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Representations of Clifford algebras and local functional equations

By

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Introduction

Let P and P^* be homogeneous polynomials in n variables of degree d with real coefficients. It is an interesting problem both in Analysis and in Number theory to find a condition on P and P^* under which they satisfy a functional equation, roughly speaking, of the form

$$\text{the Fourier transform of } |P(x)|^s = \text{Gamma factor} \times |P^*(y)|^{-n/d-s}. \tag{0.1}$$

A beautiful answer to this problem is given by the theory of prehomogeneous vector spaces due to Mikio Sato. Namely, if P and P^* are relative invariants of a regular prehomogeneous vector space and its dual, respectively, and if the characters χ and χ^* corresponding to P and P^*, respectively, satisfy the relation $\chi\chi^* = 1$, then, P and P^* satisfy a functional equation (see [9], [10], [6]).

Meanwhile, in [5], Faraut and Koranyi developed a method of constructing polynomials with the property (0.1), starting from representations of Euclidean (formally real) Jordan algebras. What is remarkable in their result is that, from representations of simple Jordan algebras of Lorentzian type, one can obtain a series of polynomials satisfying (0.1), which are not covered by the theory of prehomogeneous vector spaces (see also Clerc [4]). Thus we got to know that the class of polynomials with the property

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(0.1) is broader than the class of relative invariants of regular prehomogeneous vector spaces.

In [7], the first author gave a new construction of polynomials with the property (0.1), which includes the result of Faraut and Koranyi as a special case. Now we explain the construction briefly. Suppose that we are given homogeneous polynomials P and P^* on a real vector spaces V and its dual V^*, respectively, satisfying a functional equation of the form (0.1). Further suppose that there exists a non-degenerate quadratic mapping Q (resp. Q^*) of another real vector space W (resp. W^*) to V (resp. V^*), and Q and Q^* are dual. Then, the pullback of the functional equation for P and P^* by Q holds; namely, the pullbacks $\tilde{P}$ and $\tilde{P}^*$ of P and P^* by Q and Q^*, respectively, satisfy a functional equation of the form (0.1) and the gamma factors for the new functional equation have an explicit expression in term of those for P and P^*. A precise formulation of this result will be given in Section 1. For the proof we refer to [7].

In Section 2, we apply the general result in Section 1 to the case where $V = V^* = \mathbb{R}^n$, and $P = P^* = (x_1^2 + \cdots + x_p^2) - (x_{p+1}^2 + \cdots + x_{p+q}^2)$. Let C_p and C_q be the Clifford algebras of the positive definite quadratic forms $x_1^2 + \cdots + x_p^2$ and $x_{p+1}^2 + \cdots + x_{p+q}^2$, respectively. Then we can prove that non-degenerate self-dual quadratic mappings $Q : W \to V$ correspond to representations of the tensor product of $C_p \otimes C_q$ and, starting from representations of $C_p \otimes C_q$, we can construct quartic polynomials $\tilde{P} = P \circ Q$ satisfying functional equations of the form (0.1). Among these polynomials we find several new examples of polynomials satisfying functional equations that do not come from prehomogenous vector spaces. The non-prehomogeneous polynomials with the property (0.1) appearing in the work of Faraut, Koranyi and Clerc is a special case where the signature of the quadratic forms P is $(1, n-1)$. To prove that a given homogeneous polynomial $\tilde{P}$ does not come from a prehomogeneous vector space, it is necessary to know about the group $G_{\tilde{P}}$ of linear transformations that leave the polynomial invariant. We give a conjecture of the structure of the Lie algebras $\mathrm{Lie}(G_{\tilde{P}})$ for the pullback $\tilde{P}$ of the quadratic form P and explain some partial results.

It is natural to ask whether global zeta functions with functional equations can be associated with polynomials $\tilde{P}$ and $\tilde{P}^*$ given in [7]. For polynomials obtained from the theory of Faraut and Koranyi, this problem was solved by Achab in [1] and [2]. But her method works only for the case where the fibers $Q^{-1}(v)$ $(P(v) \neq 0)$ are compact and can not apply to our general setting. If the polynomials P and P^* are relative invariants of prehomogeneous vector spaces, then, by generalizing the method of Arakawa [3] and Suzuki [11], we can define global zeta functions for $\tilde{P}$ and prove their analytic properties (analytic continuation and functional equation) (work with K. Tamura). We shall discuss global zeta functions elsewhere.

§ 1. Pullback of local functional equations by quadratic mappings

In this section, we recall the main result of [7].

§ 1.1. Local functional equations

Let $\mathbf{V}$ be a complex vector space of dimension n with real-structure V and $\mathbf{V}^*$ the vector space dual to $\mathbf{V}$. The dual vector space V^* of the real vector space V can be regarded as a real-structure of $\mathbf{V}^*$. Let $P_1, \ldots, P_r$ (resp. $P_1^*, \ldots, P_r^*$) be homogeneous polynomial functions on $\mathbf{V}$ (resp. $\mathbf{V}^*$) defined over $\mathbb{R}$. We put

$$\boldsymbol{\Omega} = \{ v \in \mathbf{V} \mid P_1(v) \cdots P_r(v) \neq 0 \}, \quad \Omega = \boldsymbol{\Omega} \cap V,$$
$$\boldsymbol{\Omega}^* = \{ v* \in \mathbf{V}^* \mid P_1^*(v^*) \cdots P_r^*(v^*) \neq 0 \}, \quad \Omega^* = \boldsymbol{\Omega}^* \cap V.$$

We assume that

(A.1) there exists a biregular rational mapping $\phi : \boldsymbol{\Omega} \to \boldsymbol{\Omega}^*$ defined over $\mathbb{R}$.

Let

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_\nu, \quad \Omega^* = \Omega_1^* \cup \cdots \cup \Omega_\nu^*$$

be the decompositions into connected components of Ω and Ω^*. Note that (A.1) implies that the numbers of connected components of Ω and Ω^* are the same and we may assume that

$$\Omega_j^* = \phi(\Omega_j) \quad (j = 1, \ldots, \nu).$$

For an $s = (s_1, \ldots, s_r) \in \mathbb{C}^r$ with $\Re(s_1), \ldots, \Re(s_r) > 0$, we define a continuous function $|P(v)|_j^s$ on V by

$$|P(v)|_j^s = \begin{cases} \prod_{i=1}^r |P_i(v)|^{s_i}, & v \in \Omega_j, \\ 0, & v \notin \Omega_j. \end{cases}$$

The function $|P(v)|_j^s$ can be extended to a tempered distribution depending on s in $\mathbb{C}^r$ meromorphically. Similarly we define $|P^*(v^*)|_j^s$ ($s \in \mathbb{C}^r$).

We denote by $\mathcal{S}(V)$ and $\mathcal{S}(V^*)$ the spaces of rapidly decreasing functions on the real vector spaces V and V^*, respectively. For $\Phi \in \mathcal{S}(V)$ and $\Phi^* \in \mathcal{S}(V^*)$, we define the local zeta functions by setting

$$\zeta_i(s, \Phi) = \int_V |P(v)|_i^s \, \Phi(v) \, dv, \quad \zeta_i^*(s, \Phi^*) = \int_{V^*} |P^*(v^*)|_i^s \, \Phi^*(v^*) \, dv^* \quad (i = 1, \ldots, \nu).$$

It is well-known that the local zeta functions $\zeta_i(s, \Phi)$, $\zeta_i^*(s, \Phi^*)$ are absolutely convergent for $\Re(s_1), \ldots, \Re(s_r) > 0$ and have analytic continuations to meromorphic functions of s in $\mathbb{C}^r$. We assume the following:

(A.2) There exist an $A \in GL_r(\mathbb{Z})$ and a $\lambda \in \mathbb{C}^r$ such that a functional equation of the form

$$\zeta_i^*((s+\lambda)A, \hat{\Phi}) = \sum_{j=1}^{\nu} \Gamma_{ij}(s)\zeta_j(s,\Phi) \quad (i = 1, \ldots, \nu) \tag{1.1}$$

holds for every $\Phi \in \mathcal{S}(V)$, where $\Gamma_{ij}(s)$ are meromorphic functions on $\mathbb{C}^r$ not depending on Φ with $\det(\Gamma_{ij}(s)) \neq 0$ and

$$\hat{\Phi}(v^*) = \int_V \Phi(v) \exp(-2\pi\sqrt{-1}\langle v, v^*\rangle)\, dv,$$

the Fourier transform of Φ.

A lot of examples of $\{P_1, \ldots, P_r\}$ and $\{P_1^*, \ldots, P_r^*\}$ satisfying (A.1) and (A.2) can be obtained from relative invariants of regular prehomogeneous vector spaces (see [9], [10], [6]). However, in §1, we do not assume here the existence of group action that relates the polynomials to prehomogeneous vector spaces.

§ 1.2. Pullback of local functional equations

Let $\mathbf{W}$ be a complex vector space of dimension m with real structure W and $\mathbf{W}^*$ the vector space dual to $\mathbf{W}$. We consider the dual vector space W^* of W as a real structure of $\mathbf{W}^*$. Suppose that we are given quadratic mappings $Q : \mathbf{W} \to \mathbf{V}$ and $Q^* : \mathbf{W}^* \to \mathbf{V}^*$ defined over $\mathbb{R}$. The mappings $B_Q : \mathbf{W} \times \mathbf{W} \to \mathbf{V}$ and $B_{Q^*} : \mathbf{W}^* \times \mathbf{W}^* \to \mathbf{V}^*$ defined by

$$B_Q(w_1, w_2) := Q(w_1+w_2) - Q(w_1) - Q(w_2), \quad B_{Q^*}(w_1^*, w_2^*) := Q^*(w_1^*+w_2^*) - Q^*(w_1^*) - Q^*(w_2^*)$$

are bilinear. For given $v \in \mathbf{V}$ and $v^* \in \mathbf{V}^*$, the mappings $Q_{v^*} : \mathbf{W} \to \mathbb{C}$ and $Q_v^* : \mathbf{W}^* \to \mathbb{C}$ defined by

$$Q_{v^*}(w) = \langle Q(w), v^*\rangle, \quad Q_v^*(w^*) = \langle v, Q^*(w^*)\rangle$$

are quadratic forms on $\mathbf{W}$ and $\mathbf{W}^*$, which take values in $\mathbb{R}$ on W and W^*, respectively. We assume that Q and Q^* are non-degenerate and dual to each other with respect to the biregular mapping ϕ in (A.1). This means that Q and Q^* satisfy the following:

(A.3) (i) (Nondegeneracy) The open set $\tilde{\mathbf{\Omega}} := Q^{-1}(\mathbf{\Omega})$ (resp. $\tilde{\mathbf{\Omega}}^* = Q^{*-1}(\mathbf{\Omega}^*)$) is not empty and the rank of the differential of Q (resp. Q^*) at $w \in \tilde{\mathbf{\Omega}}$ (resp. $w^* \in \tilde{\mathbf{\Omega}}^*$) is equal to n. (In particular, $m \geq n$.)

(ii) (Duality) For any $v \in \mathbf{\Omega}$, the quadratic forms $Q_{\phi(v)}$ and Q_v^* are dual to each other. Namely, fix a basis of W and the basis of W^* dual to it, and denote by S_{v^*} and S_v^* the matrices of the quadratic forms Q_{v^*} and Q_v^* with respect to the bases. Then $S_{\phi(v)}$ and S_v^* $(v \in \mathbf{\Omega})$ are non-degenerate and $S_{\phi(v)} = (S_v^*)^{-1}$.

Now we collect some elementary consequences of the assumptions (A.1) and (A.3). First note that a rational function defined over $\mathbb{R}$ with no zeros and no poles on $\mathbf{\Omega}$ (resp. $\mathbf{\Omega}^*$) is a monomial of $P_1, \dots, P_r$ (resp. $P_1^*, \dots, P_r^*$). Hence the assumptions (A.1) and (A.3) (ii) imply the following lemma.

Lemma 1.1. *If we replace P_i, P_j^*, ϕ by their suitable real constant multiples (if necessary),*

(1) *there exists a $B = (b_{ij}) \in GL_r(\mathbb{Z})$ such that*

$$P_i^*(\phi(v)) = \prod_{j=1}^{r} P_j(v)^{b_{ij}} \quad (i = 1, \dots, r).$$

(2) *There exist $\kappa, \kappa^* \in \mathbb{Z}^r$ and a non-zero constant α such that*

$$\det S_v^* = \alpha^{-1} P^{\kappa}(v), \quad \det S_{v^*} = \alpha P^{*\kappa^*}(v^*).$$

(3) *The mapping ϕ is of degree -1 and there exists a $\mu \in \mathbb{Z}^r$ such that*

$$\det \left(\frac{\partial \phi(v)_i}{\partial v_j} \right) = \pm P^{\mu}(v).$$

If $P_1, \dots, P_r$ and $P_1^*, \dots, P_r^*$ are the fundamental relative invariants of a regular prehomogeneous vector space $(G, \rho, \mathbf{V})$ and its dual $(G, \rho^*, \mathbf{V}^*)$, then we have $B = A^{-1}$. Indeed, by the regularity, there exists a relative invariant P for which $\phi(v) = \operatorname{grad} \log P$ is a G-equivariant morphism satisfying (A.1). From the G-equivariance of the mapping ϕ ([8, §4, Prop. 9]), we have $B = A^{-1}$ (see [6]). It is very likely that the identity $B = A^{-1}$ always holds under the assumption (A.1) and (A.2) and, for simplicity, we assume

(A.4) $B = A^{-1}$.

Since we assumed that Ω_i (resp. Ω_i^*) are connected components, the signature of the quadratic form $Q_v^*(w^*)$ (resp. $Q_{v^*}(w)$) on W^* (resp. W) do not change when v (resp. v^*) varies on Ω_i (resp. Ω_i^*). Let p_i and q_i be the numbers of positive and negative eigenvalues of Q_v^* for $v \in \Omega_i$ and put

$$\gamma_i = \exp\left(\frac{(p_i - q_i)\pi\sqrt{-1}}{4} \right) \quad (i = 1, \dots, \nu). \tag{1.2}$$

We put

$$\tilde{P}_i(w) = P_i(Q(w)), \quad \tilde{P}_i^*(w^*) = P_i^*(Q^*(w^*)) \quad (i = 1, \dots, r)$$
$$\tilde{\Omega}_i = Q^{-1}(\Omega_i), \quad \tilde{\Omega}_i^* = {Q^*}^{-1}(\Omega_i^*) \quad (i = 1, \dots, \nu).$$

Some of $\tilde{\Omega}_i$'s and $\tilde{\Omega}_i^*$'s may be empty. We define $\left|\tilde{P}(w)\right|_i^s$ and $\left|\tilde{P}^*(w^*)\right|_i^s$ in the same manner as in §1.1. For $\Psi \in \mathcal{S}(W)$, we define the zeta functions associated with these polynomials by

$$\tilde{\zeta}_i(s,\Psi) = \int_W \left|\tilde{P}(w)\right|_i^s \Psi(w)\,dw, \quad \tilde{\zeta}_i^*(s,\Psi^*) = \int_{W^*} \left|\tilde{P}^*(w^*)\right|_i^s \Psi^*(w^*)\,dw^*.$$

We denote by $\hat{\Psi}$ the Fourier transform of Ψ:

$$\hat{\Psi}(w^*) = \int_W \Psi(w)\exp(2\pi\sqrt{-1}\langle w, w^*\rangle)\,dw.$$

Then our main result is that the functional equation (1.1) for P_i's and P_j^*'s implies a functional equation for $\tilde{P}_i$'s and $\tilde{P}_j^*$'s and the gamma factors in the new functional equation can be written explicitly. Namely, we have the following theorem.

Theorem 1.2 ([7], Theorem 4). *Under the assumptions* (A.1)–(A.4), *the zeta functions $\tilde{\zeta}_i(s,\Psi)$ and $\tilde{\zeta}_i^*(s,\Psi^*)$ satisfy the functional equation*

$$\tilde{\zeta}_i^*((s+2\lambda+\kappa/2+\mu)A, \hat{\Psi}) = \sum_{j=1}^{\nu} \tilde{\Gamma}_{ij}(s)\tilde{\zeta}_j(s,\Psi),$$

where the gamma factors $\tilde{\Gamma}_{ij}(s)$ are given by

$$\tilde{\Gamma}_{ij}(s) = 2^{-2d(s)-m/2}\,|\alpha|^{1/2}\sum_{k=1}^{\nu}\gamma_k\Gamma_{ik}(s+\lambda+\kappa/2+\mu)\Gamma_{kj}(s).$$

Here we denote by $d(s)$ $(s \in \mathbb{C}^r)$ the homogeneous degree of P^s, namely, $d(s) = \sum_{i=1}^r s_i \deg P_i$.

In the case of single variable zeta functions, namely, in the case of $r = 1$, writing $P = P_1$ and $P^* = P_1^*$, we have the following lemma.

Lemma 1.3. *Assume that $r = 1$. Then we have*

$$A = B = -1, \quad d := \deg P = \deg P^*, \quad \lambda = \frac{n}{d}, \quad \mu = -\frac{2n}{d}, \quad \kappa = \frac{m}{d}.$$

By Lemma 1.3, if $r = 1$, then the functional equation for local zeta functions takes the form

$$\tilde{\zeta}_i^*\left(-s-\frac{m}{2d}, \hat{\Psi}\right) = \sum_{j=1}^{\nu}\tilde{\Gamma}_{ij}(s)\tilde{\zeta}_j(s,\Psi),$$

$$\tilde{\Gamma}_{ij}(s) = 2^{-2ds-m/2}\,|\alpha|^{1/2}\sum_{k=1}^{\nu}\gamma_k\Gamma_{ik}\left(s+\frac{m-2n}{2d}\right)\Gamma_{kj}(s) \tag{1.3}$$

and the b-function is given by

$$\tilde{b}(s) = b(s)b\left(s + \frac{m-2n}{2d}\right), \tag{1.4}$$

where $b(s)$ and $\tilde{b}(s)$ are defined by $P^*(\partial_v)P^s(v) = b(s)P^{s-1}(v)$ and $\tilde{P}^*(\partial_w)\tilde{P}^s(w) = \tilde{b}(s)\tilde{P}^{s-1}(v)$.

Example 1.4. Let V be the vector space of real symmetric matrices of size n and put $P(v) = \det v$. Take a non-degenerate real symmetric matrix Y of size $m > n$ with arbitrary signature. Set $W = M_{m,n}(\mathbb{R})$ and define the quadratic mapping $Q : W \to V$ by $Q(w) = {}^t wYw$. Then $\tilde{P}(w) = \det({}^t wYw)$. The polynomial $\tilde{P}$ is the fundamental relative invariant of the prehomogeneous vector space $(SO(Y) \times GL(n), M_{m,n})$. If we identify the dual space of V (resp. W) with V (resp. W) via the inner product $\langle v, v^* \rangle = \operatorname{tr}({}^t vv^*)$ (resp. $\langle w, w^* \rangle = \operatorname{tr}({}^t ww^*)$, the dual of the mapping Q is given by $Q^*(w^*) = {}^t w^* Y^{-1} w^*$ and the theorem can apply to this case.

Example 1.5. In [5, Chap. 16], Faraut and Koranyi proved that, starting from a representation of a Euclidean Jordan algebra, one can construct polynomials satisfying local functional equations. Their result is covered by Theorem 1.2 (see [7, §2.2]). In [4], Clerc generalized the result of Faraut and Koranyi to several variable zeta functions, which is also covered by Theorem 1.2, and noted that, if the Euclidean Jordan algebra V is of Lorentzian type, then the polynomials $\tilde{P}$ obtained by the Faraut-Koranyi construction are *not* relative invariants of prehomogeneous vector spaces except for some low-dimensional cases (without specifying the exceptions). Let us explain this non-prehomogeneous example without referring to Jordan algebra. Let V be the $q+1$-dimensional real quadratic space of signature $(1, q)$. We fix a basis $\{e_0, e_1, \ldots, e_q\}$ of V, for which the quadratic form is given by

$$P(x_0, x_1, \ldots, x_q) = x_0^2 - x_1^2 - \cdots - x_q^2.$$

Denote by C_q the Clifford algebra of the positive definite quadratic form $x_1^2 + \cdots + x_q^2$ and consider a representation $S : C_q \to M_m(\mathbb{R})$ of C_q on an m-dimensional $\mathbb{R}$-vector space. We may assume that $S_i := S(e_i)$ $(i = 1, \ldots, q)$ are symmetric matrices. We denote by $W = \mathbb{R}^m$ the representation space of S and define a quadratic mapping $Q : W \to V$ by

$$Q(w) = ({}^t ww)e_0 + \sum_{i=1}^{q} ({}^t wS_1w, \ldots, {}^t wS_qw)e_i.$$

Then, if $\tilde{P}(w) = P(Q(w)) = ({}^t ww)^2 - \sum_{i=1}^{q} ({}^t wS_iw)^2$ does not vanish identically, Q is a self-dual non-degenerate quadratic mapping and, by Theorem 1.2, $\tilde{P}$ satisfies a local functional equation. In the next section, we generalize this construction and examine the prehomogeneity.

Remark. In [4], Clerc proved local functional equations also for zeta functions with harmonic polynomials. This part is not covered by Theorem 1.2.

§ 2. Quartic polynomials obtained from representations of Clifford algebras

Let p, q be non-negative integers and consider the quadratic form $P(x) = \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} x_{p+j}^2$ of signature (p,q) on $V = \mathbb{R}^{p+q}$. We identify V with its dual vector space via the standard inner product $(x,y) = x_1y_1 + \cdots + x_{p+q}y_{p+q}$. Put $\Omega = V \setminus \{P = 0\}$. We determine the quadratic mappings $Q : W \to V$ that are self-dual with respect to the biregular mapping $\phi : \Omega \longrightarrow \Omega$ defined by

$$\phi(v) := \frac{1}{2}\mathrm{grad} \log P(v) = \frac{1}{P(v)}(v_1, \ldots, v_p, -v_{p+1}, \ldots, -v_{p+q}).$$

By Theorem 1.2, for such a quadratic mapping Q, the complex power of the quartic polynomial $\tilde{P}(w) := P(Q(w))$ satisfies a functional equation with explicit gamma factor, unless $\tilde{P}$ vanishes identically (see [7, Lemma 6]).

For a quadratic mapping Q of $W = \mathbb{R}^m$ to $V = \mathbb{R}^{p+q}$, there exist symmetric matrices $S_1, \ldots, S_{p+q}$ of size m such that

$$Q(w) = ({}^t w S_1 w, \ldots, {}^t w S_{p+q} w).$$

For $v = (x_1, \ldots, x_{p+q}) \in \mathbb{R}^{p+q}$, we put

$$S(v) = \sum_{i=1}^{p+q} x_i S_i.$$

Then the mapping Q is self-dual with respect to ϕ if and only if

$$S(v)S(\phi(v)) = I_m \quad (v \in \Omega).$$

If we define ϵ_i to be 1 or -1 according as $i \le p$ or $i > p$, this condition is equivalent to the polynomial identity

$$\sum_{i=1}^{p} x_i^2 S_i^2 - \sum_{j=1}^{q} x_{p+j}^2 S_{p+j}^2 + \sum_{1 \le i < j \le p+q} x_i x_j \left(\epsilon_j S_i S_j + \epsilon_i S_j S_i\right) = P(x) I_m.$$

This identity holds if and only if

$$S_i^2 = I_m \ (1 \le i \le p+q),$$

$$S_i S_j = \begin{cases} S_j S_i & (1 \le i \le p < j \le p+q \text{ or } 1 \le j \le p < i \le p+q) \\ -S_j S_i & (1 \le i, j \le p \textit{ or } p+1 \le i, j \le p+q). \end{cases}.$$

This means that the mapping $S : V \to Sym_m(\mathbb{R})$ can be extended to a representation of the tensor product of the Clifford algebra C_p of $x_1^2 + \cdots + x_p^2$ and the Clifford algebra C_q of $x_{p+1}^2 + \cdots + x_{p+q}^2$.

Conversely, if we are given a representation $S : C_p \otimes C_q \to M_m(\mathbb{R})$, then the representation S is a direct sum of simple modules and a simple $C_p \otimes C_q$-module is a tensor product of simple modules of C_p and C_q. Since one can choose a basis of the representation space so that $S(\mathbb{R}^{p+q})$ is contained in $Sym_m(\mathbb{R})$, we have proved that

Theorem 2.1. *Self-dual quadratic mappings Q of $W = \mathbb{R}^m$ to the quadratic space (V, P) correspond to representations S of $C_p \otimes C_q$ such that $S(V) \subset Sym_m(\mathbb{R})$.*

The construction above is a generalization of Example 1.5 related to representations of simple Euclidean Jordan algebra of Lorentzian type. In the case $(p, q) = (1, q)$, the self-dual quadratic mappings over the quadratic space of signature $(1, q)$ correspond to representations of $C_1 \otimes C_q \cong C_q \oplus C_q$. Representations of $C_1 \otimes C_q$ can be identified with the direct sum of 2 C_q-modules W_+ and W_-. On W_+ (resp. W_-), e_0 acts as multiplication by $+1$ (resp. -1). The Lorentzian case in the Faraut-Koranyi construction is the one for which $W_- = \{0\}$.

The quartic polynomials $\tilde{P}$ ($= \tilde{P}^*$) above are conjectured not to be relative invariants of prehomogeneous vector spaces except for low-dimensional cases. It is an interesting problem to classify the prehomogeneous case.

Theorem 2.2. *If $p + q = \dim V \leq 4$, then the polynomials $\tilde{P}$ are relative invariants of prehomogeneous vector spaces.*

The prehomogeneous vector spaces appearing in the case $p + q \leq 4$ are given in the following table:

(p, q)	prehomogeneous vector space
$(1, 0)$	$(GL(1, \mathbb{R}) \times SO(k_1, k_2), \mathbb{R}^{k_1+k_2})$
$(2, 0)$	$(GL(1, \mathbb{C}) \times SO(k, \mathbb{C}), \mathbb{C}^k)$
$(1, 1)$	$(GL(1, \mathbb{R}) \times SO(k_1, k_2), \mathbb{R}^{k_1+k_2}) \oplus (GL(1, \mathbb{R}) \times SO(k_3, k_4), \mathbb{R}^{k_3+k_4})$
$(3, 0)$	$(GL(1, \mathbb{R}) \times SU(2) \times SO^*(2k), \mathbb{C}^{2k})$
$(2, 1)$	$(GL(2, \mathbb{R}) \times SO(k_1, k_2), M(2, k_1 + k_2, \mathbb{R}))$
$(4, 0)$	$(GL(1, \mathbb{H}) \times GL(1, \mathbb{H}) \times GL(k, \mathbb{H}), M(2, k, \mathbb{H}))$
$(3, 1)$	$(GL_2(\mathbb{C}) \times SU(k_1, k_2), M(2, k_1 + k_2; \mathbb{C}))$
$(2, 2)$	$(GL_2(\mathbb{R}) \times GL_2(\mathbb{R}) \times SL(k, \mathbb{R}), M(2, k; \mathbb{R})^{\oplus 2})$

(Table 1)

Here k, k_1, k_2, k_3, k_4 denote the multiplicities of simple $C_p \otimes C_q$-modules in W.

If $p+q=5$, then $\tilde{P}$ is not a relative invariant of any prehomogeneous vector spaces except the case of simple module. If $p+q=6$ and W is simple, then $\tilde{P}$ vanishes. However, if W is not simple but pure, then $\tilde{P}$ is a relative invariant of the prehomogeous vector space

$(3,3)$	$(GL_4(\mathbb{R})\times Sp(k,\mathbb{R}), M(4,2k;\mathbb{R}))\quad (k\geq 2,\ W=\text{pure})$
$(5,1)$	$(GL_2(\mathbb{H})\times Sp(k_1,k_2), M(2,k_1+k_2;\mathbb{H}))\ (k_1+k_2\geq 2,\ W=\text{pure})$

(Table $1'$)

Here a $C_p\otimes C_q$-module W is called *pure* if $W\otimes_{\mathbb{R}}\mathbb{C}$ is isotypic as a module of the subalgebra of even elements in $(C_p\otimes C_q)\otimes_{\mathbb{R}}\mathbb{C}$. Pure modules do not appear for $(p,q)=(6,0)$ and $(4,2)$.

To examine higher dimensional cases, we consider the Lie algebra $\mathfrak{g}$ of the group $G=\left\{g\in GL(W)\;\middle|\;\tilde{P}(gw)\equiv\tilde{P}(w)\right\}$ and the Lie algebra $\mathfrak{h}$ ($=\mathfrak{h}_{p,q}$) of the group $H=\{h\in GL(W)\mid Q(hw)\equiv Q(w)\}$. The Lie algebra $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Note that the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ depend on p,q and the choice of the representation of $C_p\otimes C_q$. Our problem is to determine all the cases where $(GL_1\times G, W)$ is a prehomogeneous vector space.

Conjecture 2.3. We have

$$\mathfrak{g}\cong\mathfrak{so}(p,q)\oplus\mathfrak{h}$$

except for some low dimensional cases.

The structure of $\mathfrak{h}$ can be described explicitly. By the periodicity of Clifford algebras $C_{p+8}\cong C_p\otimes M(16,\mathbb{R})$, there exists a natural correspondence between representations of $C_{p+8}\otimes C_q$ and representations of $C_p\otimes C_q$ and it can be proved that the structure of $\mathfrak{h}$ is the same for corresponding representations. This implies the isomorphisms

$$\mathfrak{h}_{p,q}\cong\mathfrak{h}_{q,p}\cong\mathfrak{h}_{p+8,q}\cong\mathfrak{h}_{p,q+8}. \tag{2.1}$$

Similarly, by $C_{p+4}\cong C_p\otimes M(2,\mathbb{H})$ and $M_2(\mathbb{H})\otimes M_2(\mathbb{H})\cong M_{16}(\mathbb{R})$, we have the isomorphism

$$\mathfrak{h}_{p,q}\cong\mathfrak{h}_{p+4,q\pm 4}. \tag{2.2}$$

Hence it is sufficient to give the structure of $\mathfrak{h}$ only for $0\leq p\leq 7$ and $0\leq q\leq 4$.

Theorem 2.4. *The Lie algebra $\mathfrak{h}$ is isomorphic to the reductive Lie algebra given* *in the following table:*

$\bar p \backslash \bar q$	0	1	2	3
0	$\mathfrak{gl}(k,\mathbb{R})$	$\mathfrak{so}(k_1,k_2)$	$\mathfrak{so}(k,\mathbb{C})$	$\mathfrak{so}^*(2k)$
1	$\mathfrak{so}(k_1,k_2)$	$\mathfrak{so}(k_1,k_2)\oplus\mathfrak{so}(k_3,k_4)$	$\mathfrak{so}(k_1,k_2)$	$\mathfrak{u}(k_1,k_2)$
2	$\mathfrak{so}(k,\mathbb{C})$	$\mathfrak{so}(k_1,k_2)$	$\mathfrak{gl}(k,\mathbb{R})$	$\mathfrak{sp}(k,\mathbb{R})$
3	$\mathfrak{so}^*(2k)$	$\mathfrak{u}(k_1,k_2)$	$\mathfrak{sp}(k,\mathbb{R})$	$\mathfrak{sp}(k_1,\mathbb{R})\oplus\mathfrak{sp}(k_2,\mathbb{R})$
4	$\mathfrak{gl}(k,\mathbb{H})$	$\mathfrak{sp}(k_1,k_2)$	$\mathfrak{sp}(k,\mathbb{C})$	$\mathfrak{sp}(k,\mathbb{R})$
5	$\mathfrak{sp}(k_1,k_2)$	$\mathfrak{sp}(k_1,k_2)\oplus\mathfrak{sp}(k_3,k_4)$	$\mathfrak{sp}(k_1,k_2)$	$\mathfrak{u}(k_1,k_2)$
6	$\mathfrak{sp}(k,\mathbb{C})$	$\mathfrak{sp}(k_1,k_2)$	$\mathfrak{gl}(k,\mathbb{H})$	$\mathfrak{so}^*(2k)$
7	$\mathfrak{sp}(k,\mathbb{R})$	$\mathfrak{u}(k_1,k_2)$	$\mathfrak{so}^*(2k)$	$\mathfrak{so}^*(2k_1)\oplus\mathfrak{so}^*(2k_2)$

(Table 2)

Here $\bar p = p \bmod 8$ *and* $\bar q = q \bmod 8$ *and* k_1, k_2, k_3, k_4, k *are the multiplicities of simple modules in* W.

Put $R_{p,q} = C_p \otimes C_q$ and let $R^+_{p,q}$ be the subalgebra of even elements in $R_{p,q}$. Then, $R^+_{p,q}$ is isomorphic to $C^+_{p,q}$, the subalgebra of even elements of the Clifford algebra $C_{p,q}$ of the quadratic form P of signature (p,q). As we can see from the table below, the structure of the Lie algebra $\mathfrak{h}$ is completely determined by the structure of $R_{p,q}$ and $R^+_{p,q}$.

<table>
<tr><th>$(R_{p,q}, R^+_{p,q})$</th><th>$(\mathbb{K},\mathbb{K}')$</th><th>$\mathfrak{h}$</th><th>$(\bar p,\bar q)$ $(\bar p \le \bar q)$</th></tr>
<tr><td rowspan="4">(T, T')</td><td>$(\mathbb{R},\mathbb{C})$</td><td>$\mathfrak{so}(k,\mathbb{C})$</td><td>$(0,2),(4,6)$</td></tr>
<tr><td>$(\mathbb{C},\mathbb{R})$</td><td>$\mathfrak{sp}(k,\mathbb{R})$</td><td>$(0,7),(2,3),(3,4),(6,7)$</td></tr>
<tr><td>$(\mathbb{C},\mathbb{H})$</td><td>$\mathfrak{so}^*(2k)$</td><td>$(0,3),(2,7),(3,6),(4,7)$</td></tr>
<tr><td>$(\mathbb{H},\mathbb{C})$</td><td>$\mathfrak{sp}(k,\mathbb{C})$</td><td>$(0,6),(2,4)$</td></tr>
<tr><td rowspan="2">$(T, T'\oplus T')$</td><td>$(\mathbb{R},\mathbb{R})$</td><td>$\mathfrak{gl}(k,\mathbb{R})$</td><td>$(0,0),(2,2),(4,4),(6,6)$</td></tr>
<tr><td>$(\mathbb{H},\mathbb{H})$</td><td>$\mathfrak{gl}(k,\mathbb{H})$</td><td>$(0,4),(2,6)$</td></tr>
<tr><td rowspan="3">$(T\oplus T, T')$</td><td>$(\mathbb{R},\mathbb{R})$</td><td>$\mathfrak{so}(k_1,k_2)$</td><td>$(0,1),(1,2),(4,5),(5,6)$</td></tr>
<tr><td>$(\mathbb{C},\mathbb{C})$</td><td>$\mathfrak{u}(k_1,k_2)$</td><td>$(1,3),(1,7),(3,5),(5,7)$</td></tr>
<tr><td>$(\mathbb{H},\mathbb{H})$</td><td>$\mathfrak{sp}(k_1,k_2)$</td><td>$(0,5),(1,4),(1,6),(2,5)$</td></tr>
<tr><td rowspan="2">$(T\oplus T, T'\oplus T')$</td><td>$(\mathbb{C},\mathbb{R})$</td><td>$\mathfrak{sp}(k_1,\mathbb{R})\oplus\mathfrak{sp}(k_2,\mathbb{R})$</td><td>$(3,3),(7,7)$</td></tr>
<tr><td>$(\mathbb{C},\mathbb{H})$</td><td>$\mathfrak{so}^*(2k_1)\oplus\mathfrak{so}^*(2k_2)$</td><td>$(3,7)$</td></tr>
<tr><td rowspan="2">$(T\oplus T\oplus T\oplus T, T'\oplus T')$</td><td>$(\mathbb{R},\mathbb{R})$</td><td>$\mathfrak{so}(k_1,k_2)\oplus\mathfrak{so}(k_3,k_4)$</td><td>$(1,1),(5,5)$</td></tr>
<tr><td>$(\mathbb{H},\mathbb{H})$</td><td>$\mathfrak{sp}(k_1,k_2)\oplus\mathfrak{sp}(k_3,k_4)$</td><td>$(1,5)$</td></tr>
</table>

(Table 2′)

Here T and T' denote the matrix algebras over $\mathbb{K}$ and $\mathbb{K}'$, respectively, of appropriate size.

Sketch of the proof of Theorem 2.4. Let $\mathcal{A}$ be the commutant of $R^+_{p,q}$ in $\mathrm{End}(W)$. For an r in $R_{p,q}$, we put $\mathcal{A}^r = \{X \in \mathcal{A} \mid {}^t X S(r) + S(r) X = 0\}$. Denote by $e_1, \dots, e_{p+q}$ the standard basis of $V = \mathbb{R}^{p+q}$. If $r = e_i r'$ with $r' \in (R^+_{p,q})^\times$, then $\mathcal{A}^r$ coincides with $\mathfrak{h}$. For an appropriate choice of r, it is not hard to calculate $\mathcal{A}^r$.

As an example, consider the case $(p,q) = (1,1)$. Then $R_{1,1}$ is isomorphic to $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ and there exist 4 inequivalent simple $R_{1,1}$-modules W_1, W_2, W_3, W_4 of dimension 1. The action of e_1, e_2 are given by

$$\begin{cases} e_1 \cdot w_1 = w_1, \\ e_2 \cdot w_1 = w_1, \end{cases} \quad \begin{cases} e_1 \cdot w_2 = -w_2, \\ e_2 \cdot w_2 = -w_2, \end{cases} \quad \begin{cases} e_1 \cdot w_3 = w_3, \\ e_2 \cdot w_3 = -w_3, \end{cases} \quad \begin{cases} e_1 \cdot w_4 = -w_4, \\ e_2 \cdot w_4 = w_4. \end{cases}$$

Since the subalgebra $R^+_{1,1}$ is generated by $e_1 e_2$, $W_1 \cong W_2$ and $W_3 \cong W_4$ as $R^+_{1,1}$-modules. Let $W = W_1^{k_1} \oplus W_2^{k_2} \oplus W_3^{k_3} \oplus W_4^{k_4}$ and identify W with the space of $k_1+k_2+k_3+k_4$-dimensional row vectors. Then we have

$$\mathcal{A} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \;\middle|\; A \in M_{k_1+k_2}(\mathbb{R}),\ B \in M_{k_3+k_4}(\mathbb{R}) \right\}.$$

Take $r = e_1$. Then the action of r on W is given by the right multiplication of the matrix

$$\begin{pmatrix} I_{k_1} & 0 & 0 & 0 \\ 0 & -I_{k_2} & 0 & 0 \\ 0 & 0 & I_{k_3} & 0 \\ 0 & 0 & 0 & -I_{k_4} \end{pmatrix}$$

and we obtain

$$\mathfrak{h} = \mathcal{A}^r = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \;\middle|\; \begin{matrix} A \in \mathfrak{so}(k_1, k_2), \\ B \in \mathfrak{so}(k_3, k_4) \end{matrix} \right\} \cong \mathfrak{so}(k_1,k_2) \oplus \mathfrak{so}(k_3,k_4).$$

If $p+q$ is relatively small, we can check Conjecture 2.3 with the aid of a symbolic calculation engine such as Maple or Mathematica and the result can be summarized in the following table.

Theorem 2.5. Under the convention

$$\begin{aligned} m_0 &= \text{minimum of the dimensions of the simple } C_p \otimes C_q\text{-modules}, \\ m &= \dim W, \\ 0 &\Longleftrightarrow \tilde{P} \equiv 0 \quad \text{(degenerate case)}, \\ \mathrm{T} &\Longleftrightarrow \mathfrak{g}_{p,q}(\rho) = \mathfrak{so}(p,q) \oplus \mathfrak{h}_{p,q}(\rho) \quad \text{(Conjecture 2.3 is true.)}, \\ \mathrm{F} &\Longleftrightarrow \mathfrak{g}_{p,q}(\rho) \supsetneqq \mathfrak{so}(p,q) \oplus \mathfrak{h}_{p,q}(\rho) \quad \text{(Conjecture 2.3 fails.)}, \\ \mathrm{pv} &\Longleftrightarrow \tilde{P} \text{ is a relative invariant of a pv}, \\ \mathrm{pure} &\Longleftrightarrow \text{(all the } R^+_{p,q} \otimes_{\mathbb{R}} \mathbb{C}\text{-simple modules in } W \otimes_{\mathbb{R}} \mathbb{C} \text{ are isomorphic)}, \\ \mathrm{mixed} &\Longleftrightarrow (W \text{ is not pure}), \end{aligned}$$

we have

$p+q$	m_0	$m=m_0$	$m=2m_0$	$m \geq 3m_0$
1	1	T, pv	T, pv	T, pv
2	1	0 0	(mixed) T, pv (pure) 0	(mixed) T, pv (pure) 0
3	2	0	F, pv	T, pv
4	4	0	F, pv	T, pv
5	8	F, pv	T	T
6	8	0	F, pv	T (pure) pv (mixed) non-pv
7	16	F, pv	T	T
8	16	F, pv	T	T
9	16	F, pv	T	T
10	16	0	F (pure) pv (mixed) non-pv	T
11	32	F, pv	T	T

(Table 3)

Almost all the non-degenerate cases in Theorem 2.5 for which Conjecture 2.3 fail are prehomogeneous cases and are given in the following table:

$p+q$	$\dim W$	$\mathfrak{g} \otimes \mathbb{C}$	$(\mathfrak{so}(p,q) \oplus \mathfrak{h}) \otimes \mathbb{C}$
3	4	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	$\mathfrak{sl}(2) \oplus \mathfrak{so}(2)$
4	8	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(2)$	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(2)$
5	8	$\mathfrak{so}(8)$	$\mathfrak{so}(5) \oplus \mathfrak{sp}(1)$
6	16 (pure)	$\mathfrak{sl}(4) \oplus \mathfrak{sl}(4)$	$\mathfrak{sl}(4) \oplus \mathfrak{sp}(2)$
6	16 (mixed)	$\mathfrak{sl}(4) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(1)$	$\mathfrak{sl}(4) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$
7	16	$\mathfrak{so}(8) \oplus \mathfrak{sl}(2)$	$\mathfrak{so}(7) \oplus \mathfrak{sl}(2)$
8	16	$\mathfrak{so}(8) \oplus \mathfrak{so}(8) \oplus \mathfrak{gl}(1)$	$\mathfrak{so}(8) \oplus \mathfrak{gl}(1)$
9	16	$\mathfrak{so}(16)$	$\mathfrak{so}(9)$
10	32 (pure)	$\mathfrak{so}(10) \oplus \mathfrak{sl}(2)$	$\mathfrak{so}(10) \oplus \mathfrak{so}(2)$
11	32	$\mathfrak{so}(12)$	$\mathfrak{so}(11)$

(Table 4)

The unique non-prehomogeneous case is

$p+q$	$\dim W$	$\mathfrak{g} \otimes \mathbb{C}$	$(\mathfrak{so}(p,q) \oplus \mathfrak{h}) \otimes \mathbb{C}$
10	32 (mixed)	$\mathfrak{so}(10) \oplus \mathfrak{gl}(1)$	$\mathfrak{so}(10)$

(Table 5)

The following is a refinement of Conjecture 2.3.

Conjecture 2.6. Conjecture 2.3 is true for $p+q \geq 12$.

Conjecture 2.6 implies that prehomogeneous cases do not appear for $p+q \geq 12$ and all the prehomogeneous cases are listed in Tables 1, $1'$ and 4.

Remark. (Added in proof) Conjectures 2.3 and 2.6 are now theorems. The proof will appear elsewhere.

References

[1] D. Achab, Représentations des algèbres de rang 2 et fonctions zêta associées, *Ann. Inst. Fourier* **45**(1995), 437–451.

[2] D. Achab, Zeta functions of Jordan algebras representations, *Ann. Inst. Fourier* **45**(1995), 1283–1303.

[3] T.Arakawa, Dirichlet series related to the Eisenstein series on the Siegel upper half-plane, *Comment. Math. Univ. St. Pauli* **27**(1978), 29–42.

[4] J.-L. Clerc, Zeta distributions associated to a representation of a Jordan algebra, Math. Z. **239**(2002), 263–276.

[5] J. Faraut and A. Koranyi, *Analysis of symmetric cones*, Oxford University Press, 1994.

[6] F. Sato, Zeta functions in several variables associated with prehomogeneous vector spaces I: Functional equations, *Tôhoku Math. J.* **34**(1982), 437–483.

[7] F. Sato, Quadratic maps and nonprehomogeneous local functional equations, *Comment. Math. Univ. St. Pauli* **56**(2007), 163–184.

[8] M. Sato and T. Kimura, A classification of irreducible prehomogeneous vector spaces and their invariants, *Nagoya Math. J.* **65**(1977), 1–155.

[9] M. Sato, Theory of prehomogeneous vector spaces (Notes taken by T.Shintani in Japanese), *Sugaku no Ayumi* **15**(1970), 85–157.

[10] M. Sato and T. Shintani, On zeta functions associated with prehomogenous vector spaces, *Ann. of Math.* **100**(1974), 131–170.

[11] T.Suzuki, Distributions with automorphy and Dirichlet series, *Nagoya Math. J.* **73**(1979), 157–169.