcup_{\alpha \in \Pi} ((w(\alpha) + \mathbb{Z}k(\alpha)a) \cup (w(2\alpha) + g(\alpha)k(\alpha)a))). \tag{4.9}$$

Since R is determined by Π, k and g,

(4.10) we also denote R by $R(\Pi, k, g)$.

Let $\alpha \in \Pi$. Let $\alpha^* := -\alpha_0(R^{\{\alpha\}}, \{\alpha\}, -k(\alpha)a)$. Then $\alpha^* = c(\alpha)\alpha + k(\alpha)a$, where

$$c(\alpha) = \begin{cases} 1 & \text{if } g(\alpha) = \emptyset, \\ 2 & \text{if } g(\alpha) = 2\mathbb{Z}+1. \end{cases} \tag{4.11}$$

Let $\mathcal{B}_+ := \{\alpha, \alpha^* | \alpha \in \Pi\}$. Then $|\mathcal{B}_+| = 2|\Pi| = 2(l+1)$. By Thereom 3.1, we have

$$R = W_{\mathcal{B}_+} \cdot \mathcal{B}_+ \text{ and } W = W_{\mathcal{B}_+} \tag{4.12}$$

(We have assumed that R is reduced).

Assume $l \geq 2$ (see (2.7)). Let α, $\beta \in \Pi$ be such that $(\beta^\vee, \alpha) = -1$. Let $\gamma = \alpha_0(R^{\{\alpha,\beta\}}, \{\alpha, \beta\}, -k(\alpha)a)$. By Lemma 4.1 (2) and Theorem 3.1, we have $g(\beta) = \emptyset$, $k_{\{\alpha,\beta\}} = k(\alpha)$ and see that $((\beta^\vee, \alpha), k(\beta)/k(\alpha), g(\alpha))$ for the rank-two reduced affine root system $R^{\{\alpha,\beta\}}$ with a base $\{\alpha, \beta, \gamma\}$ is one of the following.

$$\begin{cases} (-1,1,\emptyset) & \text{so } R^{\{\alpha,\beta\}} \text{ is } A_2^{(1)}, \text{ and } \gamma = -s_\alpha(\beta^*), \\ (-2,1,\emptyset) & \text{so } R^{\{\alpha,\beta\}} \text{ is } B_2^{(1)}, \text{ and } \gamma = -s_\alpha(\beta^*), \\ (-3,1,\emptyset) & \text{so } R^{\{\alpha,\beta\}} \text{ is } G_2^{(1)}, \text{ and } \gamma = -s_\beta s_\alpha(\beta^*), \\ (-2,2,\emptyset) & \text{so } R^{\{\alpha,\beta\}} \text{ is } D_3^{(2)}, \text{ and } \gamma = -s_\beta(\alpha^*), \\ (-3,3,\emptyset) & \text{so } R^{\{\alpha,\beta\}} \text{ is } D_4^{(3)}, \text{ and } \gamma = -s_\alpha s_\beta(\alpha^*), \\ (-2,1,2\mathbb{Z}+1) & \text{so } R^{\{\alpha,\beta\}} \text{ is } A_4^{(2)}, \text{ and } \gamma = -s_\beta(\alpha^*). \end{cases} \tag{4.13}$$

§ 4.3. List of (Π, k, g)

Theorem 4.1. *Let $R = R(\Pi, k, g)$ be as in (4.10).*

(1) *Assume $l = 1$. Let $\{\alpha_1, \alpha_0\} = \Pi$ and assume that $\{\pi(\alpha_1)\}$ is a base of $\pi(R)$ and that $k(\alpha_1) \leq k(\alpha_0)$ if $\{\pi(\alpha_0)\}$ is also a base of $\pi(R)$. Then $k(\alpha_1) = 1$ and $((\alpha_0^\vee, \alpha_1), k(\alpha_0), g(\alpha_0), g(\alpha_1))$ is exactly one of the followings:*

$$\begin{array}{l} (-2,1,\emptyset,\emptyset), \\ (-2,1,\emptyset,2\mathbb{Z}+1),\ (-2,1,2\mathbb{Z}+1,\emptyset),\ (-2,1,2\mathbb{Z}+1,2\mathbb{Z}+1), \\ (-2,2,\emptyset,\emptyset),\ (-2,2,2\mathbb{Z}+1,\emptyset), \\ (-1,1,\emptyset,\emptyset),\ (-1,1,\emptyset,2\mathbb{Z}+1),\ (-1,2,\emptyset,\emptyset),\ (-1,2,\emptyset,2\mathbb{Z}+1), \\ (-1,4,\emptyset,\emptyset). \end{array} \tag{4.14}$$

(2) *Assume $l \geq 2$. Then there exists $R(\Pi, k, g)$ such that $(W_\Pi \cdot \Pi, \mathbb{R}\Pi)$ is a rank-l reduced affine root system of any type with a base Π and $k : \Pi \to \mathbb{N}$ and $g : \Pi \to \{\emptyset, 2\mathbb{Z}+1\}$ are any maps satisfying the condition that $1 \in k(\Pi)$ and $((\alpha^\vee, \beta), k(\beta)/k(\alpha), g(\alpha))$ is the same as one of* (4.13) *for any α, $\beta \in \Pi$ with $(\beta^\vee, \alpha) = -1$.*

The statements of this theorem is well-known and, however, some of $R(\Pi, k, g)$'s are isomorphic (see [16, (6.6)] and [1, Lists 4.6, 4.25, 4.67, 4.78]). For the case $l \geq 2$, which of them are isomorphic can be read off from the statement of Theorem 6.1.

§ 5. Elliptic Lie algebras with rank ≥ 2

In this section we assume R is a reduced elliptic root system with rank ≥ 2, that is, $R \cap 2R = \emptyset$, $n = 2$ and $l \geq 2$ (see (2.7)). We have assumed the rank $l \geq 2$ mainly because we use the fact (5.7) below. We fix a fundamental-set $\Pi \cup \{a\}$ of R.

§ 5.1. Useful lemma

The following lemma is useful.

Lemma 5.1. *Let $\mathcal{V}'$ be a 2-dimensional $\mathbb{C}$-linear space having a non-degenerate symmetric bilinear form $(\,,\,):\mathcal{V}'\times\mathcal{V}'\to\mathbb{C}$. Let γ_1, $\gamma_2\in(\mathcal{V}')^\times$. Let $\mathfrak{a}$ be a Lie algebra over $\mathbb{C}$ generated by $\bar{h}_\gamma$ ($\gamma\in\mathcal{V}'$), $\bar{E}_1$, $\bar{E}_2$, $\bar{F}_1$, $\bar{F}_2$ and satisfying the equations $\bar{h}_{x\gamma+x'\gamma'}=x\bar{h}_\gamma+x'\bar{h}_{\gamma'}$, $[\bar{h}_\gamma,\bar{h}_{\gamma'}]=0$, $[\bar{h}_\gamma,\bar{E}_i]=(\gamma,\gamma_i)\bar{E}_i$, $[\bar{h}_\gamma,\bar{F}_i]=-(\gamma,\gamma_i)\bar{F}_i$, and $[\bar{E}_i,\bar{F}_i]=\delta_{ij}\bar{h}_{\gamma_i^\vee}$, for x, $x'\in\mathbb{C}$, γ, $\gamma'\in\mathcal{V}'$, and $i\in\{1,2\}$.*

(1) *For $k\in\mathbb{N}$, we have*

$$\begin{aligned}[\mathrm{ad}(\bar{E}_1)^k(\bar{E}_2),\mathrm{ad}(\bar{F}_1)^k(\bar{F}_2)]\\ =k!(\textstyle\prod_{m=1}^{k-1}((\gamma_1^\vee,\gamma_2)+m))(k(\gamma_1,\gamma_2^\vee)\bar{h}_{\gamma_1^\vee}+(\gamma_1^\vee,\gamma_2)\bar{h}_{\gamma_2^\vee}).\end{aligned}\tag{5.1}$$

(2) *Let $m:=(\gamma_1^\vee,\gamma_2)$. Assume $m\in\mathbb{Z}_-$. Assume that $\bar{h}_{\gamma_1^\vee}$ and $\bar{h}_{\gamma_2^\vee}$ are linearly independent. Assume $\mathrm{ad}(\bar{E}_1)^r(\bar{E}_2)=\mathrm{ad}(\bar{F}_1)^r(\bar{F}_2)=0$ for some $r\in\mathbb{N}$. Let*

$$\bar{n}=n(\bar{E}_1,\bar{F}_1):=\exp(\mathrm{ad}\bar{E}_1)\exp(-\mathrm{ad}\bar{F}_1)\exp(\mathrm{ad}\bar{E}_1).\tag{5.2}$$

Then we have

$$\begin{aligned}&\mathrm{ad}(\bar{E}_1)^{1-m}(\bar{E}_2)=\mathrm{ad}(\bar{F}_1)^{1-m}(\bar{F}_2)=0,\\ &\bar{n}(\bar{h}_\gamma)=\bar{h}_\gamma-(\gamma_1,\gamma)\bar{h}_{\gamma_1^\vee},\ \bar{n}(\bar{E}_1)=-\bar{F}_1,\ \bar{n}(\bar{F}_1)=-\bar{E}_1,\\ &\bar{n}((\mathrm{ad}\bar{E}_1)^i\bar{E}_2)=\tfrac{(-1)^i i!}{(-m-i)!}(\mathrm{ad}\bar{E}_1)^{-m-i}\bar{E}_2\neq 0,\\ &\bar{n}((\mathrm{ad}\bar{F}_1)^i\bar{F}_2)=\tfrac{(-1)^{m-i} i!}{(-m-i)!}(\mathrm{ad}\bar{F}_1)^{-m-i}\bar{F}_2\neq 0,\end{aligned}\tag{5.3}$$

for $0\le i\le -m$ and $\gamma\in\mathcal{V}'$.

We can get (5.1) directly and get (5.3) by using a representation theory of sl_2.

§ 5.2. Definition of elliptic Lie algebras with rank ≥ 2

Let $\mathcal{A}:=\{(\alpha,\beta)\in\Pi\times\Pi\,|\,(\alpha,\beta^\vee)=-1\}$. Let $\mathcal{B}:=\mathcal{B}_+\cup(-\mathcal{B}_+)$, and $\mathcal{B}^{2,\prime}:=\{(\mu,\nu)\in\mathcal{B}\times\mathcal{B}\,|\,\mu\neq\nu\neq-\mu\}$. For $(\mu,\nu)\in\mathcal{B}^{2,\prime}$, let $x_{\mu,\nu}=1-((\mu^\vee,\nu)-|(\mu^\vee,\nu)|)/2$. Let $\mathcal{V}^\mathbb{C}=\mathbb{C}\otimes_\mathbb{R}\mathcal{V}$, so $\mathcal{V}^\mathbb{C}$ is a $l+2$-dimensional $\mathbb{C}$-linear space. We identify $\mathcal{V}$ with the $\mathbb{R}$-linear subspace $1\otimes\mathcal{V}$ of $\mathcal{V}^\mathbb{C}$; we extend $(\,,\,)$ to the symmetric bilinear form on $\mathcal{V}^\mathbb{C}$ in a standard way. We say that a map $\omega:\mathcal{A}\to\mathbb{C}^\times$ is a *tuning* if $\omega(\alpha,\beta)\omega(\beta,\alpha)=1$ whenever $(\alpha^\vee,\beta)=-1$. Denote ω_1 by the tuning with $\omega_1(\alpha,\beta)=1$ for all $(\alpha,\beta)\in\mathcal{A}$, and moreover, if $W_\Pi\cdot\Pi$ is $A_l^{(1)}$, then for $q\in\mathbb{C}^\times$, denote ω_q by the tuning with $\omega_q(\alpha_i,\alpha_{i+1})=1$ ($0\le i\le l$) and $\omega_q(\alpha_l,\alpha_0)=q$, where the numbering of the elements of Π is the same as that of the Dynkin diagram of $A_l^{(1)}$ in Subsection 3.2.

Definition 5.1. Let k and g be as in Theorem 4.1 (2). Let $\omega:\mathcal{A}\to\mathbb{C}^\times$ be a tuning. Let $\mathfrak{g}^\omega=\mathfrak{g}(\Pi,k,g,\omega)$ be the Lie algebra over $\mathbb{C}$ defined by generators:

$$h_\sigma\ (\sigma\in\mathcal{V}^\mathbb{C}),\quad E_\mu\ (\mu\in\mathcal{B}),\tag{5.4}$$

and relations:

(SR1) $xh_\sigma + yh_\tau = h_{x\sigma+y\tau}$ if x, $y \in \mathbb{C}$ and σ, $\tau \in \mathcal{V}^{\mathbb{C}}$,
(SR2) $[h_\sigma, h_\tau] = 0$ if σ, $\tau \in \mathcal{V}^{\mathbb{C}}$,
(SR3) $[h_\sigma, E_\mu] = (\sigma, \mu)E_\mu$ if $\sigma \in \mathcal{V}^{\mathbb{C}}$ and $\mu \in \mathcal{B}$,
(SR4) $[E_\mu, E_{-\mu}] = h_{\mu^\vee}$ if $\mu \in \mathcal{B}_+$,
(SR5) $(\mathrm{ad}E_\mu)^{x_{\mu,\nu}} E_\nu = 0$ if $(\mu, \nu) \in \mathcal{B}^{2,\prime}$,
(SR6) $c(\alpha)(\mathrm{ad}E_{\alpha^*})^{\frac{k(\beta)}{k(\alpha)}} E_\beta = \omega(\alpha, \beta)(\mathrm{ad}E_\alpha)^{c(\alpha)\frac{k(\beta)}{k(\alpha)}} E_{\beta^*}$ if $(\alpha, \beta) \in \mathcal{A}$,
(SR7) $(-1)^{c(\alpha)+1}c(\alpha)(\mathrm{ad}E_{-\alpha^*})^{\frac{k(\beta)}{k(\alpha)}} E_{-\beta} = \frac{1}{\omega(\alpha,\beta)}(\mathrm{ad}E_{-\alpha})^{c(\alpha)\frac{k(\beta)}{k(\alpha)}} E_{-\beta^*}$ if $(\alpha, \beta) \in \mathcal{A}$,
(SR8) $(\mathrm{ad}E_\alpha)^i(\mathrm{ad}E_{\alpha^*})^{\frac{k(\beta)}{k(\alpha)}-i} E_\beta = 0$ if $(\alpha, \beta) \in \mathcal{A}$ and $1 \leq i \leq \frac{k(\beta)}{k(\alpha)} - 1$,
(SR9) $(\mathrm{ad}E_{-\alpha})^i(\mathrm{ad}E_{-\alpha^*})^{\frac{k(\beta)}{k(\alpha)}-i} E_{-\beta} = 0$ if $(\alpha, \beta) \in \mathcal{A}$ and $1 \leq i \leq \frac{k(\beta)}{k(\alpha)} - 1$.

We call $\mathfrak{g}(\Pi, k, g, \omega)$ an *elliptic Lie algebra*, see Introduction. Let $\mathfrak{g} = \mathfrak{g}(\Pi, k, g) := \mathfrak{g}^{\omega_1}$.

We have

Lemma 5.2. *If $W_\Pi \cdot \Pi$ is not $A_l^{(1)}$ (resp. is $A_l^{(1)}$), then there is an isomorphism φ from $\mathfrak{g}^\omega$ to $\mathfrak{g}$ (resp. to $\mathfrak{g}^{\omega_q}$ for some $q \in \mathbb{C}^\times$) such that $\varphi(h_\sigma) = h_\sigma$ ($\sigma \in \mathcal{V}^{\mathbb{C}}$) and $\varphi(E_\mu) \in \mathbb{C}^\times E_\mu$ ($\mu \in \mathcal{B}$).*

Proof. Using (5.1), we can modify (SR6-7) by taking non-zero scalar products of E_μ's. □

Let $\mathfrak{h}^\omega = \mathfrak{h}^\omega(\Pi, k, g, \omega) := \{h_\sigma \in \mathfrak{g}^\omega | \sigma \in \mathcal{V}^{\mathbb{C}}\}$, and $\mathfrak{h} = \mathfrak{h}(\Pi, k, g) := \mathfrak{h}^{\omega_1}$.

Since all equations in (SR1-9) are $\mathbb{Z}R$-homogeneous, where $R = R(\Pi, k, g)$, we can regard $\mathfrak{g}^\omega$ as the $\mathbb{Z}R$-graded Lie algebra $\mathfrak{g}^\omega = \oplus_{\sigma\in\mathbb{Z}R}\mathfrak{g}^\omega_\sigma$ (that is $[\mathfrak{g}^\omega_\sigma, \mathfrak{g}^\omega_{\sigma'}] \subset \mathfrak{g}^\omega_{\sigma+\sigma'}$) such that $E_\mu \in \mathfrak{g}^\omega_\mu$ for all $\mu \in \mathcal{B}$. Note $\mathfrak{h}^\omega \subset \mathfrak{g}^\omega_0$. For each $\mu \in \mathcal{B}_+$, we can define n_μ to be $n(E_\mu, E_{-\mu})$ (see (5.2)) as an automorphism of $\mathfrak{g}^\omega$, so $n_\mu(\mathfrak{g}^\omega_\sigma) = \mathfrak{g}^\omega_{s_\mu(\sigma)}$. Let $\mathcal{R}^\omega = \{\sigma \in \mathbb{Z}R | \dim \mathfrak{g}^\omega_\sigma \neq 0\}$. Then we have

$$W_{\mathcal{B}_+} \cdot \mathcal{R}^\omega = \mathcal{R}^\omega. \tag{5.5}$$

Let S a non-empty proper connected subset of Π. Let $\mathfrak{g}^{\omega,S}$ be the Lie algebra over $\mathbb{C}$ defined by the generators h_σ ($\sigma \in \mathbb{C}S \oplus \mathbb{C}a$), $E_{\pm\alpha}$, $E_{\pm\alpha^*}$ ($\alpha \in S$) and the same relations as those in (SR1-9). Let $\iota^{\omega,S} : \mathfrak{g}^{\omega,S} \to \mathfrak{g}^\omega$ be the homomorphism sending the generators to those denoted by the same symbols. Let $\mathfrak{g}^{\omega,S}_\sigma = (\iota^{\omega,S})^{-1}(\mathfrak{g}^\omega_\sigma)$ for $\sigma \in \mathbb{Z}R^S$, so $\mathfrak{g}^{\omega,S} = \oplus_{\sigma\in\mathbb{Z}R^S}\mathfrak{g}^{\omega,S}_\sigma$. Let $\mathfrak{g}^S = \mathfrak{g}^{\omega_1,S}$, and $\mathfrak{g}^S_\sigma = \mathfrak{g}^{\omega_1,S}_\sigma$. Let $\mathcal{R}^{\omega,S} = \{\sigma \in \mathbb{Z}R^S | \dim \mathfrak{g}^{\omega,S}_\sigma \neq 0\}$.

Let $\alpha \in \Pi$. Then $\mathfrak{g}^{\omega,\{\alpha\}} = \mathfrak{g}^{\{\alpha\}}$, since $\mathfrak{g}^{\omega,\{\alpha\}}$ is defined by using (SR1-5). By Serre's relations (SR1-5), $\mathfrak{g}^{\omega,\{\alpha\}}$ is (the derived algebra of) an affine Lie algebra with $\mathcal{R}^{\omega,\{\alpha\}} =$

$R^{\{\alpha\}} \cup \mathbb{Z}k(\alpha)a$, where the affine root system $R^{\{\alpha\}}$ is $A_1^{(1)}$ or $A_2^{(1)}$. Hence $\dim \mathfrak{g}_0^{\omega,\{\alpha\}} = 2$, and $\dim \mathfrak{g}_\lambda^{\omega,\{\alpha\}} = 1$ ($\lambda \in \mathcal{R}^{\omega,\{\alpha\}} \setminus \{0\}$). Note $\mathcal{R}^{\omega,\{\alpha\}} \setminus \{0\} = R^{\{\alpha\}} \cup \mathbb{Z}^\times k(\alpha)a$.

Lemma 5.3. *There is a homomorphism χ^ω from $\mathfrak{g}^\omega$ to a Lie algebra $\mathfrak{b}^\omega$ such that $\dim \chi^\omega(\mathfrak{h}^\omega) = l+2$, $\dim \chi^\omega(\iota^{\omega,\{\alpha\}}(\mathfrak{g}_\lambda^{\omega,\{\alpha\}})) = 1$ for all $\alpha \in \Pi$ and all $\lambda \in R^{\{\alpha\}} \cup \mathbb{Z}^\times k(\alpha)a$, and*

$$\begin{aligned} &\chi^\omega(\mathfrak{h}^\omega + \textstyle\sum_{\alpha\in\Pi}\sum_{\lambda\in R^{\{\alpha\}}\cup\mathbb{Z}^\times k(\alpha)a} \iota^{\omega,\{\alpha\}}(\mathfrak{g}_\lambda^{\omega,\{\alpha\}})) \\ &\quad = \chi^\omega(\mathfrak{h}^\omega) \oplus \textstyle\bigoplus_{\alpha\in\Pi}\bigoplus_{\lambda\in R^{\{\alpha\}}\cup\mathbb{Z}^\times k(\alpha)a} \chi^\omega(\iota^{\omega,\{\alpha\}}(\mathfrak{g}_\lambda^{\omega,\{\alpha\}})). \end{aligned} \tag{5.6}$$

(If $\omega = \omega_1$, then $\mathfrak{b}^\omega$ is given as an 'affinization' $\mathfrak{a} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ of (the derived algebra of) an affine Lie algebra $\mathfrak{a}$, see [19, Proposition 3.1].)

Proof. If $\omega = \omega_1$, then we can define $\chi = \chi^{\omega_1}$ in a way entirely similar to that of [19, Proposition 3.1], inspired by so-called an 'unfolding process' of a Dynkin diagram of a reduced affine root system, and we see by checking each case directly that such χ has the property (5.6). The existence of a χ^{ω_q} is well-known (see [6]). Then this lemma follows from Lemma 5.2. □

For each $\alpha \in \Pi$, let $[R^{\{\alpha\}}]^+ := R^{\{\alpha\}} \cap (\mathbb{N}\alpha + \mathbb{Z}k(\alpha)a)$, and $[R^{\{\alpha\}}]^- := -[R^{\{\alpha\}}]^+$. Note that $R^{\{\alpha\}} = [R^{\{\alpha\}}]^+ \cup [R^{\{\alpha\}}]^-$.

Lemma 5.4. *For each $(\alpha, \beta) \in \mathcal{A}$,*

(5.7) *$\mathfrak{g}^{\omega,\{\alpha,\beta\}}$ is (the derived algebra of) an affine Lie algebra with the affine root system $R^{\{\alpha,\beta\}}$,*

which implies $\mathcal{R}^{\omega,\{\alpha,\beta\}} = R^{\{\alpha,\beta\}} \cup \mathbb{Z}k(\alpha)a$. In particular, for each $(\alpha', \beta') \in \Pi \times \Pi$ with $\alpha' \neq \beta'$, we have

$$[\iota^{\omega,\{\alpha'\}}(\mathfrak{g}_\lambda^{\omega,\{\alpha'\}}), \iota^{\omega,\{\beta'\}}(\mathfrak{g}_\mu^{\omega,\{\beta'\}})] = 0 \tag{5.8}$$

for all $(\lambda, \mu) \in ([R^{\{\alpha'\}}]^+ \times [R^{\{\beta'\}}]^-) \cup ([R^{\{\alpha'\}}]^- \times [R^{\{\beta'\}}]^+)$.

Proof. Note first that h_α, h_β and h_a are linearly independent in $\mathfrak{g}^{\omega,\{\alpha,\beta\}}$, which follows from Lemma 5.3. Let $\gamma \in R^{\{\alpha,\beta\}}$ be as in (4.13). If γ is expressed as $-s_{\gamma_1} \dots s_{\gamma_{r-1}}(\gamma_r^*)$ in (4.13) with $\gamma_i \in \{\alpha, \beta\}$, then we let $E_{\pm\gamma} := n_{\gamma_1} \dots n_{\gamma_{r-1}}(E_{\mp\gamma_r^*}) \in \mathfrak{g}_{\pm\gamma}^{\omega,\{\alpha,\beta\}}$. Let $\gamma_{r+1} \in \{\alpha, \beta\} \setminus \{\gamma_r\}$. By (SR6-7) and (5.3), we have $n_{\pm\gamma_r^*}(E_{\pm\gamma_{r+1}}) = n_{\pm\gamma_r}(E_{\pm\gamma_{r+1}^*})$. Hence $\mathfrak{g}^{\omega,\{\alpha,\beta\}}$ is generated by $E_{\pm\alpha}$, $E_{\pm\beta}$ and $E_{\pm\gamma}$. We show

$$[E_{\pm\alpha}, E_{\mp\gamma}] = [E_{\pm\beta}, E_{\mp\gamma}] = 0. \tag{5.9}$$

If $R^{\{\alpha,\beta\}} \neq A_4^{(2)}$, we have this in the same way as in [19, §2.3]. Assume $R^{\{\alpha,\beta\}} = A_4^{(2)}$. We write $X \sim Y$ if $X \in \mathbb{C}^\times Y$. By (5.3) and (SR6),

$$E_{-\gamma} \sim [E_\beta, [E_\beta, E_{\alpha^*}]] \sim [E_\beta, [E_\alpha, [E_\alpha, E_{\beta^*}]]] \tag{5.10}$$

Then $[E_\beta, E_{-\gamma}] = 0$ follows from (SR5). We have

$$\begin{aligned}[E_{-\gamma}, E_\alpha] &\sim [[E_\beta,[E_\alpha,[E_\alpha,E_{\beta^*}]]],E_\alpha] \quad \text{(by (5.10))} \\ &\sim [[E_\beta,E_\alpha],[E_\alpha,[E_\alpha,E_{\beta^*}]]]] \quad \text{(by (SR5))} \\ &\sim [[E_\beta,E_\alpha],[E_\beta,E_{\alpha^*}]]] \quad \text{(by (SR6))} \\ &\sim n_\beta([E_\alpha,[E_\beta,E_{\alpha^*}]]) \quad \text{(by (5.3))} \\ &\sim n_\beta([E_\alpha,[E_\alpha,[E_\alpha,E_{\beta^*}]]]) \quad \text{(by (SR6))} \\ &= 0 \quad \text{(by (SR5))}.\end{aligned}$$

The remaining equalities of (5.9) can be shown similarly. Hence by (5.3) and (SR5), the above generators satisfy Serre's relations. Hence (5.7) holds, as desired. □

For $i \in \mathbb{N}$, let $(\mathfrak{n}^{\omega,\pm})^{(i)}$ be the $\mathbb{C}$-linear subspaces of $\mathfrak{g}^\omega$ defined by $(\mathfrak{n}^{\omega,\pm})^{(1)} := \oplus_{\alpha\in\Pi}\oplus_{\lambda\in[R^{\{\alpha\}}]^\pm}\iota^{\omega,\{\alpha\}}(\mathfrak{g}_\lambda^{\omega,\{\alpha\}})$ (see Lemme 5.3), and $(\mathfrak{n}^{\omega,\pm})^{(i)} := [(\mathfrak{n}^{\omega,\pm})^{(1)}, (\mathfrak{n}^{\omega,\pm})^{(i-1)}]$ inductively for $i \geq 2$. Let $\mathfrak{n}^{\omega,\pm}$ be the two Lie subalgebras of $\mathfrak{g}^\omega$ defined by $\mathfrak{n}^{\omega,\pm} := \sum_{i=1}^\infty (\mathfrak{n}^{\omega,\pm})^{(i)}$. Let $\mathfrak{n}_\sigma^{\omega,\pm} = \mathfrak{g}_\sigma^\omega \cap \mathfrak{n}^{\omega,\pm}$. Then $\mathfrak{n}^{\omega,\pm} = \oplus_{\sigma\in(\mathbb{Z}_\pm\Pi\oplus\mathbb{Z}a)\setminus\mathbb{Z}a}\mathfrak{n}_\sigma^{\omega,\pm}$. For each $\alpha \in \Pi$, since $\iota^{\omega,\{\alpha\}}$ is a Lie algebra homomorphism (preserving $\mathbb{Z}\Pi \oplus \mathbb{Z}a$-grading), we have $\mathfrak{n}_\mu^{\omega,\pm} = \mathfrak{n}_\mu^{\omega,\pm} \cap (\mathfrak{n}^{\omega,\pm})^{(1)} = \iota^{\omega,\{\alpha\}}(\mathfrak{g}_\mu^{\omega,\{\alpha\}})$ for all $\mu \in (\mathbb{Z}_\pm\alpha \oplus \mathbb{Z}a) \setminus \mathbb{Z}a$. Moreover, by (5.8), we have

$$(5.11)\quad [(\mathfrak{n}^{\omega,+})^{(1)}, (\mathfrak{n}^{\omega,-})^{(1)}] \subset (\mathfrak{n}^{\omega,+})^{(1)} + (\mathfrak{n}^{\omega,-})^{(1)} + \sum_{\alpha\in\Pi}\sum_{\sigma\in\mathbb{Z}k(\alpha)a}\iota^{\omega,\{\alpha\}}(\mathfrak{g}_\sigma^{\omega,\{\alpha\}}).$$

Hence by Lemma 5.3 and (5.7), we have

$$(5.12)\qquad \mathfrak{g}^\omega = \mathfrak{h}^\omega \oplus \mathfrak{n}^{\omega,+} \oplus \mathfrak{n}^{\omega,-} \oplus (\bigoplus_{\alpha\in\Pi}\bigoplus_{\sigma\in\mathbb{Z}^\times k(\alpha)a}\iota^{\omega,\{\alpha\}}(\mathfrak{g}_\sigma^{\omega,\{\alpha\}})),$$

$\dim\mathfrak{h}^\omega = l+2$, and $\dim\mathfrak{n}_\lambda^{\omega,\pm} = \dim\iota^{\omega,\{\alpha\}}(\mathfrak{g}_\sigma^{\omega,\{\alpha\}}) = 1$ for $\alpha \in \Pi$, $\lambda \in [R^{\{\alpha\}}]^\pm$ and $\sigma \in \mathbb{Z}^\times k(\alpha)a$. By (3.22), we have

$$(5.13)\quad \begin{cases} R = W_\Pi \cdot \bigcup_{\alpha\in\Pi}[R^{\{\alpha\}}]^+, \\ (\mathbb{Z}R)^\times \setminus R \\ = W_\Pi \cdot (\bigcup_{\alpha\in\Pi}(\mathbb{N}\alpha\oplus\mathbb{Z}a)\setminus[R^{\{\alpha\}}]^+) \cup ((\mathbb{Z}R)^\times \setminus (\mathbb{Z}_+\Pi\cup\mathbb{Z}_-\Pi)\oplus\mathbb{Z}a). \end{cases}$$

Then by (5.5), using a standard argument as in [10], [18], together with the automorphisms n_μ ($\mu \in \mathcal{B}_+$), we have

Theorem 5.1. *We have* $(\mathcal{R}^\omega)^\times = R$, $\dim\mathfrak{g}_\mu^\omega = 1$, $[\mathfrak{g}_\mu^\omega, \mathfrak{g}_{-\mu}^\omega] = \mathbb{C}h_{\mu^\vee}$ $(\mu \in R)$, $\mathfrak{g}_0^\omega = \mathfrak{h}^\omega$, $\dim\mathfrak{h}^\omega = l+2$, $(\mathcal{R}^\omega)^0 \subset \mathbb{Z}\delta\oplus\mathbb{Z}a$, *and* $\dim\mathfrak{g}_{ma}^\omega = |\{\alpha\in\Pi | m \in \mathbb{Z}k(\alpha)\}|$ $(m \in \mathbb{Z}^\times)$.

By the following theorem, we can compute $\dim \mathfrak{g}^{\omega}_{\lambda}$ for $\lambda \in \mathbb{Z}\delta \oplus \mathbb{Z}a$.

Theorem 5.2. *Let $\Pi' \cup \{a'\}$ be a fundamental-set of R. Then there exist a tuning η for $\Pi' \cup \{a'\}$ and an isomorphism $f : \mathfrak{g}(\Pi', k', g', \eta) \to \mathfrak{g}^{\omega}$ such that $f(\mathfrak{g}'^{,\eta}_{\lambda}) = \mathfrak{g}^{\omega}_{\lambda}$ for all $\lambda \in \mathbb{Z}\Pi \oplus \mathbb{Z}a$, where $\mathfrak{g}'^{,\eta} := \mathfrak{g}(\Pi', k', g', \eta)$. In particular, we have*

$$\dim \mathfrak{g}^{\omega}_{ma'} = |\{\alpha' \in \Pi' | m \in \mathbb{Z}k'(\alpha')\}| \text{ for } m \in \mathbb{Z}^{\times}. \tag{5.14}$$

Proof. Let $\mathcal{B}_+{}' = \{\alpha', (\alpha')^* | \alpha \in \Pi'\}$ and $\mathcal{B}' = \mathcal{B}_+{}' \cup -\mathcal{B}_+{}'$. By (SR1-9), Theorem 5.1 and (5.3), for some η, we have a homomorphism f of the statement such that $f(\mathfrak{g}'^{,\eta}_{\mu'}) = \mathfrak{g}^{\omega}_{\mu'}$ for all $\mu' \in \mathcal{B}'$. Since $\mathfrak{g}'^{,\eta}$ is generated by $\mathfrak{g}'^{,\eta}_{\mu'}$ ($\mu' \in \mathcal{B}'$), we have $f(\mathfrak{g}'^{,\eta}_{\lambda}) \subset \mathfrak{g}^{\omega}_{\lambda}$ for all $\lambda \in \mathbb{Z}R = \mathbb{Z}\Pi' \oplus \mathbb{Z}a'$. Since $R = W_{\mathcal{B}_+{}'} \cdot \mathcal{B}_+{}'$ by (4.12), using $n(E_{\mu'}, E_{-\mu'}) \in \mathrm{Aut}(\mathfrak{g}'^{,\eta})$ ($\mu' \in \mathcal{B}'$), by Theorem 5.1, we have $f(\mathfrak{g}'^{,\eta}_{\beta}) = \mathfrak{g}^{\omega}_{\beta}$ for all $\beta \in R$. Since $E_{\mu} \in f(\mathfrak{g}'^{,\eta})$ for all $\mu \in \mathcal{B}$, we have $f(\mathfrak{g}'^{,\eta}) = \mathfrak{g}^{\omega}$, so $f(\mathfrak{g}'^{,\eta}_{\lambda}) = \mathfrak{g}^{\omega}_{\lambda}$ for all $\lambda \in \mathbb{Z}R$. By the same argument, for some tuning ω' for $\Pi \cup \{a\}$, we have an epimorphism $f' : \mathfrak{g}^{\omega'} = \mathfrak{g}(\Pi, k, g, \omega') \to \mathfrak{g}'^{,\eta}$ such that $f'(\mathfrak{g}^{\omega'}_{\lambda}) = \mathfrak{g}'^{,\eta}_{\lambda}$ for all $\lambda \in \mathbb{Z}R$. Hence $\dim \mathfrak{g}^{\omega'}_{\lambda} \geq \dim \mathfrak{g}^{\omega}_{\lambda}$ for all $\lambda \in \mathbb{Z}R$, so $(\mathcal{R}^{\omega})^0 \subset (\mathcal{R}^{\omega'})^0$. Assume that $W_{\Pi} \cdot \Pi$ is not $A^{(1)}_l$. By Lemma 5.2, we have $\dim \mathfrak{g}^{\omega'}_{\lambda} = \dim \mathfrak{g}_{\lambda} = \dim \mathfrak{g}^{\omega}_{\lambda}$ for all $\lambda \in \mathbb{Z}R$, so $(\mathcal{R}^{\omega})^0 = (\mathcal{R}^{\omega'})^0$. Hence $f \circ f'$ is an isomorphism, so is f. Assume that $W_{\Pi} \cdot \Pi$ is $A^{(1)}_l$. Assume $\varphi : \mathfrak{g}(\Pi, k, g, \omega_{q_1}) \to \mathfrak{g}(\Pi, k, g, \omega_{q_2})$ is an epimorphism such that $\varphi(\mathfrak{g}(\Pi, k, g, \omega_{q_1})_{\lambda}) = \mathfrak{g}(\Pi, k, g, \omega_{q_2})_{\lambda}$ for all $\lambda \in \mathbb{Z}R$. For $\gamma \in \mathcal{B}_+$, let $c_{\gamma} \in \mathbb{C}^{\times}$ be such that $\varphi(E_{\gamma}) = c_{\gamma} E_{\gamma}$ ($E_{\gamma} \neq 0$ by Lemma 5.3). For $\alpha \in \Pi$, let $d_{\alpha} = c_{\alpha}/c_{\alpha^*}$. By (SR6), we have $\omega_{q_2}(\alpha, \beta) = \omega_{q_1}(\alpha, \beta) d_{\alpha}/d_{\beta}$ (the element of (SR6) is not zero by Lemma 5.3 and (5.1)). Hence $d_{\alpha_i} = d_{\alpha_{i+1}}$ for $0 \leq i \leq l$. Since $\omega_{q_2}(\alpha_l, \alpha_0) = \omega_{q_1}(\alpha_l, \alpha_0)$, we have $q_1 = q_2$. Then by the same argument as above, we conclude that f is an isomorphism.

The last statement follows from Theorem 5.1. □

By the same argument as that for the proof of Theorem 5.2, we have

Theorem 5.3. *Let $\mathfrak{t} = \oplus_{\lambda \in \mathbb{Z}\Pi \oplus \mathbb{Z}a} \mathfrak{t}_{\lambda}$ be a $\mathbb{Z}\Pi \oplus \mathbb{Z}a$-graded Lie algebra with $\mathcal{T} := \{\lambda \in \mathbb{Z}R | \dim \mathfrak{t}_{\lambda} \neq 0\}$ satisfying the conditions* (i)-(iv) *below.*

(i) $\mathcal{T}^{\times} = R$, *and* $\dim \mathfrak{t}_{\mu} = 1$ *for all* $\mu \in R$.

(ii) $\mathfrak{t}$ *is generated by* $\mathfrak{t}_{\mu}$*'s with all* $\mu \in R$.

(iii) $[\mathfrak{t}_0, \mathfrak{t}_0] = \{0\}$.

(iv) *There exists a* $\mathbb{C}$*-linear epimorphism* $j : \mathcal{V}^{\mathbb{C}} \to \mathfrak{t}_0$ *satisfying the following conditions* (iv-i) *and* (iv-ii).

(iv-i) $[j(\sigma), X] = (\sigma, \lambda) X$ *for all* $\sigma \in \mathcal{V}^{\mathbb{C}}$, *all* $\lambda \in \mathbb{Z}\Pi \oplus \mathbb{Z}a$, *and all* $X \in \mathfrak{t}_{\lambda}$.

(iv-ii) $[\mathfrak{t}_{\beta}, \mathfrak{t}_{-\beta}] = \mathbb{C} j(\beta^{\vee})$ *for all* $\beta \in R$.

Then there exist a tuning ω for $\Pi \cup \{a\}$ and an epimorphism $f : \mathfrak{g}^\omega \to \mathfrak{t}$ such that $f(\mathfrak{g}^\omega_\lambda) = \mathfrak{t}_\lambda$ for all $\lambda \in \mathbb{Z}\Pi \oplus \mathbb{Z}a$. (Therefore $\mathfrak{t}$ is generated by $\mathfrak{t}_\nu$'s with $\nu \in \mathcal{B}^{2,\prime}$.)

§ 6. List of $\dim \mathfrak{g}_{m\delta+ra}$

In this section we use the notation as follows. For a $\mathbb{Z}$-module X, $r \in \mathbb{Z}$ and x, $y \in X$, let $x \equiv_r y$ means $x - y \in rX$. Recall that $l = |\Pi| - 1 \geq 2$, and see Subsection 3.2 for the numbering of the elements α_i $(0 \leq i \leq l)$ of Π. Let $\delta = \delta(\Pi)$. Fix $\gamma_1 \in \Pi_{\rm sh} \setminus \{\alpha_0\}$. Fix $\gamma_2 \in \Pi_{\rm lg} \setminus \{\alpha_0\}$ if $R_{\rm lg} \neq \emptyset$. Let $M := \mathbb{Z}\delta \oplus \mathbb{Z}a$. We also denote $m\delta + ra \in M$ with m, $r \in \mathbb{Z}$ by $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]$. Let $R = R(\Pi, k, g)$ be as in (4.10). Let $L_{\rm sh}$, $L_{\rm lg}$ and $L_{\rm ex}$ be the subsets of M such that $\gamma_1 + L_{\rm sh} = R \cap (\gamma_1 + M)$, $\gamma_2 + L_{\rm lg} = R \cap (\gamma_2 + M)$ (if $R_{\rm lg} \neq \emptyset$), and $2\gamma_1 + L_{\rm ex} = R \cap (2\gamma_1 + M)$ (if $R_{\rm ex} \neq \emptyset$). Let $\Pi' := \Pi \setminus \{\alpha_0\}$, so $\pi(\Pi')$ is a base of $\pi(R)$. By Lemma 2.1, we have $R_{\rm sh} = W_{\Pi'} \cdot \gamma_1 + L_{\rm sh}$, $R_{\rm lg} = W_{\Pi'} \cdot \gamma_2 + L_{\rm lg}$ and $R_{\rm ex} = W_{\Pi'} \cdot 2\gamma_1 + L_{\rm ex}$. Let $\mathfrak{g}^\omega := \mathfrak{g}(\Pi, k, g, \omega)$, and $\mathfrak{g} := \mathfrak{g}^{\omega_1}$.

Remark 6.1. (Due to Kaiming Zhao) Here we would like to mention that a map from M to $\{0, 1, \ldots, t-1\}$ which is periodic modulo t on any line in M is not necessarily meant to be periodic modulo tM. This indicates that we have to be very careful when calculating $\dim \mathfrak{g}^\omega_{m\delta+ra}$ because (5.14) does not immediately imply that $\dim \mathfrak{g}^\omega_{m\delta+ra}$ is periodic, although we finally see that this is true.

Let $f : M \to \mathbb{Z}_+$ be a map such that $m\mathbb{Z} + r\mathbb{Z} = f(\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right])\mathbb{Z}$, where $f(\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right])$ is a g.c.d. of m and r if $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. By definition, $f(h\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]) = h \cdot f(\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right])$ for all $h \in \mathbb{Z}$ and all $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right] \in M$. Let $t \in \mathbb{N}$ be such that $t \geq 2$. Define the map $f_t : M \to \{0, 1, \ldots, t-1\}$ by $f_t(\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]) \equiv_t f(\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right])$. Then $f_t((h_1 t + h_2)\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]) = f_t(h_2\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right])$ for all $h_1 \in \mathbb{Z}$, all $h_2 \in \{0, 1, \ldots, t-1\}$ and all $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right] \in M$. Now assume that $t = 25$ and $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right] = \left[\begin{smallmatrix} 40 \\ 200 \end{smallmatrix}\right]$. Then $f(\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]) = 40$ and $f(\left[\begin{smallmatrix} m+t \\ r \end{smallmatrix}\right]) = 5$. Hence $f_t(\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]) = 15 \neq 5 = f_t(\left[\begin{smallmatrix} m+t \\ r \end{smallmatrix}\right])$, as desired.

Now we have the following theorem.

Theorem 6.1. *Assume $\mathfrak{g}^\omega = \mathfrak{g}$ if $W_\Pi \cdot \Pi$ is not $A^{(1)}_l$ (see Lemma 5.2). Then $\dim \mathfrak{g}^\omega_\sigma$ with $\sigma \in M \setminus \{0\}$ are listed below.*

(1) *Assume that $W_\Pi \cdot \Pi$ is $X^{(1)}_l$ with $X = A, \ldots, G$, and $k(\alpha) = 1$ and $g(\alpha) = \emptyset$ for all $\alpha \in \Pi$, so $L_{\rm sh} = M$, $R_{\rm ex} = \emptyset$, and $L_{\rm lg} = M$ if $R_{\rm lg} \neq \emptyset$ (so $X = B, C, F$ or G). Then we have $\dim \mathfrak{g}^\omega_\sigma = l + 1$ for all $\sigma \in M \setminus \{0\}$.*

(2) *Assume $W_\Pi \cdot \Pi$ is $X^{(1)}_l$ with $X = B, C, F$ or G. Let $r = (\gamma_2, \gamma_2)/(\gamma_1, \gamma_1)$. Assume that $k(\alpha) = (\alpha, \alpha)/(\gamma_1, \gamma_1)$ and $g(\alpha) = \emptyset$ for all $\alpha \in \Pi$, so $L_{\rm sh} = M$, $L_{\rm lg} = \mathbb{Z}\delta \oplus \mathbb{Z}ra$, and $R_{\rm ex} = \emptyset$. Then we have $\dim \mathfrak{g}_{\sigma_1} = l + 1$ for all $\sigma_1 \in L_{\rm lg} \setminus \{0\}$, and $\dim \mathfrak{g}_{\sigma_2} = |\Pi_{\rm sh}|$ for all $\sigma_2 \in M \setminus L_{\rm lg}$. (This R is isomorphic to $R(\Pi_1, k_1, g_1)$ for which $W_{\Pi_1} \cdot \Pi_1$ is $D^{(2)}_{l+1}$, $A^{(2)}_{2l-1}$, $E^{(2)}_6$ $(l = 4)$, or $D^{(3)}_4$ $(l = 2)$ respectively, and $k_1(\alpha) = 1$, $g_1(\alpha) = \emptyset$ $(\alpha \in \Pi)$.)*

(3) *Assume* $W_\Pi \cdot \Pi$ *is* $D^{(2)}_{l+1}$, $A^{(2)}_{2l-1}$, $E^{(2)}_6$ *(*$l=4$*), or* $D^{(3)}_4$ *(*$l=2$*). Let* $r = (\gamma_2,\gamma_2)/(\gamma_1,\gamma_1)$. *Assume that* $k(\alpha) = (\alpha,\alpha)/(\gamma_1,\gamma_1)$ *and* $g(\alpha)=\emptyset$ *for all* $\alpha\in\Pi$, *so* $L_{\rm sh}=M$, $L_{\rm lg}=rM$, *and* $R_{\rm ex}=\emptyset$. *Then we have* $\dim\mathfrak{g}_{\sigma_1} = l+1$ *for all* $\sigma_1\in L_{\rm lg}\setminus\{0\}$, *and* $\dim\mathfrak{g}_{\sigma_2} = |\Pi_{\rm sh}|$ *for all* $\sigma_2 \in M\setminus rM$.

(4) *Assume* $W_\Pi\cdot\Pi$ *is* $D^{(2)}_{l+1}$, *and* $k(\alpha_0)=2$, $k(\alpha_1)=1$, $k(\beta)=2$ *(*$\beta\in\Pi_{\rm lg}$*),* $g(\alpha)=\emptyset$ *(*$\alpha\in\Pi$*), so* $L_{\rm sh}=\{0,\delta,a\}+2M$, $L_{\rm lg}=2M$, *and* $R_{\rm ex}=\emptyset$. *Then we have* $\dim\mathfrak{g}_{\sigma_1}=l+1$ *for all* $\sigma_1\in 2M\setminus\{0\}$, *and* $\dim\mathfrak{g}_{\sigma_2}=1$ *for all* $\sigma_2\in M\setminus 2M$.

(5) *Assume* $W_\Pi\cdot\Pi$ *is* $D^{(2)}_{l+1}$, *and* $k(\alpha_0)=2$, $g(\alpha_0)=2\mathbb{Z}+1$, $k(\alpha_1)=1$, $g(\alpha_1)=\emptyset$, $k(\beta)=2$, $g(\beta)=\emptyset$ *(*$\beta\in\Pi_{\rm lg}$*), so* $L_{\rm sh}=\{0,\delta,a\}+2M$, $L_{\rm lg}=2M$ *and* $\frac{1}{2}L_{\rm ex}=\delta+a+2M$. *Then we have* $\dim\mathfrak{g}_{\sigma_1}=l+1$ *for all* $\sigma_1\in 2M\setminus\{0\}$, *and* $\dim\mathfrak{g}_{\sigma_2}=1$ *for all* $\sigma_2\in M\setminus 2M$.

(6) *Assume* $W_\Pi\cdot\Pi$ *is* $D^{(2)}_{l+1}$, *and* $k(\alpha_0)=1$, $g(\alpha_0)=2\mathbb{Z}+1$, $k(\alpha_1)=1$, $g(\alpha_1)=2\mathbb{Z}+1$, $k(\beta)=1$, $g(\beta)=\emptyset$ *(*$\beta\in\Pi_{\rm lg}$*), so* $L_{\rm sh}=M$, $L_{\rm lg}=\{0,a\}+2M$, *and* $L_{\rm ex}=a+2M$. *Then we have* $\dim\mathfrak{g}_{\sigma_1}=l+1$ *for all* $\sigma_1\in L_{\rm lg}\setminus\{0\}$, *and* $\dim\mathfrak{g}_{\sigma_2}=1$ *for all* $\sigma_2\in M\setminus L_{\rm lg}$. *(This* R *is isomorphic to* $R(\Pi_2,k_2,g_2)$ *for which* $W_{\Pi_2}\cdot\Pi_2$ *is* $A^{(2)}_{2l}$, *and* $k_2(\alpha)=1$, $g_2(\alpha)=\emptyset$ *(*$\alpha\in\Pi_{\rm sh}$*),* $k_2(\beta)=2$, $g_2(\beta)=\emptyset$ *(*$\beta\in\Pi_{\rm lg}\cup\Pi_{\rm ex}$*).)*

(7) *Assume* $W_\Pi\cdot\Pi$ *is* $A^{(2)}_{2l}$, *and* $k(\alpha)=1$ *(*$\alpha\in\Pi$*),* $g(\alpha_1)=2\mathbb{Z}+1$, $g(\beta)=\emptyset$ *(*$\beta\in\Pi_{\rm lg}\cup\Pi_{\rm ex}$*), so* $L_{\rm sh}=L_{\rm lg}=M$, *and* $L_{\rm ex}=\{\delta,\delta+a,a\}+2M$. *Then we have* $\dim\mathfrak{g}_\sigma=l+1$ *for all* $\sigma\in M\setminus\{0\}$.

(8) *Assume* $W_\Pi\cdot\Pi$ *is* $B^{(1)}_l$, *and* $k(\alpha)=1$ *(*$\alpha\in\Pi$*),* $g(\alpha_1)=2\mathbb{Z}+1$, $g(\beta)=\emptyset$ *(*$\beta\in\Pi_{\rm lg}$*), so* $L_{\rm sh}=L_{\rm lg}=M$, *and* $L_{\rm ex}=a+2M$. *Let* $M'=\{0,a\}+2M$. *Then we have* $\dim\mathfrak{g}_{\sigma_1}=l+1$ *for all* $\sigma_1\in M'\setminus\{0\}$, *and* $\dim\mathfrak{g}_{\sigma_2}=1$ *for all* $\sigma_2\in M\setminus M'$. *(This* R *is isomorphic to* $R(\Pi_3,k_3,g_3)$ *for which* $W_{\Pi_3}\cdot\Pi_3$ *is* $A^{(2)}_{2l}$, *and* $k_3(\alpha)=1$, $g_3(\alpha)=\emptyset$ *(*$\alpha\in\Pi_{\rm sh}\cup\Pi_{\rm lg}$*),* $k_3(\beta)=2$, $g_3(\beta)=\emptyset$ *(*$\beta\in\Pi_{\rm ex}$*).)*

(9) *Assume* $W_\Pi\cdot\Pi$ *is* $A^{(2)}_{2l}$, *and* $k(\alpha)=1$, $g(\alpha)=\emptyset$ *(*$\alpha\in\Pi$*), so* $L_{\rm sh}=L_{\rm lg}=M$, *and* $L_{\rm ex}=\{\delta,\delta+a\}+2M$. *Then we have* $\dim\mathfrak{g}_{\sigma_1}=l+1$ *for all* $\sigma_1\in M\setminus(L_{\rm ex}\cup\{0\})$, *and* $\dim\mathfrak{g}_{\sigma_2}=l$ *for all* $\sigma_2\in L_{\rm ex}$. *(This* R *is isomorphic to* $R(\Pi_4,k_4,g_4)$ *for which* $W_{\Pi_4}\cdot\Pi_4$ *is* $A^{(2)}_{2l}$, *and* $k_4(\alpha_1)=1$, $g_4(\alpha_1)=2\mathbb{Z}+1$, $k_4(\alpha_0)=2$, $g_4(\alpha_0)=\emptyset$, $k_4(\beta)=1$, $g_4(\beta)=\emptyset$ *(*$\beta\in\Pi_{\rm lg}$*).)*

(10) *Assume* $W_\Pi\cdot\Pi$ *is* $D^{(2)}_{l+1}$, *and* $k(\alpha)=1$ *(*$\alpha\in\Pi$*),* $g(\alpha_0)=2\mathbb{Z}+1$, $g(\beta)=\emptyset$ *(*$\beta\in\Pi_{\rm lg}\cup\{\alpha_1\}$*). Then*

$$L_{\rm sh}=M,\ L_{\rm lg}=\{0,a\}+2M \text{ and } L_{\rm ex}=\{2\delta+a,2\delta+3a\}+4M, \tag{6.1}$$

and we have

$$\dim\mathfrak{g}_{p\delta+za}=\begin{cases} l+1 & \text{if } p\equiv_4 0 \text{ and } \begin{bmatrix}p\\z\end{bmatrix}\neq\begin{bmatrix}0\\0\end{bmatrix},\\ 1 & \text{if } p\equiv_2 1,\\ l & \text{if } p\equiv_4 2 \text{ and } z\equiv_2 0,\\ l+1 & \text{if } p\equiv_4 2 \text{ and } z\equiv_2 1.\end{cases} \tag{6.2}$$

(This R *is isomorphic to* $R(\Pi_5,k_5,g_5)$ *for which* $W_{\Pi_5}\cdot\Pi_5$ *is* $A^{(2)}_{2l}$, $k(\alpha)=(\alpha,\alpha)/(\gamma_1,\gamma_1)$, $g_5(\alpha)=\emptyset$ *(*$\alpha\in\Pi$*).)*

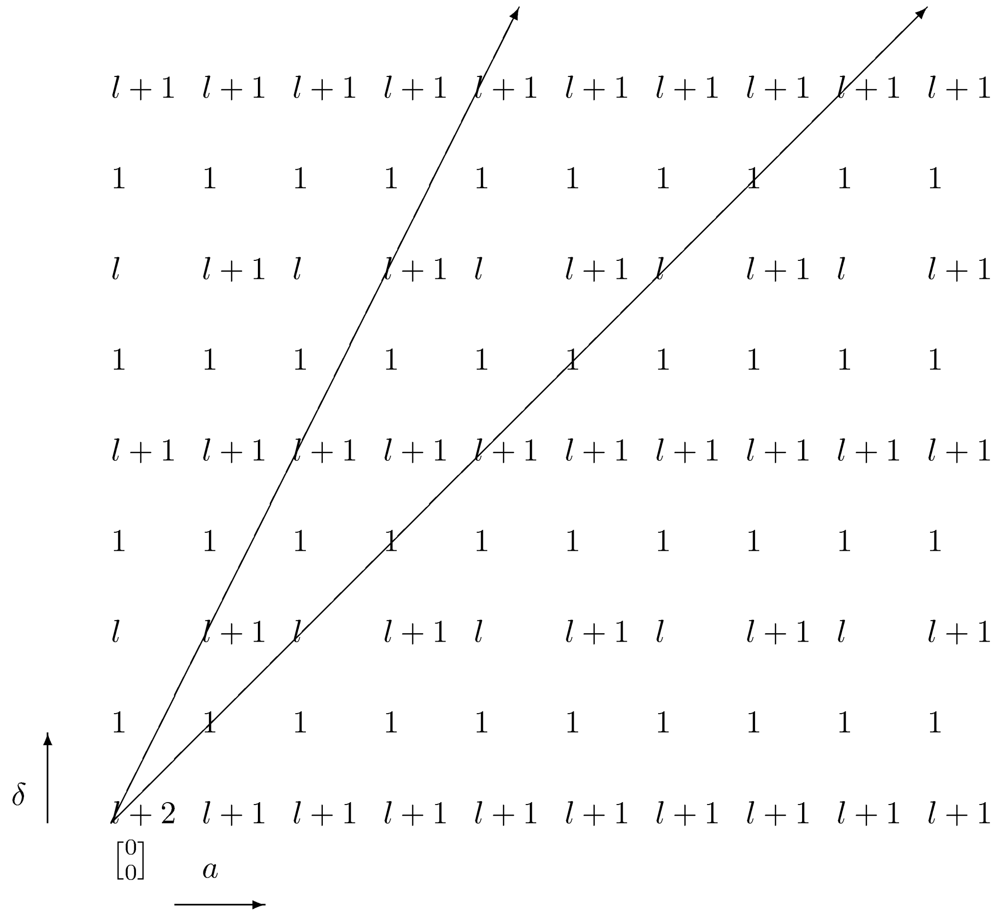


Figure 1. $\dim \mathfrak{g}_{m\delta+ra}$ in (6.2)

(11) *Assume* $W_\Pi \cdot \Pi$ *is* $C_l^{(1)}$, *and* $k(\alpha_0) = 2$, $k(\alpha_l) = 1$, $k(\beta) = 1$ $(\beta \in \Pi_{\rm sh})$, $g(\alpha) = \emptyset$ $(\alpha \in \Pi)$, *so* $L_{\rm sh} = M$, $L_{\rm lg} = \{0, \delta, a\} + 2M$, *and* $R_{\rm ex} = \emptyset$. *Then we have* $\dim \mathfrak{g}_{\sigma_1} = l+1$ *for all* $\sigma_1 \in 2M \setminus \{0\}$, *and* $\dim \mathfrak{g}_{\sigma_2} = l$ *for all* $\sigma_2 \in M \setminus 2M$.

(At this moment, we do not see why $\dim \mathfrak{g}_{p\delta+za}$ are periodic modulo tM for some $t \in \mathbb{N}$. Maybe one of reasons is that $\mathfrak{g}$ may be realized as a 'fixed point' Lie algebra, see also [3], [20].)

Proof. We only prove (10), since (1)-(9), (11) are similarly treated.

Assume $(\alpha_1, \alpha_1) = 1$. Define $\varepsilon_i \in \mathcal{V}$ $(1 \leq i \leq l)$ by $\varepsilon_1 := \alpha_1$ and $\varepsilon_j := \alpha_j + \varepsilon_{j-1}$ $(2 \leq j \leq l)$. Then $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, and $\alpha_0 = \delta - \varepsilon_1$. Moreover, we have

$$(6.3) \quad \begin{aligned} W_\Pi \cdot \alpha_1 &= \cup_{\epsilon \in \{-1,1\}, 1 \leq i \leq l} \epsilon \varepsilon_i + 2\mathbb{Z}\delta, \\ W_\Pi \cdot \alpha_r &= \cup_{\epsilon_1, \epsilon_2 \in \{-1,1\}, 1 \leq i < j \leq l} \epsilon_1 \varepsilon_i + \epsilon_2 \varepsilon_j + 2\mathbb{Z}\delta \ (2 \leq r \leq l), \\ W_\Pi \cdot \alpha_0 &= \cup_{\epsilon \in \{-1,1\}, 1 \leq i \leq l} \epsilon \varepsilon_i + (2\mathbb{Z}+1)\delta. \end{aligned}$$

Then by (4.9), we have

$$
\begin{aligned}
R = & \cup_{\epsilon\in\{-1,1\},1\leq i\leq l}\epsilon\varepsilon_i + 2\mathbb{Z}\delta + \mathbb{Z}a \\
& \cup \cup_{\epsilon_1,\epsilon_2\in\{-1,1\},1\leq i<j\leq l}\ \epsilon_1\varepsilon_i + \epsilon_2\varepsilon_j + 2\mathbb{Z}\delta + \mathbb{Z}a \\
& \cup \cup_{\epsilon\in\{-1,1\},1\leq i\leq l}\ \epsilon\varepsilon_i + (2\mathbb{Z}+1)\delta + \mathbb{Z}a \\
& \cup \cup_{\epsilon\in\{-1,1\},1\leq i\leq l}\ 2(\epsilon\varepsilon_i + (2\mathbb{Z}+1)\delta) + (2\mathbb{Z}+1)a \\
= & \cup_{\epsilon\in\{-1,1\},1\leq i\leq l}\epsilon\varepsilon_i + M \\
& \cup \cup_{\epsilon_1,\epsilon_2\in\{-1,1\},1\leq i<j\leq l}\ \epsilon_1\varepsilon_i + \epsilon_2\varepsilon_j + \{0,a\} + 2M \\
& \cup \cup_{\epsilon\in\{-1,1\},1\leq i\leq l}\ 2\epsilon\varepsilon_i + \{2\delta+a, 2\delta+3a\} + 4M.
\end{aligned} \tag{6.4}
$$

Hence we have (6.1), as desired.

Let $\Pi' \cup \{a'\}$ be a fundamental-set of R. Let $\delta' := \delta(\Pi')$, so $\{\delta', a'\}$ is a $\mathbb{Z}$-basis of M.

Assume $a' \equiv_4 a = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Then $\delta' \equiv_4 \delta = \left[\begin{smallmatrix}1\\y\end{smallmatrix}\right]$, where we replace Π' with $-\Pi'$ if necessary. Let $\delta'' = \delta' - ya'$. Then $\{\delta'', a'\}$ is a $\mathbb{Z}$-basis of M. Since $\delta'' \equiv_4 \delta = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \equiv_2 \delta = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, we have $L_{\rm lg} = \{0, a'\} + 2M$ and $L_{\rm ex} = \{2\delta'' + a', 2\delta'' + 3a'\} + 4M$. Hence we have the root system isomorphism $f_1 : \mathbb{R}R \to \mathbb{R}R$ (cf. (2.4)) such that $f_1(\alpha_j) = \alpha_j$ $(1 \leq j \leq l)$, $f_1(\delta) = \delta''$ and $f_1(a) = a'$. Then by Theorem 5.2, we have $\dim \mathfrak{g}_{ma'} = l+1$ for $m \in \mathbb{Z}^\times$.

Assume $a' \equiv_4 \delta = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. Let $R_5 = R(\Pi_5, k_5, g_5)$ be as in the statement. Let $\mathfrak{g}' := \mathfrak{g}(\Pi_5, k_5, g_5)$. Define the $\mathbb{R}$-linear isometry $f_2 : \mathbb{R}R_5 \to \mathbb{R}R$ by $f_2(\alpha_j) = \alpha_j$ $(1 \leq j \leq l)$, $f_2(\delta) = 2\delta - a$ and $f_2(a) = \delta$. Note that $f_2(L_{\rm sh}) = f_2(M) = M = L_{\rm sh}$, $f_2(L_{\rm lg}) = f_2(\{0,\delta\} + 2M) = L_{\rm lg}$ and $f_2(L_{\rm ex}) = f_2(\{\delta, 3\delta\} + 4M) = L_{\rm ex}$. Hence f_2 is a root system isomorphism. Let $a'' := f_2^{-1}(a')$. Then $a'' \equiv_4 a$. By the same argument as above, as for $\dim \mathfrak{g}'_{ma''}$, we have the same equalities as in (6.5) below. Then Theorem 5.2 implies that

$$
\dim \mathfrak{g}_{ma'} = \begin{cases} l+1 & \text{if } m \neq 0 \text{ and } m \equiv_4 0, \\ 1 & \text{if } m \equiv_2 1, \\ l & \text{if } m \equiv_4 2. \end{cases} \tag{6.5}
$$

For other a''s, we can utilize the root system isomorphisms $f_i : \mathbb{Z}R \to \mathbb{Z}R$ $(3 \leq i \leq 5)$ defined by $f_i(\alpha_j) = \alpha_j$ for all $1 \leq j \leq l$, and $f_3(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]) = \left[\begin{smallmatrix}-1\\0\end{smallmatrix}\right]$, $f_3(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]) = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, $f_4(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $f_4(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]) = \left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right]$, $f_5(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]) = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, $f_5(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]) = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Let $R_6 = R(\Pi_6, k_6, g_6)$ be such that $W_{\Pi_6} \cdot \Pi_6$ is $D^{(2)}_{l+1}$, $k_6(\alpha_i) = 1$ for $0 \leq i \leq l$, and $g_6(\alpha_0) = \emptyset$, $g_6(\alpha_1) = 2\mathbb{Z}+1$ and $g_6(\alpha_j) = \emptyset$ for $2 \leq j \leq l-1$. Then we can also use the root system isomorphism $f_6 : \mathbb{Z}R_6 \to \mathbb{Z}R$ defined by $f_6(\alpha_j) = \alpha_j$ $(1 \leq j \leq l)$, $f_6(\delta) = \delta$ and $f_6(a) = 2\delta + a$.

Finally we have

Case-1. If $a' \equiv_4 \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0\\3\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]$, then we have $\dim \mathfrak{g}_{ma'} = l+1$ for $m \in \mathbb{Z}^\times$.

Case-2. If $a' \equiv_4 \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}3\\3\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right]$, then the same as (6.5) holds.

Let $\lambda = p\delta + za = \begin{bmatrix} p \\ z \end{bmatrix} = ma'$ with p, $z \in \mathbb{Z}$ and $m \in \mathbb{Z}^{\times}$. Let $\begin{bmatrix} x \\ y \end{bmatrix} = a'$, so $x\mathbb{Z} + y\mathbb{Z} = \mathbb{Z}$.

Assume that $p \equiv_4 0$. If $x \equiv_2 1$, then $m \equiv_4 0$, so $\dim \mathfrak{g}_\lambda = l + 1$. If $x \equiv_2 0$, then $y \equiv_2 1$, so Case-1 implies $\dim \mathfrak{g}_\lambda = l + 1$.

Assume that $p \equiv_4 2$ and $z \equiv_2 0$. If $x \equiv_2 0$, then $y \equiv_2 1$, so $m \equiv_2 0$, so $p \equiv_4 0$, contradiction. Hence $x \equiv_2 1$, so $m \equiv_4 2$, so Case-2 implies $\dim \mathfrak{g}_\lambda = l$.

Assume that $p \equiv_4 2$ and $z \equiv_2 1$. Then $m \equiv_2 1$, $y \equiv_2 1$ and $x \equiv_2 0$, so Case-1 implies $\dim \mathfrak{g}_\lambda = l + 1$.

Assume that $p \equiv_2 1$. Then $m \equiv_2 1$ and $x \equiv_2 1$, so Case-2 implies $\dim \mathfrak{g}_\lambda = 1$.

Thus we have (6.2), as desired. This completes the proof. □

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