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Logarithmic derivative and the Capelli identities

By

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§ 1. Introduction

Let T_{ij} be variables, and $\partial/\partial T_{ij}$ the corresponding partial differential operators for $1 \leq i, j \leq m$. Define $m \times m$ matrices T and $\partial/\partial T$ with entries in polynomial coefficient differential operators as

$$T = \Big(T_{ij}\Big)_{1\leq i,j\leq m}, \qquad \frac{\partial}{\partial T} = \left(\frac{\partial}{\partial T_{ij}}\right)_{1\leq i,j\leq m}.$$

Then the Capelli identity [1, 2, 3] is written as

$$\det {}^tT \det \frac{\partial}{\partial T} = \det \left({}^tT \frac{\partial}{\partial T} + \begin{pmatrix} m-1 & & & O \\ & m-2 & & \\ & & \ddots & \\ O & & & 0 \end{pmatrix} \right),$$

where det is the column-determinant defined by $\det A = \sum_\sigma \operatorname{sgn}(\sigma) A_{\sigma(1)1} \cdots A_{\sigma(m)m}$. One of applications of the Capelli identity is a computation of the b-function $b_f(s)$ of $f(T) = \det T$. The b-function is the polynomial in s determined by

$$f(\partial)(f^{s+1}) = b_f(s) f^s,$$

and given explicitly by

$$b_f(s) = (s+1)(s+2)\cdots(s+m), \tag{1.1}$$

where $f(\partial)$ is the constant coefficient differential operator obtained by substituting the corresponding partial differential operators to the variables in f.

From a viewpoint of representation theory of Lie algebras, the Capelli identity can be interpreted as a formula which expresses invariant differential operator (LHS) by an

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image of the center of the universal enveloping algebra of the general linear Lie algebra (RHS). In this article, we focus on more naive interpretation that the Capelli identity is a non-commutative version of the product formula of determinants. The following formula is a special case of our first main theorem (Theorem 4.4), which is an analogue of the Capelli identity in this sense:

$$\det \frac{\partial}{\partial T}\ \det {}^tT\ \det \frac{\partial}{\partial T} \cdots \det {}^tT\ \det \frac{\partial}{\partial T} = \det\left(\frac{\partial}{\partial T}{}^tT\frac{\partial}{\partial T}\cdots {}^tT\frac{\partial}{\partial T}\right). \tag{1.2}$$

Remark that there is no diagonal shift on the right-hand side, which is an important point in the original Capelli identity. Remark also that the entries of the matrix on the right-hand side commute with each other (Proposition 4.2).

The first result and its proof are related to the b-function of the prehomogeneous vector space associated to the equioriented quiver of type A:

$$\overset{n_0}{\bullet} \leftarrow \overset{n_1}{\bullet} \leftarrow \cdots \leftarrow \overset{n_l}{\bullet},$$

where $n_0 = n_l$ and every n_r is greater than or equal to n_0. A vertex labeled n_i expresses the vector space $\mathbb{C}^{n_i}$, and we associate $GL(n_i;\mathbb{C})$ to the vertex. An arrow from one vertex labeled n_i to another labeled n_j expresses $\mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})$. Thus this quiver gives the prehomogeneous vector space (G, V), where

$$\begin{aligned} G &= GL(n_0;\mathbb{C}) \times \cdots \times GL(n_l;\mathbb{C}), \\ V &= \mathrm{Mat}(n_0, n_1;\mathbb{C}) \oplus \mathrm{Mat}(n_1, n_2;\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}(n_{l-1}, n_l;\mathbb{C}), \end{aligned}$$

and the action is given for $(g_0, \dots, g_l) \in G$ and $(X^{(1)}, \dots, X^{(l)}) \in V$ by

$$(g_0, g_1, \dots, g_l).(X^{(1)}, X^{(2)}, \dots, X^{(l)}) = (g_0X^{(1)}g_1^{-1}, g_1X^{(2)}g_2^{-1}, \dots g_{l-1}X^{(l)}g_l^{-1}).$$

The polynomial ψ on V defined by

$$\psi = \det(X^{(1)}X^{(2)} \cdots X^{(l)}), \qquad (X^{(1)}, X^{(2)}, \dots, X^{(l)}) \in V,$$

which is nonzero by $n_r \geq n_0$ for all r, is a relative invariant of the prehomogeneous vector space (G, V). When every n_r $(2 \leq r \leq l-1)$ is greater than n_0, this relative invariant ψ is fundamental, that is, it does not have another relative invariant as its factor. Using the result of F. Sato and Sugiyama [5], Sugiyama computed the b-functions of the prehomogeneous vector spaces associated to the quivers of type A with any n_r and any orientations of arrows [6]. As a special case of his result, the b-function of the above relative invariant ψ is given by

$$b_\psi(s) = (s + n_1)^{(n_0)}(s + n_2)^{(n_0)} \cdots (s + n_l)^{(n_0)},$$

where $a^{(b)}$ is the descending factorial power defined by

$$a^{(b)} = a(a-1)(a-2)\cdots(a-b+1).$$

In the second main theorem (Theorem 5.1) we give another way of computing the b-function by means of the Capelli identity of our first main theorem.

§ 2. Logarithmic derivative

We localize the Weyl algebra $\mathbb{C}[T_{ij}, \partial/\partial T_{ij}\ ;\ 1 \leq i,j \leq m]$ by the nonzero polynomial $f = \det T$ and work with the algebra $\mathbb{C}[T_{ij}, \partial/\partial T_{ij}, f^{-1}\ ;\ 1 \leq i,j \leq m]$. We define differential operators with parameters following Noumi-Umeda-Wakayama [4].

Definition 2.1. Define $g = \log\det T$, and

$$\frac{\partial}{\partial T_{ij}}(u) = \frac{\partial}{\partial T_{ij}} + u\frac{\partial g}{\partial T_{ij}}$$

for $1 \leq i,j \leq m$. If we define $\partial g/\partial T = (\partial g/\partial T_{ij})_{1\leq i,j\leq m}$, which is equal to ${}^tT^{-1}$, then we also write

$$\frac{\partial}{\partial T}(u) = \frac{\partial}{\partial T} + u\frac{\partial g}{\partial T}.$$

Remark that we have

$$\frac{\partial}{\partial T_{ij}}(u) = f^{-u}\frac{\partial}{\partial T_{ij}}f^{u}, \qquad \frac{\partial}{\partial T}(u) = f^{-u}\frac{\partial}{\partial T}f^{u}, \tag{2.1}$$

and it follows that the entries of the matrix $(\partial/\partial T)(u)$ commute with each other. In addition, it follows from (1.1) and (2.1) that

$$\det\left(\frac{\partial}{\partial T}(u)\right)(f^{s+1}) = (s+u+1)(s+u+2)\cdots(s+u+m)f^{s} \tag{2.2}$$

We have the Capelli identity with parameter u using the operators defined in Definition 2.1.

Proposition 2.2 (Noumi-Umeda-Wakayama[4]). *We have*

$$\det {}^tT \det\frac{\partial}{\partial T}(u) = \det\left({}^tT\frac{\partial}{\partial T}(u) + \begin{pmatrix} m-1 & & & O \\ & m-2 & & \\ & & \ddots & \\ O & & & 0 \end{pmatrix}\right)$$
$$= \det\left({}^tT\frac{\partial}{\partial T} + \begin{pmatrix} u+m-1 & & & O \\ & u+m-2 & & \\ & & \ddots & \\ O & & & u \end{pmatrix}\right).$$

Proof. In [4], this formula was obtained by the classical limit of the q-analogue of the Capelli identity. However, to prove only this formula is easy. The proof for the first equality reduces to the case of the original Capelli identity, since the commutation relations of T_{ij} and $(\partial/\partial T_{kl})(u)$ are the same as that of T_{ij} and $\partial/\partial T_{kl}$. Second equality holds since $\partial g/\partial T = {}^tT^{-1}$. □

Let us define

$$T = X^{(1)}X^{(2)}\cdots X^{(l)}.$$

Note that T_{ij}'s are algebraically independent even when they are written in the coordinate functions of $X^{(r)}$ $(1 \le r \le l)$ thanks to the condition $n_r \ge n_0 = n_l$. We also define differential operators with parameters in terms of $X^{(r)}$.

Definition 2.3. Let $g = \log\det T = \log\det(X^{(1)}X^{(2)}\cdots X^{(l)})$, and we define

$$\frac{\partial}{\partial X_{ij}^{(r)}}(u) = \frac{\partial}{\partial X_{ij}^{(r)}} + u\frac{\partial g}{\partial X_{ij}^{(r)}} \qquad \begin{pmatrix} 1 \le i \le n_{r-1} \\ 1 \le j \le n_r \end{pmatrix}$$

for $1 \le r \le l$. We again use the notation in Definition 2.1 like $\partial g/\partial X^{(r)}$. Then we have

$$\frac{\partial}{\partial X^{(r)}}(u) = \frac{\partial}{\partial X^{(r)}} + u\frac{\partial g}{\partial X^{(r)}} = f^{-u}\frac{\partial}{\partial X^{(r)}}f^u. \tag{2.3}$$

In view of (2.3), the entries of $(\partial/\partial X^{(r)})(u)$ $(1 \le r \le l)$ commute with each other for the same parameter u.

§ 3. Chain rules

Let $T = X^{(1)}X^{(2)}\cdots X^{(l)}$ as in the previous section. Then we have chain rules for derivatives of a function in T by the variables $X_{ij}^{(r)}$.

Lemma 3.1 (chain rules). *Let $1 \le r \le l$, and u be a complex number.*
(1) For a function $\phi = \phi(T)$ in T, we have

$$\frac{\partial \phi}{\partial X^{(r)}}(u) = {}^t(X^{(1)}X^{(2)}\cdots X^{(r-1)})\frac{\partial \phi}{\partial T}(u)\,{}^t(X^{(r+1)}X^{(r+2)}\cdots X^{(l)}),$$

where $(\partial\phi/\partial X^{(r)})(u)$ means the matrix $\partial\phi/\partial X^{(r)} + u(\partial g/\partial X^{(r)})\phi$, and so on. In particular, when $r = 1$ or l, this formula becomes

$$\frac{\partial \phi}{\partial X^{(1)}}(u) = \frac{\partial \phi}{\partial T}(u)\,{}^t(X^{(2)}X^{(3)}\cdots X^{(l)}),$$
$$\frac{\partial \phi}{\partial X^{(l)}}(u) = {}^t(X^{(1)}X^{(2)}\cdots X^{(l-1)})\frac{\partial \phi}{\partial T}(u).$$

(2) Set $S = X^{(1)}X^{(2)}\cdots X^{(r)}$. For a function $\phi = \phi(S)$ in S, we have

$$\frac{\partial \phi}{\partial X^{(r)}}(u) = {}^t(X^{(1)}X^{(2)}\cdots X^{(r-1)})\frac{\partial \phi}{\partial S}(u).$$

Proof. (1) If we have the formula for $u = 0$, then the formula for general u is obtained by the conjugation $f^{-u}(\partial\phi/\partial X^{(r)})f^u$. Therefore we have only to prove the formula for $u = 0$. Fix r $(1 \le r \le l)$, and set

$$X = X^{(1)}X^{(2)}\cdots X^{(r-1)}, \qquad Y = X^{(r)}, \qquad Z = X^{(r+1)}X^{(r+2)}\cdots X^{(l)}.$$

Then $T_{ab} = \sum_{i,j} X_{ai}Y_{ij}Z_{jb}$, and we have

$$\begin{aligned}\left(\frac{\partial \phi}{\partial X^{(r)}}\right)_{(i,j)} = \left(\frac{\partial \phi}{\partial Y}\right)_{(i,j)} &= \sum_{1\le a,b\le n_0} \frac{\partial \phi}{\partial T_{ab}}\frac{\partial T_{ab}}{\partial Y_{ij}} \\ &= \sum_{1\le a,b\le n_0} \frac{\partial \phi}{\partial T_{ab}} X_{ai}Z_{jb} \qquad = \left({}^tX\frac{\partial \phi}{\partial T}\,{}^tZ\right)_{(i,j)}.\end{aligned}$$

(2) It suffices to prove the assertion when $u = 0$. By the same notation as above, $S_{ab} = \sum_i X_{ai}Y_{ib}$, and we have

$$\begin{aligned}\left(\frac{\partial \phi}{\partial X^{(r)}}\right)_{(i,j)} = \left(\frac{\partial \phi}{\partial Y}\right)_{(i,j)} &= \sum_{\substack{1\le a\le n_0\\ 1\le b\le n_r}} \frac{\partial \phi}{\partial S_{ab}}\frac{\partial S_{ab}}{\partial Y_{ij}} \\ &= \sum_{\substack{1\le a\le n_0\\ 1\le b\le n_r}} \frac{\partial \phi}{\partial S_{ab}} X_{ai}\delta_{bj} \qquad = \left({}^tX\frac{\partial \phi}{\partial S}\right)_{(i,j)}.\end{aligned}$$

□

Remark 3.2 (chain rules without parameters). We need the condition $n_r \le n_0 = n_l$ for any r in Lemma 3.1. Otherwise $f = \det T$ becomes zero, and $\partial g/\partial T$ is undefined. However, we do not need this condition at all, when the parameter u is equal to zero in Lemma 3.1. Its proof is the same as in the proof of Lemma 3.1.

Although $\partial g/\partial T$ is the transposed inverse of the matrix T, the matrix $\partial g/\partial X^{(r)}$ is not the transposed inverse of the matrix $X^{(r)}$ in general, even if it is a square matrix. However, we have the following left and right inverses.

Lemma 3.3 (left and right inverses)**.** *Let $1 \le r \le l$. Then we have*

$$\frac{\partial g}{\partial X^{(1)}}\frac{\partial g}{\partial X^{(2)}}\cdots\frac{\partial g}{\partial X^{(r)}}\,{}^t(X^{(1)}X^{(2)}\cdots X^{(r)}) = 1_{n_0},$$

and

$$ {}^t(X^{(r)}X^{(r+1)}\cdots X^{(l)})\frac{\partial g}{\partial X^{(r)}}\frac{\partial g}{\partial X^{(r+1)}}\cdots\frac{\partial g}{\partial X^{(l)}} = 1_{n_0}. $$

In particular, $\partial g/\partial X^{(r)}$ is equal to ${}^t(X^{(r)})^{-1}$ when $n_0 = n_1 = \cdots = n_l$.

Proof. We prove the first formula by induction on r, and the second one is proved similarly. When $r = 1$, we set $X = X^{(1)}$, $Y = X^{(2)}X^{(3)}\cdots X^{(r)}$, and $f = \det T = \det(XY)$. We have

$$ \left(\frac{\partial g}{\partial X}\,{}^tX\right)_{(i,j)} = \sum_{a=1}^{n_1}\frac{\partial g}{\partial X_{ia}}X_{ja} = \sum_{a=1}^{n_1}X_{ja}\frac{\partial f}{\partial X_{ia}}\cdot\frac{1}{f}. \tag{3.1} $$

Since f is of total degree one with respect to the variables $X_{j1}, X_{j2}, \ldots, X_{j,n_1}$ in each row of X, both sides of (3.1) are equal to 1 when $i = j$. In addition, since f is alternating with respect to the rows of X, (3.1) is equal to 0 when $i \neq j$. Hence $(\partial g/\partial X)\,{}^tX = 1_{n_0}$, and the formula is proved for $r = 1$.

Let $r \geq 2$, and we use the notation $X^{(a,b)} = X^{(a)}X^{(a+1)}\cdots X^{(b)}$. It follows from Lemma 3.1 (1) (chain rules) and the formula $(\partial g/\partial T)\,{}^tT = 1_{n_0}$ that

$$ \begin{aligned} &\frac{\partial g}{\partial X^{(1)}}\cdots\frac{\partial g}{\partial X^{(r)}}\,{}^tX^{(1,r)} \\ &\quad= \frac{\partial g}{\partial X^{(1)}}\cdots\frac{\partial g}{\partial X^{(r-1)}}\cdot{}^tX^{(1,r-1)}\frac{\partial g}{\partial T}\,{}^tX^{(r+1,l)}\cdot{}^tX^{(1,r)} \\ &\quad= \frac{\partial g}{\partial X^{(1)}}\cdots\frac{\partial g}{\partial X^{(r-1)}}\,{}^tX^{(1,r-1)}, \end{aligned} $$

which is equal to 1_{n_0} by the hypothesis of induction. Hence the desired formula is proved. □

We have the following formulas for change of variables.

Lemma 3.4 (change of variables). *For a function $\phi = \phi(T)$ in T and a complex number u, we have*

$$ \frac{\partial\phi}{\partial T}(u) = \frac{\partial g}{\partial X^{(1)}}\cdots\frac{\partial g}{\partial X^{(r-1)}}\cdot\frac{\partial\phi}{\partial X^{(r)}}(u)\cdot\frac{\partial g}{\partial X^{(r+1)}}\cdots\frac{\partial g}{\partial X^{(l)}}. $$

Remark that the right-hand side is independent of r.

Proof. We use the notation $X^{(a,b)} = X^{(a)}X^{(a+1)}\cdots X^{(b)}$. By Lemma 3.1 (1) (chain rule), we have

$$ \frac{\partial\phi}{\partial X^{(r)}}(u) = {}^tX^{(1,r-1)}\,\frac{\partial\phi}{\partial T}(u)\,{}^tX^{(r+1,l)}. $$

By multiplying $(\partial g/\partial X^{(1)}) \cdots (\partial g/\partial X^{(r-1)})$ from the left to the above equation, and $(\partial g/\partial X^{(r+1)}) \cdots (\partial g/\partial X^{(l)})$ from the right, we have the lemma thanks to Lemma 3.3. $\square$

§ 4. Product formula

Let $T = X^{(1)}X^{(2)} \cdots X^{(l)}$, and we use the notation $X^{(a,b)} = X^{(a)}X^{(a+1)} \cdots X^{(b)}$ as in the previous sections. In this section we prove the product formula which is a generalization of the Capelli identity. A special case of the product formula is already stated in (1.2). We need some lemmas.

Lemma 4.1. *For complex numbers $u_1, \dots, u_l$, we have the equation*

$$\begin{aligned}&\frac{\partial}{\partial X^{(1)}}(u_1)\frac{\partial}{\partial X^{(2)}}(u_2) \cdots \frac{\partial}{\partial X^{(l)}}(u_l)\\ &\quad= \frac{\partial}{\partial T}(u_1 + n_1 - n_0)^t T \frac{\partial}{\partial T}(u_2 + n_2 - n_0)^t T \cdots {}^t T \frac{\partial}{\partial T}(u_l + n_l - n_0),\end{aligned}$$

as differential operators acting on functions in T.

Proof. First we prove the following equation as differential operators acting on functions in T:

$$\frac{\partial}{\partial T}(u - n_0)^t T = \frac{\partial}{\partial X^{(1,r)}}(u - n_r)^t X^{(1,r)} \qquad (1 \le r \le l). \tag{4.1}$$

Set $X = X^{(1,r)}$ and $Y = X^{(r+1,l)}$. By Remark 3.2 (chain rule without parameters), we have $\partial\phi/\partial X = (\partial\phi/\partial T)^t Y$. By multiplying ${}^t X$ from the right followed by transposing the equation, we obtain $X^t(\partial/\partial X) = T^t(\partial/\partial T)$. As for the left-hand side, we have

$$\begin{aligned}\left(X^t\left(\frac{\partial}{\partial X}\right)\right)_{(i,j)} &= \sum_{a=1}^{n_r} X_{ia}\frac{\partial}{\partial X_{ja}} = \sum_{a=1}^{n_r}\left(\frac{\partial}{\partial X_{ja}}X_{ia} - \delta_{ij}\right)\\ &= \left(\frac{\partial}{\partial X}\,{}^t X\right)_{(j,i)} - n_r\delta_{ij} = \left(\frac{\partial}{\partial X}(-n_r)\,{}^t X\right)_{(j,i)},\end{aligned}$$

since $\partial g/\partial X$ is the left inverse of ${}^t X$ by Lemma 3.3. Hence the left-hand side turns out to be equal to ${}^t(\partial/\partial X(-n_r)^t X)$. The right-hand side similarly turns out to be equal to ${}^t(\partial/\partial T(-n_0)^t T)$. Therefore we have $(\partial/\partial X)(u - n_r)^t X = (\partial/\partial T)(u - n_0)^t T$ by conjugation of f^{-u} and f^u, and hence (4.1) is proved.

Thanks to (4.1), we have

$$\begin{aligned}&\frac{\partial}{\partial T}(u_1 - n_0)^t T \cdot \frac{\partial}{\partial T}(u_2 - n_0)^t T \cdots {}^t T \cdot \frac{\partial}{\partial T}(u_l - n_0)\\ &\quad= \frac{\partial}{\partial X^{(1,1)}}(u_1 - n_1)^t X^{(1,1)} \cdot \frac{\partial}{\partial X^{(1,2)}}(u_2 - n_2)^t X^{(1,2)} \cdots\\ &\qquad \cdots \frac{\partial}{\partial X^{(1,l-1)}}(u_{l-1} - n_{l-1})^t X^{(1,l-1)} \cdot \frac{\partial}{\partial X^{(1,l)}}(u_l - n_l),\end{aligned}$$

where the last factor is by $T = X^{(1,l)}$ and $n_0 = n_l$. It follows from Lemma 3.1 (2) (chain rule) that the expression above is equal to

$$\frac{\partial}{\partial X^{(1)}}(u_1 - n_1)\frac{\partial}{\partial X^{(2)}}(u_2 - n_2)\cdots\frac{\partial}{\partial X^{(l)}}(u_l - n_l).$$

Thus the lemma is proved. □

Proposition 4.2. *Let $u_1, u_2, \ldots, u_l$ be complex numbers, and T_{ij} independent variables ($1 \le i, j \le n_0$). The entries of the matrix*

$$\frac{\partial}{\partial T}(u_1)\,{}^tT\frac{\partial}{\partial T}(u_2)\,{}^tT\cdots{}^tT\frac{\partial}{\partial T}(u_l) \tag{4.2}$$

commute with each other.

Proof. Suppose that $n_0 = n_1 = \cdots = n_l$ in Lemma 4.1. Then the matrix of this proposition is the right-hand side of Lemma 4.1.

As remarked in Definition 2.3, for each r the entries of $(\partial/\partial X^{(r)})(u_r)$ commute with each other. Moreover, each $\partial g/\partial X^{(r)}$ is the transposed inverse of $X^{(r)}$ as in Lemma 3.3. In particular, $(\partial/\partial X^{(r)})(u_r)$ contains variables only in $X^{(r)}$, and it follows that the entries of $(\partial/\partial X^{(r)})(u_r)$ commute with the entries of $(\partial/\partial X^{(r')})(u_{r'})$. Thus the left-hand side of Lemma 4.1 is a matrix with commutative entries.

Therefore the entries of Expression (4.2) commute with each other when $T = X^{(1,l)}$ with all n_r being equal. Finally, since the entries of $X^{(1,l)}$ are algebraically independent as mentioned before Definition 2.3, we have the proposition. □

In the following, we use $|A|$ for the column-determinant of A for simplicity. However, all the matrices to take determinants have commutative entries in the rest of the article.

Lemma 4.3. *If $n_0 = n_1 = \cdots = n_l$, then we have*

$$\begin{aligned}\left|\frac{\partial}{\partial T}(u_1)\right| |{}^tT| \left|\frac{\partial}{\partial T}(u_2)\right| |{}^tT| \cdots |{}^tT| \left|\frac{\partial}{\partial T}(u_l)\right| \\ = \left|\frac{\partial}{\partial X^{(1)}}(u_1)\right| \left|\frac{\partial}{\partial X^{(2)}}(u_2)\right| \cdots \left|\frac{\partial}{\partial X^{(l)}}(u_l)\right|.\end{aligned} \tag{4.3}$$

Proof. For a function $\phi = \phi(T)$ in T and $1 \le r \le l$, it follows from Lemma 3.4 that

$$\frac{\partial\phi}{\partial T}(u) = \frac{\partial g}{\partial X^{(1,r-1)}}\,\frac{\partial\phi}{\partial X^{(r)}}(u)\,\frac{\partial g}{\partial X^{(r+1,l)}}. \tag{4.4}$$

Since all n_r are equal, $\partial g/\partial X^{(1,r-1)}$ and $\partial g/\partial X^{(r+1,l)}$ are equal to the transposed inverses of the square matrices $X^{(1,r-1)}$ and $X^{(r+1,l)}$, respectively. In particular, their entries do not contain coordinate functions of $X^{(r)}$. Hence we can remove the function ϕ from (4.4), and the determinant of the resulting equation factors into three determinants as follows:

$$\left|\frac{\partial}{\partial T}(u)\right| = |{}^t(X^{(1,r-1)})^{-1}| \left|\frac{\partial}{\partial X^{(r)}}(u)\right| |{}^t(X^{(r+1,l)})^{-1}|.$$

By substituting the above equation to $(\partial/\partial T)(u_r)$ $(1 \leq r \leq l)$ on the left-hand side of (4.3), the determinants $|{}^tT|$, $|{}^t(X^{(1,r-1)})^{-1}|$ and $|{}^t(X^{(r+1,l)})^{-1}|$ $(1 \leq r \leq l)$ cancel out, and we obtain the assertion. □

Theorem 4.4 (product formula). *Let $u_1, u_2, \ldots, u_l$ be complex numbers, and T_{ij} independent variables. We have*

$$\left|\frac{\partial}{\partial T}(u_1)\, {}^tT \frac{\partial}{\partial T}(u_2)\, {}^tT \cdots {}^tT \frac{\partial}{\partial T}(u_l)\right| = \left|\frac{\partial}{\partial T}(u_1)\right| |{}^tT| \left|\frac{\partial}{\partial T}(u_2)\right| |{}^tT| \cdots |{}^tT| \left|\frac{\partial}{\partial T}(u_l)\right|.$$

Proof. Let $T = X^{(1)}X^{(2)} \cdots X^{(l)}$ with $n_0 = n_1 = \cdots = n_l$. Note that T_{ij}'s are still algebraically independent, since $n_r \geq n_0$ for all r. The matrix on the left-hand side is equal to $(\partial/\partial X^{(1)})(u_1)(\partial/\partial X^{(2)})(u_2) \cdots (\partial/\partial X^{(l)})(u_l)$ by Lemma 4.1. Its determinant factors into l determinants as

$$\left|\frac{\partial}{\partial X^{(1)}}(u_1)\right| \left|\frac{\partial}{\partial X^{(2)}}(u_2)\right| \cdots \left|\frac{\partial}{\partial X^{(l)}}(u_l)\right|, \tag{4.5}$$

since the entries of the matrices in this expression commute with each other for the same reason as in the proof of Proposition 4.2. By Lemma 4.3, (4.5) is equal to the right-hand side of the theorem, and we proved the theorem. □

§ 5. b-Functions associated to quivers of type A

Let $T = X^{(1)}X^{(2)} \cdots X^{(l)}$, and $|A|$ denote the column-determinant as in the previous sections. In this section we consider the prehomogeneous vector space associated to an equioriented quiver of type A:

$$\overset{n_0}{\bullet} \leftarrow \overset{n_1}{\bullet} \leftarrow \cdots \leftarrow \overset{n_l}{\bullet} \qquad (n_0 = n_l \text{ and } n_r \geq n_0 \text{ for all } r),$$

and calculate the b-function of the relative invariant

$$\psi = \det(X^{(1)}X^{(2)} \cdots X^{(l)})$$

of the prehomogeneous vector space. This b-function is first obtained as a special case of Sugiyama's result [6], which uses the result of F. Sato and Sugiyama [5]. Our calculation gives a way of computing by means of Capelli identities.

Theorem 5.1 (b-function). *The b-function $b_\psi(s)$ of ψ is given by*

$$b_\psi(s) = (s+n_1)^{(n_0)}(s+n_2)^{(n_0)}\cdots(s+n_l)^{(n_0)},$$

where $a^{(b)}$ is defined by $a^{(b)} = a(a-1)(a-2)\cdots(a-b+1)$.

Proof. We have

$$\psi(\partial)(\psi^{s+1}) = \left|\frac{\partial}{\partial X^{(1)}}\frac{\partial}{\partial X^{(2)}}\cdots\frac{\partial}{\partial X^{(l)}}\right|\left(\left|X^{(1)}X^{(2)}\cdots X^{(l)}\right|^{s+1}\right). \tag{5.1}$$

Using Lemma 4.1 and Theorem 4.4, we have

$$\begin{aligned}
&\text{Eq. (5.1)}\\
&= \left|\frac{\partial}{\partial T}(n_1-n_0)\,{}^tT\frac{\partial}{\partial T}(n_2-n_0)\,{}^tT\cdots{}^tT\frac{\partial}{\partial T}(n_l-n_0)\right|(|T|^{s+1})\\
&= \left|\frac{\partial}{\partial T}(n_1-n_0)\right|\left|{}^tT\right|\left|\frac{\partial}{\partial T}(n_2-n_0)\right|\left|{}^tT\right|\cdots\left|{}^tT\right|\left|\frac{\partial}{\partial T}(n_l-n_0)\right|(|T|^{s+1}).
\end{aligned}$$

First we apply the determinant $|(\partial/\partial T)(n_l-n_0)|$ to $|T|^{s+1}$, and obtain $(s+n_l)^{(n_0)}|T|^s$ by (2.2). Second we apply $|{}^tT|$ to it, and obtain $(s+n_l)^{(n_0)}|T|^{s+1}$, and so on. Finally we obtain

$$\psi(\partial)(\psi^{s+1}) = (s+n_1)^{(n_0)}(s+n_2)^{(n_0)}\cdots(s+n_l)^{(n_0)}\psi^s,$$

and hence the b-function is $b_\psi(s) = (s+n_1)^{(n_0)}(s+n_2)^{(n_0)}\cdots(s+n_l)^{(n_0)}$. □

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