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On the linearity of causal automorphisms of symmetric cones

By

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Abstract

We study the unique extension of local causal automorphisms of the Shilov boundary of an irreducible bounded symmetric domain of tube type. As an application, we show the linearity of causal automorphisms of symmetric cones in a simple Euclidean Jordan algebra.

Introduction

Let $\mathfrak{g}_1$ be a simple Euclidean Jordan algebra of degree r. Then each element $X \in \mathfrak{g}_1$ has a signature (K[3]), which is uniquely determined by X and is a pair (p, q) of non-negative integers p and q with $p + q \leq r$. Let $\Omega_{p,q}$ be the totality of elements $X \in \mathfrak{g}_1$ with signature (p, q). The structure group $\mathrm{Str}(\mathfrak{g}_1)$ of $\mathfrak{g}_1$ is a subgroup of $\mathrm{GL}(\mathfrak{g}_1)$ having the Jordan determinant as a relative invariant polynomial. The identity component of the structure group $\mathrm{Str}(\mathfrak{g}_1)$ is denoted by $\mathrm{Str}^0(\mathfrak{g}_1)$. Each $\Omega_{p,q}$ is a $\mathrm{Str}^0(\mathfrak{g}_1)$-orbit, and we have the $\mathrm{Str}^0(\mathfrak{g}_1)$-orbit decomposition (i.e. Sylvester's law of inertia)

$$\mathfrak{g}_1 = \coprod_{p+q \leq r} \Omega_{p,q}. \tag{$*$}$$

$\Omega_{p,q}$ is open if and only if $p + q = r$. $\Omega_{r,0} (= -\Omega_{0,r})$ is a Riemannian symmetric convex cone, and $\Omega_{i,r-i} (i \neq 0, r)$ is an affine symmetric non-convex cone. The (linear) automorphism group $G(\Omega_{r,0})$ of $\Omega_{r,0}$ is defined to be the subgroup of elements of $\mathrm{GL}(\mathfrak{g}_1)$ leaving $\Omega_{r,0}$ stable. It is known that $G(\Omega_{r,0})$ is an open subgroup of $\mathrm{Str}(\mathfrak{g}_1)$. Note that $G(\Omega_{r,0})$ leaves the orbit decomposition $(*)$ stable.

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Let Ω denote any one of open cones $\Omega_{i,r-i}$, for simplicity. The closure C of $\Omega_{r,0}$ is a causal cone (cf. **1.2**). By attaching the parallel transport of C to each point of $\mathfrak{g}_1$, we have a parallel cone field $\mathcal{C}_{\mathfrak{g}_1}$ on $\mathfrak{g}_1$, which is a causal structure on $\mathfrak{g}_1$ with model cone C (cf. Definition 1.2). The restriction $\mathcal{C}_\Omega$ of $\mathcal{C}_{\mathfrak{g}_1}$ to Ω is a causal structure on Ω.

The purpose of this paper is to announce the following result (Theorem 3.3): Any diffeomorphism of Ω leaving the causal structure $\mathcal{C}_\Omega$ invariant is necessarily a linear map belonging to $G(\Omega_{r,0})$, provided that $\dim \mathfrak{g}_1 \geq 3$.

The proof heavily depends on the Liouville-type theorem (Theorem 2.1) for the causal structure on the Shilov boundary M^- of the symmetric tube domain D over the symmetric cone $\Omega_{r,0}$. M^- is a causal flag manifold of the holomorphic automorphism group $G(D)$ of D, and a symmetric cone Ω is realized in M^- as a causal open submanifold. Theorem 2.1 guarantees that the causal automorphism group of Ω is a subgroup of $G(D)$. We then conclude that this subgroup acts linearly on $\mathfrak{g}_1$ (Theorem 3.3).

In §1, we interpret a causal structure $\mathcal{C}$ as a G-structure $Q(\mathcal{C})$ and mention the coincidence of the automorphism groups of both structures (Theorem 1.6). In **1.3**, we give a graded Lie algebra approach to the Sylvester's law of inertia (Proposition 1.7). Note that two groups $G(\Omega_{r,0})$ and $G_0(D)$ are identical.

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The detailed account of this paper is included in our paper [6].

§ 1. Preliminaries

1.1. Let M be an n-dimensional smooth manifold. By a *frame* on M at $p \in M$ we mean a basis $(u_1, \cdots, u_n)$ of the tangent space $T_p(M)$. Let $(e_1, \cdots, e_n)$ be the standard basis of $\mathbb{R}^n$. We will identify a frame $(u_1, \cdots, u_n)$ at a point p with the linear isomorphism u of $\mathbb{R}^n$ onto $T_p(M)$ defined by $u(e_i) = u_i,\ 1 \leq i \leq n$. The totality $F(M)$ of frames on M is called the *frame bundle* of M, which is a principal bundle over M with structure group $\mathrm{GL}(n, \mathbb{R})$. The fiber of $F(M)$ over $p \in M$ is the totality $\mathrm{Isom}(\mathbb{R}^n, T_p(M))$ of linear isomorphisms of $\mathbb{R}^n$ onto $T_p(M)$.

Let G be a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$. A subbundle Q of $F(M)$ with G as the structure group is called a *G-structure* on M.

Definition 1.1. Let (M, Q) and (M', Q') be two manifolds with G-structures Q and Q', respectively. Let f be a diffeomorphism of M onto M'. Note that f is lifted up to the bundle isomorphism $\overline{f}$ of $F(M)$ onto $F(M')$, which induces f on M. f is called an *isomorphism* of (M, Q) onto (M', Q'), if $\overline{f}(Q) = Q'$ is satisfied.

An *automorphism* of a manifold (M, Q) with a G-structure Q is defined analogously.

The *automorphism group* $\mathrm{Aut}(M, Q)$ of (M, Q) is defined as

$$\mathrm{Aut}(M, Q) = \{f \in \mathrm{Diff}(M) : \overline{f}(Q) = Q\}, \tag{1.1}$$

where $\mathrm{Diff}(M)$ denotes the diffeomorphism group of M. The group $\mathrm{Aut}(M, Q)$ may be infinite-dimensional, in general.

1.2. Let V be a real vector space, and let C be a closed convex cone in V with vertex at the origin $0 \in V$. Then C is said to be a *causal cone*, if the interior of C is not empty and if the relation $C \cap (-C) = (0)$ is valid. For a causal cone C in V, one can define the (linear) *automorphism group* as follows:

$$\mathrm{Aut}(C) = \{g \in \mathrm{GL}(V) : gC = C\}. \tag{1.2}$$

Definition 1.2. (Faraut [1], Hilgert-Ólafsson [2]) Let M be an n-manifold and let $T(M)$ be the tangent bundle of M with standard fiber $\mathbb{R}^n$. Let C be a causal cone in $\mathbb{R}^n$, and let $\mathcal{C} = \{C_p\}_{p \in M}$ be a family of causal cones $C_p \subset T_p(M)$. Then $\mathcal{C}$ is called a *causal structure on M with model cone C*, if there exists a family of local trivialization $\{(U_i, \phi_i)\}_{i \in I}$ of $T(M)$ over M satisfying the condition

$$\phi_i(p, C) = C_p, \ p \in U_i, \ i \in I, \tag{1.3}$$

where $\phi_i : U_i \times \mathbb{R}^n \to T(M)|_{U_i}$ is a vector bundle isomorphism over U_i. The pair $(M, \mathcal{C})$ is called a *causal manifold*.

The condition (1.3) assures the smoothness of the assignment $p \mapsto C_p$.

Definition 1.3. Let $(M, \mathcal{C})$ and $(M', \mathcal{C}')$ be two causal manifolds with model cone C, where $\mathcal{C} = \{C_p\}_{p \in M}$ and $\mathcal{C}' = \{C'_q\}_{q \in M'}$. A diffeomorphism $f : M \to M'$ is said to be a *causal isomorphism*, if $f_* C_p = C'_{f(p)}$ is valid for $p \in M$.

Similarly, we can define a *causal automorphism* of $(M, \mathcal{C})$. The causal automorphism group of $(M, \mathcal{C})$, where $\mathcal{C} = \{C_p\}_{p \in M}$, is defined as

$$\mathrm{Aut}(M, \mathcal{C}) = \{g \in \mathrm{Diff}(M) : g_* C_p = C_{g(p)}, \ p \in M\}, \tag{1.4}$$

which may be infinite-dimensional, in general.

Under the situation in Definition 1.2, it can be seen by using (1.3) that the transition functions of $F(M)$ are $\mathrm{Aut}(C)$-valued with respect to the open covering $\{U_i\}_{i \in I}$, which implies that there exists a subbundle $Q(\mathcal{C})$ of $F(M)$ with structure group $\mathrm{Aut}(C)$. $Q(\mathcal{C})$ is given in the coordinate-free way as follows.

Lemma 1.4. (K[6]) *Let $\mathcal{C} = \{C_p\}_{p\in M}$ be a causal structure on M with model cone C. Then the* $\mathrm{Aut}(C)$*-structure $Q(\mathcal{C})$ on M is given intrinsically as*

$$Q(\mathcal{C}) = \{u \in F(M) : u(C) = C_p,\ p \in M\}. \tag{1.5}$$

We say that $Q(\mathcal{C})$ is the $\mathrm{Aut}(C)$*-structure associated to a causal structure* $\mathcal{C}$. The reverse process is also valid.

Lemma 1.5. (K[6]) *Let C be a causal cone in $\mathbb{R}^n$, $n = \dim M$, and let Q be an* $\mathrm{Aut}(C)$*-structure on M. Then $\mathcal{C}(Q) := \{u(C) : u \in Q\}$ is a causal structure on M with model cone C.*

Theorem 1.6. (K[6]) *Let C be a causal cone in $\mathbb{R}^n$. Let $(M, \mathcal{C})$ and $(M', \mathcal{C}')$ be two causal manifolds with the model cone C. Then a diffeomorphism $f : M \to M'$ is a causal isomorphism, if and only if f is an isomorphism $(M, Q(\mathcal{C})) \to (M', Q(\mathcal{C}'))$ of* $\mathrm{Aut}(C)$*-structures. As a special case we have the following coincidence of the two automorphism groups*

$$\mathrm{Aut}(M, \mathcal{C}) = \mathrm{Aut}(M, Q(\mathcal{C})). \tag{1.6}$$

1.3. Let D be an irreducible bounded symmetric domain of tube type, and let $G(D)$ be the full holomorphic automorphism group of D. The Lie algebra $\mathfrak{g} := \mathrm{Lie}\, G(D)$ is simple of hermitian type, and is expressed as a 3-graded Lie algebra (abbreviated to 3-GLA):

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1\,. \tag{1.7}$$

Let $Z \in \mathfrak{g}_0$ be the characteristic element of $\mathfrak{g}$, that is, a unique element satisfying $\mathrm{ad}\, Z|_{\mathfrak{g}_k} = k1$, and let τ be a grade-reversing (i.e. $\tau(\mathfrak{g}_k) = \mathfrak{g}_{-k}$) Cartan involution of $\mathfrak{g}$. Note that $G(D)$ is a normal subgroup of the Lie algebra automorphism group $\mathrm{Aut}\, \mathfrak{g}$ with index 2. Let $G_0(D)$ be the subgroup of $G(D)$ consisting of all grade-preserving automorphisms. $G_0(D)$ coincides with the centralizer $C(Z)$ of Z in $G(D)$. We have that $\mathrm{Lie}\, G_0(D) = \mathfrak{g}_0$. Also note that $G_0(D)$ acts on $\mathfrak{g}_1$ linearly. Let us consider two maximal parabolic subgroups $U^{\pm}(D) := G_0(D) \exp \mathfrak{g}_{\pm 1}$ of $G(D)$, which are opposite to each other. The flag manifold

$$M^- = G(D)/U^-(D) \tag{1.8}$$

is the Shilov boundary of D with respect to a suitable choice of invariant complex structures of D (Koranyi-Wolf [7]). M^- is expressed as a Riemannian symmetric space (called a *symmetric R-space*) of a maximal compact subgroup of $G(D)$. Let r be the rank of M^-.

Proposition 1.7. (K[5]) *Under the above situation, there exists a* $3r$*-dimensional graded subalgebra* $\mathfrak{a}$ *of the GLA* $\mathfrak{g}$*:*

$$\mathfrak{a} = \mathfrak{a}_{-1} + \mathfrak{a}_0 + \mathfrak{a}_1$$

satisfying the following conditions

(i) $\mathfrak{a}$ *is a direct sum of pairwise commutative* $\mathfrak{sl}(2,\mathbb{R})$*-triples* $\langle E_{-i}, \check{\beta}_i, E_i\rangle$, $1 \leq i \leq r$, *where* $E_{-i} = -\tau(E_i)$.

(ii) $\mathfrak{a}_{\pm 1} = \sum_{i=1}^{r} \mathbb{R}E_{\pm i}$, $\mathfrak{a}_0 = \sum_{i=1}^{r} \mathbb{R}\check{\beta}_i$.

(iii) (Sylvester's law of inertia)
Let

$$o_{p,q} := \sum_{i=1}^{p} E_i - \sum_{j=p+1}^{p+q} E_j \in \mathfrak{a}_1 \subset \mathfrak{g}_1, \quad 0 \leq p+q \leq r,$$

and let us consider the $G_0(D)$*-orbits* $\Omega_{p,q} := G_0(D)o_{p,q}$ *in* $\mathfrak{g}_1$. *Then we have the* $G_0(D)$*-orbit decomposition:*

$$\mathfrak{g}_1 = \coprod_{p+q \leq r} \Omega_{p,q}.$$

$\Omega_{p,q}$ *is open, if and only if* $p+q = r$. *Furthermore,* $\Omega_{r,0}$ *(*$=-\Omega_{0,r}$*) is a Riemannian symmetric convex cone, which are the non-compact dual of* M^-. $\Omega_{i,r-i}$ $(i \neq 0, r)$ *are affine symmetric non-convex cones.*

Example 1.8. Let $D = \{Z = X + iY : X, Y \in H(r,\mathbb{R}), Y > 0\}$ be the Siegel upper half-plane of degree r, where $H(r,\mathbb{R})$ denotes the space of real symmetric matrices of degree r. Then one has $G(D) = \mathrm{Sp}(r,\mathbb{R})$, $\mathfrak{g} = \mathfrak{sp}(r,\mathbb{R})$, $\mathfrak{g}_0 = \mathfrak{gl}(r,\mathbb{R})$, $\mathfrak{g}_{\pm 1} = H(r,\mathbb{R})$, $G_0(D) = \mathrm{GL}(r,\mathbb{R})$, $U^-(D) = \mathrm{GL}(r,\mathbb{R}) \exp H(r,\mathbb{R})$, $M^- = G(D)/U^-(D) = U(r)/O(r)$. Furthermore we have that $o_{p,q} = \mathrm{diag}(I_p, I_q, 0)$, where I_k denotes the unit matrix of degree k. Hence the open $G_0(D)$-orbits are given by

$$\Omega_{i,r-i} = \mathrm{GL}(r,\mathbb{R})/O(i, r-i) = H_{i,r-i}(r,\mathbb{R}),$$

where $H_{i,r-i}(r,\mathbb{R})$ denotes the subset of $H(r,\mathbb{R})$ consisting of elements of signature $(i, r-i)$. We also have the equality $\Omega_{p,q} = H_{p,q}(r,\mathbb{R})$, $0 \leq p+q < r$. Note that the linear automorphism group $\mathrm{Aut}(H_{r,0}(r,\mathbb{R}))$ is $G_0(D)(= \mathrm{GL}(r,\mathbb{R}))$.

§ 2. Liouville-type theorems for the causal flag manifolds M^-

Let $C := \overline{\Omega}_{r,0}$ be the closure of the symmetric convex cone $\Omega_{r,0} \subset \mathfrak{g}_1$. Then C is a causal cone. Moreover, considering the equality $C = \coprod_{p \leq r} \Omega_{p,0}$, we have that

$$\mathrm{Aut}(C) = G_0(D). \tag{2.1}$$

Let o^- denote the origin of the coset space $M^- = G(D)/U^-(D)$. In view of (1.7), the tangent space $T_{o^-}(M^-)$ can be identified with $\mathfrak{g}_1$. As was done in K[4], one can extend the cone C at o^- to the $G(D)$-invariant causal structure $\mathcal{C}$ on the whole M^- (with model cone C). The following theorem is an analogue of the Liouville theorems for the conformal transformation group of the pseudo-Euclidean spaces.

Theorem 2.1. *Let D be an irreducible bounded symmetric domain of tube type, and let $G(D)$ be the full holomorphic automorphism group of D. Let $M^- = G(D)/U^-(D)$ be the Shilov boundary of D expressed as the flag manifold, and let $\mathcal{C}$ be the $G(D)$-invariant causal structure on M^- with model cone $C := \overline{\Omega}_{r,0}$. Suppose that* $\dim M^- = \dim_{\mathbb{C}} D \geq 3$. *Then we have*

$$\mathrm{Aut}(M^-, \mathcal{C}) = G(D), \tag{2.2}$$

as the transformation groups on M^-. Furthermore, let U be a connected open set in M^-, and let f be a local causal transformation on M^- defined on U. Then f extends to the causal automorphism defined on the whole M^- induced by a unique element $a \in G(D)$.

Proof. (Sketch) By Theorem 1.6, the equivalency of the causal structures with model cone C reduces to that of $G_0(D)$- structures. The latter equivalence problem was perfectly solved by constructing the Cartan connection, by Tanaka [10]. Let ω be the $\mathfrak{g}$-valued left invariant Maurer-Cartan form of $G(D)$. We regard $G(D)$ as the total space of the principal $U^-(D)$-bundle over $M^- = G(D)/U^-(D)$. Then ω is the Cartan connection corresponding to the $G_0(D)$-structure $Q(\mathcal{C})$. In more detail, under the dimension assumption of M^-, there is a one-to-one correspondence between (local) automorphisms of the $G_0(D)$-structure $Q(\mathcal{C})$ on M^- and (local) bundle automorphisms of the principal bundle $G(D) \to M^-$ leaving ω invariant (Tanaka [10]). Thus the problem of extending the local causal transformation f is reduced to that of extending a local diffeomorphism on $G(D)$ leaving ω invariant. This is solved by using the classical result (e.g. Sternberg [8]) on the extension of local automorphisms of the Maurer-Cartan structure $(G(D), \omega)$. □

Remark. The relation (2.2) was first obtained in Kaneyuki [4], by a slightly different method. The crucial part of the proof is to show that the linear isotropy group of $\mathrm{Aut}(M^-, \mathcal{C})$ at the origin o^- is $G_0(D)$.

Corollary 2.2. *Let N be a connected open submanifold of $(M^-, \mathcal{C})$, and let $\mathcal{C}_N$ be the restriction of $\mathcal{C}$ to N. Suppose that* $\dim M^- \geq 3$. *Then we have*

$$\mathrm{Aut}(N, \mathcal{C}_N) = \{g \in \mathrm{Aut}(M^-, \mathcal{C}) = G(D) : gN = N\} \tag{2.3}$$

§ 3. Linearity of causal automorphisms of symmetric cones $\Omega_{i,r-i}$

For simplicity, we will denote an arbitrary symmetric cone $\Omega_{i,r-i}, (0 \le i \le r)$ by Ω. Let us recall the causal cone $C = \overline{\Omega}_{r,0}$ in $\mathfrak{g}_1$ with vertex at 0. Attaching the cone $C_X := C + X$ to an arbitrary point $X \in \mathfrak{g}_1$, we obtain a parallel cone field

$$\mathcal{C}_{\mathfrak{g}_1} := \{C_X\}_{X \in \mathfrak{g}_1},$$

which is the causal structure on $\mathfrak{g}_1$ with model cone C.

Now we need the other parabolic subgroup $U^+(D)$. We consider the affine representation of $U^+(D)$ on $\mathfrak{g}_1$. Choose an element $u = (\exp A)g \in (\exp \mathfrak{g}_1)G_0(D) = U^+(D)$. For each point $X \in \mathfrak{g}_1$, we define the image point $u \cdot X$ as

$$u \cdot X := (\operatorname{Ad} g)X + A.$$

Under this action, $\mathfrak{g}_1$ is expressed as the homogeneous space

$$\mathfrak{g}_1 = U^+(D)/G_0(D).$$

The causal structure $\mathcal{C}_{\mathfrak{g}_1}$ is $U^+(D)$-invariant. Moreover $U^+(D)$ coincides with the *affine causal automorphism group* $\operatorname{Aff}(\mathfrak{g}_1, \mathcal{C}_{\mathfrak{g}_1})$, that is, the group of all affine transformations of $\mathfrak{g}_1$ leaving $\mathcal{C}_{\mathfrak{g}_1}$ invariant. The restriction of $\mathcal{C}_{\mathfrak{g}_1}$ to Ω is denoted by $\mathcal{C}_\Omega$, which is a $G_0(D)$-invariant causal structure on Ω with model cone C.

Next we consider the map ξ of $\mathfrak{g}_1$ into M^- defined by

$$\xi(X) = (\exp X)o^-, \ X \in \mathfrak{g}_1 .$$

It is well-known that ξ is an open dense embedding of $\mathfrak{g}_1$ into M^-. Furthermore ξ is $U^+(D)$-equivariant. Identifying $\mathfrak{g}_1$ and Ω with their ξ-images, we have the following

Lemma 3.1. *$\mathcal{C}_{\mathfrak{g}_1}$ and $\mathcal{C}_\Omega$ coincides with the restrictions of $\mathcal{C}$ to $\mathfrak{g}_1$ and Ω, respectively, that is, $(\mathfrak{g}_1, \mathcal{C}_{\mathfrak{g}_1})$ and $(\Omega, \mathcal{C}_\Omega)$ are open causal submanifolds of $(M^-, \mathcal{C})$.*

Therefore, as an immediate consequence of Corollary 2.2, we have

Lemma 3.2. *Suppose that* $\dim \mathfrak{g}_1 = \dim \Omega \ge 3$. *Then we have*

$$\begin{aligned} \operatorname{Aut}(\mathfrak{g}_1, \mathcal{C}_{\mathfrak{g}_1}) &= \{f \in G(D) : f(\mathfrak{g}_1) = \mathfrak{g}_1\}, \\ \operatorname{Aut}(\Omega, \mathcal{C}_\Omega) &= \{f \in G(D) : f(\Omega) = \Omega\}. \end{aligned} \tag{3.1}$$

In the next theorem, we determine the causal automorphism groups of symmetric cones, which turn out to be one and the same linear group, independent of i.

Theorem 3.3. *Let Ω be any one of the symmetric cones $\Omega_{i,r-i}$ $(0 \leq i \leq r)$ in $\mathfrak{g}_1$. Suppose that* $\dim \mathfrak{g}_1 = \dim \Omega \geq 3$. *Then we have*

$$\begin{aligned} &\mathrm{Aut}(\mathfrak{g}_1, \mathcal{C}_{\mathfrak{g}_1}) = U^+(D) = \mathrm{Aff}(\mathfrak{g}_1, \mathcal{C}_{\mathfrak{g}_1}), \\ &\mathrm{Aut}(\Omega, \mathcal{C}_\Omega) = G_0(D). \end{aligned}$$

Proof. (Sketch) We will only give the outline of the proof of the second equality. Put $L := \mathrm{Aut}(\Omega, \mathcal{C}_\Omega) \subset G(D)$, and let $\mathfrak{l} := \mathrm{Lie}\, L$. $\mathfrak{l}$ has the structure of a graded subalgebra of $\mathfrak{g}$. We then conclude that $\mathfrak{l}$ is either one of the parabolic subalgebras $\mathfrak{u}^{\pm} = \mathfrak{g}_0 + \mathfrak{g}_{\pm 1}$ or the Levi-subalgebra $\mathfrak{g}_0$. It is easy to see that the case $\mathfrak{l} = \mathfrak{u}^+$ is excluded. Also the case $\mathfrak{l} = \mathfrak{u}^-$ is excluded by using Takeuchi [9]. The case $\mathfrak{l} = \mathfrak{g}_0$ actually occurs. $\square$

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